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THE SOLUTION OF A SPECIAL ARITHMETIC EQUATION

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ABSTRACT. Writing $\phi^l(n)=\phi(\phi^{l-1}(n))$, the integers n which are solutions of $n=k\phi^l(n)$ are considered. The set of all such n is completely characterized.

1. **Introduction.** If $\phi(n)$ denotes the Euler function, it can easily be shown that the only solutions of the equation

$$n=k\phi(n) \tag{1.1}$$

for some k, an integer, are $n=2^\alpha$ for $k=2$, and $n=2^\alpha\cdot 3^\beta$ for $k=3$. In this note a generalization of this result is obtained. Iterating the Euler ϕ-function, we write $\phi^l(n)=\phi(\phi^{l-1}(n))$, and for a fixed l we investigate the integers n which are solutions of

$$n=k\phi^l(n) \tag{1.2}$$

for some integer k.

Letting $S(l)$ denote the set of all solutions for some k, of (1.2), a solution is called *primitive* if

(i) $n\in S(l)$
(ii) n cannot be written as $n=2^\alpha\hat{n}$, $\alpha>0$, where $\hat{n}\in S(l)$.

The set of all primitive solutions is denoted by $P(l)$.

It is proved that apart from a finite number of exceptions, which are determined, the set $P(l)$ consists of integers of the form $n=2\cdot 3^\beta$, $\beta>l$. A lemma establishing the relationship between the sets $P(l)$ and $S(l)$ then provides us with all the solutions. In particular, it easily follows that $P(1)=\{1,2,3,2\cdot 3^\beta,\beta>1\}$ and $S(1)=\{2^\alpha, 2^\alpha\cdot 3^\beta\}$ which is precisely the known result for (1.1).

2. **Preliminary Lemmas.** The relationship between the sets $P(l)$ and $S(l)$ is provided by

LEMMA 2.1.

$$S(l)=\bigcup_{\alpha=0}^{\infty} 2^\alpha P(l) \tag{2.1}$$

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Proof. If $n \in P(l)$, then for some $k = k^*$

$$2^\alpha n = 2^\alpha k^* \phi^l(n). \tag{2.2}$$

Since $\phi^l(2^\alpha n) = 2^\nu \phi^l(n)$ where $\nu \le \alpha$, (2.1) implies that $2^\alpha n$ is also a solution of (1.2), with $k = 2^{\alpha-\nu}k^*$. Thus

$$\bigcup_{\alpha=0}^{\infty} 2^\alpha P(l) \subset S(l). \tag{2.3}$$

The inclusion of $S(l)$ in the union on the right of (2.1) is clear, and (2.1) follows from (2.3).

By Lemma 2.1, it suffices to consider only the primitive solutions of (1.2). In order to describe these solutions we introduce the following known definition [1].

Since $\phi(n)$ is defined as the number of positive integers not exceeding n which are relatively prime to n, it follows that for $n > 1$, $\phi(n) < n$, which in turn implies that if $\phi^{l-1}(n) > 1$,

$$\phi^l(n) < \phi^{l-1}(n). \tag{2.4}$$

If $n = \prod p_i^{\alpha_i}$ then $\phi(n) = \prod p_i^{\alpha_i - 1}(p_i - 1)$, and clearly for $n > 2$, $\phi(n)$ is even. Thus from (2.4) it follows that by making l large enough we must arrive at

$$\phi^l(n) = 2. \tag{2.5}$$

When (2.5) holds we say n is of class l and write this as $C(n) = l$, where we define $C(1) = C(2) = 0$.

The proof of the main theorem relies on the following known results which are stated here without proofs, [1].

LEMMA 2.2.

$$C(xy) = C(x) + C(y) + \varepsilon(xy) \tag{2.6}$$

where

$$\varepsilon(xy) = \begin{cases} 1 & \text{if } 2 \mid x \text{ and } 2 \mid y \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.3. *For two positive integers x and y, if x is a multiple of y then*

$$\phi(xy) = y\phi(x). \tag{2.7}$$

3. Main Result.

THEOREM 3.1. *The set of primitive solutions of* (1.2) *consists of*

(i) *odd n, where $C(n) < l$*
(ii) *$n = 2s$ where s is odd and $C(n) = C(s) = l$*
(iii) *$n = 2 \cdot 3^\beta$ where $\beta > l$ and $C(n) > l$.*

Proof. To establish (i) and (ii) note that $C(n)<l$ implies $\phi^l(n)=1$, while $C(n)=l$ implies $\phi^l(n)=2$. Substituting this into (1.2) we have $n=k$ or $n=2k$ respectively, which gives rise to the primitive solutions as described above.

