Bull. Aust. Math. Soc. **94** (2016), 155–164
doi:10.1017/S0004972715001859

THE GREEN–OSHER INEQUALITY IN RELATIVE GEOMETRY

YUNLONG YANG✉ and DEYAN ZHANG

(Received 19 October 2015; accepted 17 December 2015; first published online 17 February 2016)

Abstract

In this paper we give a proof of the Green–Osher inequality in relative geometry using the minimal convex annulus, including the necessary and sufficient condition for the case of equality.

2010 *Mathematics subject classification*: primary 52A40; secondary 52A38, 52A10.

Keywords and phrases: Bonnesen-style inequalities, Green–Osher inequalities, minimal convex annulus, relative geometry, Steiner polynomials.

1. Introduction

We denote by $\mathbb{R}^n$ the usual n-dimensional Euclidean space with the canonical inner product $\langle\cdot\,,\cdot\rangle$. A bounded closed convex set K in $\mathbb{R}^n$ is called a *convex body* if it has nonempty interior. When $n = 2$, it is called a *convex domain*. The volume of a set $M \subset \mathbb{R}^n$ is denoted by $V(M)$. The *Minkowski sum* of convex bodies K and L, and the *Minkowski scalar product* of K for $t > 0$ are, respectively, defined by

$$K + L = \{x + y \mid x \in K, y \in L\}$$

and

$$tK = \{tx \mid x \in K\}.$$

Minkowski found the following fundamental formula: the volume of the linear combination of convex bodies $K_1, \ldots, K_m$ with nonnegative coefficients $t_1, \ldots, t_m$ is a homogeneous polynomial of degree n with respect to $t_1, \ldots, t_m$, that is,

$$V(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1,\ldots,i_n=1}^{m} V(K_{i_1}, \ldots, V_{K_{i_n}})t_{i_1} \cdots t_{i_n}. \tag{1.1}$$

The coefficient $V(K_{i_1}, \ldots, K_{i_n})$ is the *mixed volume* of $K_{i_1}, \ldots, K_{i_n}$, and it is nonnegative and symmetric in the indices and dependent only on $K_{i_1}, \ldots, K_{i_n}$. For a convex body K

© 2016 Australian Mathematical Publishing Association Inc. 0004-9727/2016 $16.00

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

and the n-dimensional unit ball B_n, the *Steiner polynomial* is a special case of (1.1):

$$V(K + tB_n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i. \tag{1.2}$$

The coefficient $W_i(K)$ is called the ith *quermassintegral*, and it is the mixed volume of $n - i$ copies of K and i copies of B_n. Similar to (1.2), for a fixed convex body E, the volume of the Minkowski sum $K + tE$ gives the *relative Steiner polynomial* of K with respect to E:

$$V(K + tE) = \sum_{i=0}^{n} \binom{n}{i} W_i(K, E) t^i, \tag{1.3}$$

where the coefficient $W_i(K, E)$ is called the ith *relative quermassintegral of K with respect to E*.

The (relative) Steiner polynomial appears in many problems. In dimension three, Hernández Cifre and Saorín [11] discussed *the missing boundary of the Blaschke diagram* through the locations of the roots of the Steiner polynomial (1.2) for $n = 3$. More detailed results on the locations of the roots of the (relative) Steiner polynomial can be found in [10, 12]. Bonnesen-style inequalities are discussed in [14, 17].

Let K be a convex domain with perimeter L and area A and let r_{in} and r_{out} be the inradius and outradius of K, respectively. The *Bonnesen inequality* (see [1, 2]) is

$$A - Ls + \pi s^2 \leq 0, \quad s \in [r_{\text{in}}, r_{\text{out}}]. \tag{1.4}$$

Using this and symmetrisation, Gage [4] successfully proved an inequality for the total squared curvature for convex curves. Following his work, Green and Osher [8] obtained a generalised formula with respect to the curvature of all C^2 convex curves in the plane. These inequalities play a critical role in the curve evolution problem (see, for example, [5, 13]). For a fixed convex domain E, Böröczky *et al.* [3] rediscovered the generalised case of (1.4) in relative geometry, that is,

$$A_K - 2W(K, E)s + A_E s^2 \leq 0, \quad s \in [R_{\text{in}}, R_{\text{out}}], \tag{1.5}$$

where A_K and A_E are the areas of K and E, $W(K, E)$ is the relative quermassintegral of K with respect to E and R_{in} and R_{out} are the inradius and outradius of K with respect to E. Equality occurs in (1.5) when $s = R_{\text{in}}$ if and only if K is the Minkowski sum of a dilation of E and a line segment, and equality in (1.5) holds when $s = R_{\text{out}}$ if and only if E is the Minkowski sum of a dilation of K and a line segment. Peri *et al.* [15] proved a stronger result:

$$A_K - 2W(K, E)s + A_E s^2 \leq 0, \quad s \in [R_{\text{in}}(x_0), R_{\text{out}}(x_0)], \tag{1.6}$$

where x_0 is the centre of the minimal convex annulus of K with respect to E. (The definitions of $R_{\text{in}}(x_0)$ and $R_{\text{out}}(x_0)$ can be found in Section 2.) Inequalities which contain only support functions led to further advances in the curve evolution problem in relative geometry (see [6, 7]) and the log-Brunn–Minkowski problem (cf. [3]).

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

In this paper, inspired by the impressive work in [15], we give a simplified proof of the Green–Osher inequality in relative geometry using the minimal convex annulus, including the necessary and sufficient condition for the case of equality. In Section 2, we present some basic concepts about convex domains. In Section 3, we derive the Green–Osher inequality in relative geometry.

2. Preliminaries

Let K be a convex domain. A line l is called a *support line* of K if it passes through at least one boundary point of K and if the entire convex domain K lies on one side of l. Take a point O inside K as the origin of our frame. Let $l(\theta)$ be the support line of K in the direction $\mathbf{u}(\theta) = (\cos\theta, \sin\theta)$, where θ is the oriented angle from the positive x-axis to the perpendicular line of $l(\theta)$. The *support function* of K is defined to be

$$p(\theta) = \sup_{x\in K}\langle x, \mathbf{u}(\theta)\rangle, \quad \mathbf{u}(\theta) \in S^1.$$

It is easy to see that $p(\theta)$ is the signed distance of the support line $l(\theta)$ of K with exterior normal vector $\mathbf{u}(\theta)$ from the origin. Clearly, p, as a function of θ, is single-valued and 2π-periodic. For a fixed convex domain E with support function $\gamma(\theta)$, if $p(\theta)$ and $\gamma(\theta)$ are continuously differentiable, then

$$W(K, E) = \frac{1}{2}\int_0^{2\pi} (p(\theta)\gamma(\theta) - p'(\theta)\gamma'(\theta))\, d\theta.$$

If $p(\theta)$ and $\gamma(\theta)$ are C^2, then

$$W(K, E) = \frac{1}{2}\int_0^{2\pi} p(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = \frac{1}{2}\int_0^{2\pi} \gamma(\theta)(p(\theta) + p''(\theta))\, d\theta. \tag{2.1}$$

The *relative curvature* of K with respect to E is given by

$$\kappa(\theta) = \frac{\gamma(\theta) + \gamma''(\theta)}{p(\theta) + p''(\theta)}$$

and the *relative curvature radius* of K with respect to E is

$$\rho(\theta) = \frac{p(\theta) + p''(\theta)}{\gamma(\theta) + \gamma''(\theta)}.$$

For $n = 2$, (1.3) turns into

$$A(K + tE) = A_K + 2W(K, E)t + A_E t^2, \quad t \geq 0.$$

From the mixed area inequality, $W(K, E)^2 - A_K A_E \geq 0$. Denote by t_1, t_2 ($t_1 \geq t_2$) the two roots of the relative Steiner polynomial of K with respect to E, that is,

$$t_1 = -\frac{W(K, E)}{A_E} + \frac{\delta}{A_E}, \quad t_2 = -\frac{W(K, E)}{A_E} - \frac{\delta}{A_E},$$

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

where $\delta = \sqrt{W(K,E)^2 - A_K A_E} \geq 0$. Let

$$R_{\rm in} = \max\{r > 0 \mid x + rE \subseteq K, \exists\, x \in K\}$$

and

$$R_{\rm out} = \min\{r > 0 \mid x + rE \supseteq K, \exists\, x \in K\}$$

be the *inradius* and *outradius* of K with respect to E, respectively. For $x \in K$, set

$$R_{\rm in}(x) = \max\{r \geq 0 \mid x + rE \subseteq K\}$$

and

$$R_{\rm out}(x) = \min\{r > 0 \mid x + rE \supseteq K\}.$$

The *convex annulus of centre* x is defined by

$$A_x(E) = \{y \in \mathbb{R}^2 \mid y \in x + R_{\rm out}(x)E \text{ and } y \notin \mathrm{int}(x + R_{\rm in}(x)E)\}.$$

When the convex annulus $A_x(E)$ contains K and $R_{\rm out}(x) - R_{\rm in}(x)$ attains its minimum, the corresponding convex annulus is called the *minimal convex annulus* of K with respect to E. If E is smooth and strictly convex, then the minimal convex annulus of K with respect to E has a unique centre (see [16]) and the centre is denoted by x_0.