This leaves only the possibility that $C(n)>l$. In this case, n is even since $\phi^l(n)$ is even. If $n=2m$ where m is even, (2.6) gives $C(n)=C(2)+C(m)+1>l$, or $C(m)\geq l$, and thus by (2.7) $\phi^l(n)=2\phi^l(m)$. Inserting this into (1.2) yields

$$2m=2k\phi^l(m),$$

which implies that m is also a solution of (1.2), contradicting the fact that n is primitive. Thus in this case we may assume that $n=2n'$ where $(n',2)=1$ and $C(2n')>l$, and that (1.2) is equivalent to

$$2n'=k\phi^l(2n') \tag{3.1}$$

where k must be odd and $2\parallel\phi^l(2n')$.

If

$$2n'=2\prod_{\substack{i=1\\p_i\neq 2}}^{r} p_i^{\alpha_i} \quad\text{then}\quad \phi(2n')=\prod_{\substack{i=1\\p_i\neq 2}}^{r} p_i^{\alpha_i-1}(p_i-1),$$

and clearly 2^r divides $\phi(2n')$. Writing $\phi(2n')=2^r\cdot T$, we analyze (3.1) according to the parity of T.

If $T=2T'$ then $\phi(2n')=2^r\cdot 2T'$ and

$$\phi^l(2n')=\begin{cases}2^r\phi^{l-1}(2T') & \text{if}\quad \phi^{l-1}(2T')>1\\ 2^\nu,\ \nu\leq r & \text{if}\quad \phi^{l-1}(2T')=1\end{cases}. \tag{3.2}$$

This follows by repeated application of the fact that for m even, $\phi(2^rm)=2^r\phi(m)$, (which is itself a consequence of Lemma 2.3).

Inserting (3.2) in (3.1) yields

$$2n'=k\phi^l(2n')=\begin{cases}2^rk\phi^{l-1}(2T') & \text{if}\quad \phi^{l-1}(2T')>1\\ 2^\nu k,\ \nu\leq r & \text{if}\quad \phi^{l-1}(2T')=1\end{cases}. \tag{3.3}$$

From (3.3) we see that if $\phi^{l-1}(2T')>1$ then $2\mid\phi^{l-1}(2T')$ so that $r=0$, $n'=1$ and $C(2n')=0$, which contradicts $C(2n')>l\geq 0$. On the other hand if $\phi^{l-1}(2T')=1$ it follows that $\nu=1$, implying that $\phi^l(2n')=2$ or $C(2n')=l$, again a contradiction. Thus T cannot be even.

For T odd, we have $\phi(2n')=2^r\cdot T=2^{r-1}\cdot 2T$ and

$$\phi^l(2n')=\begin{cases}2^{r-1}\phi^{l-1}(2T) & \text{if}\quad \phi^{l-1}(2T)>1\\ 2^\nu,\ \nu\leq r-1 & \text{if}\quad \phi^{l-1}(2T)=1\end{cases}. \tag{3.4}$$

Inserting (3.4) in (3.1) yields

$$2n'=k\phi^l(2n')=\begin{cases}2^{r-1}k\cdot\phi^{l-1}(2T) & \text{if}\quad \phi^{l-1}(2T)>1\\ 2^\nu k,\ \nu\leq r-1 & \text{if}\quad \phi^{l-1}(2T)=1\end{cases}. \tag{3.5}$$

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From (3.5) it follows that if $\phi^{l-1}(2T)>1$, $r=1$ so that $n'=p^\alpha$ and (3.1) becomes

$$2p^\alpha = k\phi^l(2p^\alpha). \tag{3.6}$$

If $p \nmid \phi^l(2p^\alpha)$ then $\phi^l(2p^\alpha)=2$ implying $C(2p^\alpha)=l$, which is not this case. If $p \mid \phi^l(2p^\alpha)$, it follows that $\alpha>l$ and since $\phi^l(2p^\alpha)=p^{\alpha-l}(p-1)\cdot M$ where $(M, p)=1$, (3.6) implies that $M=1$ and $p-1=2$ or $p=3$. Thus the primitive solutions in this case are $n=2\cdot 3^\beta$ where $\beta>l$. However, if $\phi^{l-1}(2T)=1$, then (3.5) implies that $\nu=1$ and $\phi^l(2n')=2$ or $C(2n')=l$, again not this case, and the proof of the theorem is completed.

Thus in conclusion we see that apart from the finite number of exceptions for which $C(n)\le l$, the set $P(l)$ consists only of the integers $n=2\cdot 3^\beta$ where $\beta>l$.

In particular, for $l=1$, 2 we have $P(1)=\{1, 2, 3, 2\cdot 3^\beta, \beta>l\}$ and $P(2)=\{1, 3, 2\cdot 5, 2\cdot 7, 2\cdot 3^2, 2\cdot 3^\beta, \beta>2\}$, and the corresponding solution sets are $S(1)=\{2^\alpha, 2^\alpha\cdot 3^\beta\}$ and $S(2)=\{2^\alpha, 2^{\alpha+1}\cdot 5, 2^{\alpha+1}\cdot 7, 2^\alpha\cdot 3^\beta\}$.

Bibliography

1. Harold N. Shapiro, *An Arithmetic Function Arising from the ϕ Function*, American Mathematical Monthly, Vol. L, No. 1, 1943 pp. 18–30.

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