Definition 2.1 [8]. Consider

$$\sup\left\{\int_I \rho(\theta)\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta \,\middle|\, I \subset S^1, \int_I \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = A_E\right\}.$$

Let I_1 denote the smallest subset of S^1 with measure A_E and realising the above supremum, and let I_2 be its complement. There exists an $a \in \mathbb{R}^+$ such that

$$I_1 \subseteq \{\theta \mid \rho(\theta) \geq a\}, \quad I_2 \subseteq \{\theta \mid \rho(\theta) \leq a\}.$$

Set

$$\rho_i = \frac{1}{A_E}\int_{I_i} \rho(\theta)\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta, \quad i = 1, 2.$$

Then

$$\rho_1 + \rho_2 = \frac{2W(K,E)}{A_E} \quad \text{and} \quad \rho_1 \geq \rho_2$$

and there is a $b \geq 0$ such that

$$\rho_1 = \frac{W(K,E)}{A_E} + b \quad \text{and} \quad \rho_2 = \frac{W(K,E)}{A_E} - b.$$

3. The Green–Osher inequality in relative geometry

We will provide a different proof of the Green–Osher inequality in relative geometry, using the next proposition and the method of [15].

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

PROPOSITION 3.1. *Let K, E be two convex domains with E symmetric. If K, E are smooth and strictly convex, and K, E are not homothetic, then*

$$-t_1 < R_{\text{in}}(x_0) < \frac{W(K,E)}{A_E} < R_{\text{out}}(x_0) < -t_2,$$

where x_0 is the centre of the minimal convex annulus of K with respect to E.

To prove the above proposition, we need the following lemma, which is a direct consequence of [15, Lemmas 1 and 2].

LEMMA 3.2. *Let K, E be two smooth and strictly convex domains with E symmetric and let x_0 be the centre of the minimal convex annulus of K with respect to E. If $a, b \in \partial K \cap \partial(x_0 + R_{\text{in}}(x_0)E)$ and $A, B \in \partial K \cap \partial(x_0 + R_{\text{out}}(x_0)E)$ are such that the intersection of the segments $[a, b]$ and $[A, B]$ is not empty, then there exists a line l satisfying:*

(i) *$l \cap K$ is a line segment with x_0 as its midpoint;*
(ii) *the points a and b lie on different sides of l, and so do A and B.*

PROOF OF PROPOSITION 3.1. Let $p(\theta)$ and $\gamma(\theta)$ be the support functions of K and E. If K is centrally symmetric, then $R_{\text{in}} = R_{\text{in}}(x_0)$ and $R_{\text{out}} = R_{\text{out}}(x_0)$. It follows from (1.5) that

$$A_K - 2W(K,E)R_{\text{in}}(x_0) + A_E R_{\text{in}}^2(x_0) = A_K - 2W(K,E)R_{\text{in}} + A_E R_{\text{in}}^2 < 0$$

and

$$A_K - 2W(K,E)R_{\text{out}}(x_0) + A_E R_{\text{out}}^2(x_0) = A_K - 2W(K,E)R_{\text{out}} + A_E R_{\text{out}}^2 < 0,$$

which implies the result.

Suppose that K is not centrally symmetric. By Lemma 3.2, there exists a line l through x_0 such that $l \cap K$ is a segment with midpoint x_0 and the pairs a, b and A, B lie in different regions l^+, l^-, where l^+ and l^- are two closed half-planes separated by l (with the points a, b, A, B as in Lemma 3.2). Suppose that l cuts K into the two regions K^+, K^-, respectively lying in l^+, l^-. Consider the two regions K_1 and K_2 obtained from K^+ and K^- by a symmetry with centre x_0. As K_1 and K_2 are not necessarily convex, denote the convex hulls of K_1 and K_2 by K_1' and K_2', with support functions $p_1(\theta)$ and $p_2(\theta)$, respectively. By the symmetrisation procedure, it is clear that $R_{\text{in}}(x_0)$ and $R_{\text{out}}(x_0)$ are the same for K, K_1' and K_2'. For $i = 1, 2$,

$$A_{K_i'} - 2W(K_i', E)s + A_E s^2 < 0, \quad s \in [R_{\text{in}}(x_0), R_{\text{out}}(x_0)]. \tag{3.1}$$

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

Let $w(\theta)$, $w_1(\theta)$ and $w_2(\theta)$ be the width functions of K, K_1' and K_2'. Since K_1' and K_2' are symmetric with respect to x_0, $w_i(\theta) = 2p_i(\theta)$ for $i = 1, 2$. It follows from the construction of K_1' and K_2' that $p_1(\theta) + p_2(\theta) \le w(\theta)$ (cf. [15, page 353 (6)]) and, then, from the symmetry of W and (2.1),

$$W(K, E) = \frac{1}{2}\int_0^{2\pi} p(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta = \frac{1}{4}\int_0^{2\pi} w(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta.$$

Thus,

$$\begin{aligned} W(K_1', E) + W(K_2', E) &= \frac{1}{4}\int_0^{2\pi} (w_1(\theta) + w_2(\theta))(\gamma(\theta) + \gamma''(\theta))\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (p_1(\theta) + p_2(\theta))(\gamma(\theta) + \gamma''(\theta))\,d\theta \\ &\le \frac{1}{2}\int_0^{2\pi} w(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta = 2W(K, E). \end{aligned}$$

Together with (3.1) and the fact $A_{K_1'} + A_{K_2'} \ge A_{K_1} + A_{K_2} = 2A_K$, this yields

$$A_K - 2W(K, E)s + A_E s^2 < 0, \qquad s \in [R_{\text{in}}(x_0), R_{\text{out}}(x_0)].$$

Hence, $-t_1 < R_{\text{in}}(x_0) < W(K, E)/A_E < R_{\text{out}}(x_0) < -t_2$. □

The next proposition also plays a role in the proof of the Green–Osher inequality in relative geometry. We deal with it by means of the minimal convex annulus.

Proposition 3.3. *If K, E are two smooth and strictly convex domains and E is symmetric, then*

$$\rho_1 \ge -t_2. \tag{3.2}$$

Moreover, if K and E are not homothetic, then

$$\rho_1 > -t_2. \tag{3.3}$$

Proof. Let $p(\theta)$ and $\gamma(\theta)$ be the support functions of K and E. It is well known that the centre, x_0, of the minimal convex annulus of K with respect to E is unique when E is smooth and strictly convex (cf. [16]). From (1.6) and the mixed area inequality $W(K, E)^2 - A_K A_E \ge 0$, it follows that

$$-t_1 \le R_{\text{in}}(x_0) \le R_{\text{out}}(x_0) \le -t_2.$$

Choose x_0 as the origin; then $R_{\text{in}}(x_0)\gamma(\theta) \le p(\theta) \le R_{\text{out}}(x_0)\gamma(\theta)$, which implies that

$$-\frac{\delta}{A_E}\gamma(\theta) \le p(\theta) - \frac{W(K, E)}{A_E}\gamma(\theta) \le \frac{\delta}{A_E}\gamma(\theta), \qquad \delta = \sqrt{W(K, E)^2 - A_K A_E} \ge 0.$$

On I_1, $\rho(\theta) - a \ge 0$. Combined with the above inequality, this yields

$$-\Big(p(\theta) - \frac{W(K, E)}{A_E}\gamma(\theta)\Big)(\rho(\theta) - a) \le \frac{\delta}{A_E}\gamma(\theta)(\rho(\theta) - a).$$

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

By integrating this over the interval I_1,

$$-\frac{1}{A_E}\int_{I_1}\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)(\gamma(\theta)+\gamma''(\theta))\,d\theta \le \frac{\delta}{A_E}(\rho_1-a). \tag{3.4}$$

Similarly, $\rho(\theta)-a\le 0$ on I_2, so

$$-\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)\le -\frac{\delta}{A_E}\gamma(\theta)(\rho(\theta)-a)$$

and, integrating this over the interval I_2, gives

$$-\frac{1}{A_E}\int_{I_2}\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)(\gamma(\theta)+\gamma''(\theta))\,d\theta \le -\frac{\delta}{A_E}(\rho_2-a). \tag{3.5}$$

From (3.4) and (3.5),

$$-\frac{1}{A_E}\int_0^{2\pi}\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)(\gamma(\theta)+\gamma''(\theta))\,d\theta \le \frac{2b\delta}{A_E}.$$

The left-hand side can be simplified to

$$\frac{2(W(K,E)^2-A_KA_E)}{A_E^2}=\frac{2\delta^2}{A_E^2};$$

thus, $b\ge \delta/A_E\ge 0$, that is, $\rho_1\ge -t_2$.

If K and E are not homothetic, by Proposition 3.1,

$$-t_1<R_{\text{in}}(x_0)<R_{\text{out}}(x_0)<-t_2.$$

Since $R_{\text{in}}(x_0)\gamma(\theta)\le p(\theta)\le R_{\text{out}}(x_0)\gamma(\theta)$,

$$-\frac{\delta}{A_E}\gamma(\theta)<p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)<\frac{\delta}{A_E}\gamma(\theta),\quad \delta=\sqrt{W(K,E)^2-A_KA_E}>0.$$

For I_1 and I_2, $\rho(\theta)\equiv a$ holds on at most one interval, unless K and E are homothetic. Without loss of generality, assume that $\rho(\theta)>a$ on a subinterval I_1' of I_1. On I_1', $\rho(\theta)>a$ and

$$-\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)<\frac{\delta}{A_E}\gamma(\theta)(\rho(\theta)-a).$$

Integrating this expression over the interval I_1 yields

$$-\frac{1}{A_E}\int_{I_1}\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)(\gamma(\theta)+\gamma''(\theta))\,d\theta<\frac{\delta}{A_E}(\rho_1-a),$$

which, together with (3.5), gives

$$-\frac{1}{A_E}\int_0^{2\pi}\Big(p(\theta)-\frac{W(K,E)}{A_E}\gamma(\theta)\Big)(\rho(\theta)-a)(\gamma(\theta)+\gamma''(\theta))\,d\theta<\frac{2b\delta}{A_E}.$$

By a similar argument, $b>\delta/A_E>0$, which implies that $\rho_1>-t_2$. □

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

In order to deal with the equality case of the Green–Osher inequality in relative geometry, we will need the following lemma.

Lemma 3.4. *Let K, E be two smooth and strictly convex domains. If K and E are not homothetic, then*

$$\rho_1 > \rho_2.$$

Proof. By Definition 2.1, $\rho_1 \geq \rho_2$. To prove this lemma, it is enough to prove that K and E are homothetic when $\rho_1 = \rho_2$. If $\rho_1 = \rho_2$, then, for any $I \subset S^1$ and $\int_I \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = A_E$,

$$\int_I \rho(\theta)\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = W(K, E). \tag{3.6}$$

Set

$$A = \left\{\theta \,\middle|\, \rho(\theta) > \frac{W(K,E)}{A_E}\right\}, \quad B = \left\{\theta \,\middle|\, \rho(\theta) < \frac{W(K,E)}{A_E}\right\}, \quad C = S^1 \setminus (A \cup B).$$

Then $\int_A \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta < A_E$ and $\int_B \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta < A_E$.

Next, we have to prove that $A = \emptyset$ and $B = \emptyset$. If $A \neq \emptyset$, then there exists an interval $C' \subset C$ such that $\int_{A\cup C'} \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = A_E$ or $\int_{B\cup C'} \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = A_E$. Without loss of generality, set $\int_{A\cup C'} \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta = A_E$; then

$$\int_{A\cup C'} \rho(\theta)\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta > \frac{W(K,E)}{A_E} m(A) + \frac{W(K,E)}{A_E}(A_E - m(A)) = W(K, E),$$

where $m(A) = \int_A \gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta$, which contradicts (3.6). Similarly, it can be shown that $B = \emptyset$. □

Theorem 3.5. *Let K, E be two smooth and strictly convex domains and E symmetric. If $p(\theta)$ and $\gamma(\theta)$ are the support functions of K and E, $\rho(\theta)$ is the relative curvature radius of K with respect to E and $F(x)$ is a strictly convex function on $(0, +\infty)$, then*

$$\frac{1}{2A_E} \int_0^{2\pi} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta \geq \frac{1}{2}(F(-t_1) + F(-t_2)), \tag{3.7}$$

where t_1 and t_2 are the two roots of the relative Steiner polynomial of K with respect to E, and the equality in (3.7) holds if and only if K and E are homothetic.

Proof. Applying Jensen's inequality to I_i $(i = 1, 2)$ yields

$$\frac{1}{A_E} \int_{I_i} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta \geq F(\rho_i).$$

So,

$$\frac{1}{2A_E} \int_0^{2\pi} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\, d\theta \geq \frac{1}{2}(F(\rho_1) + F(\rho_2)).$$

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

Here, $\rho_1 = W(K,E)/A_E + b, \rho_2 = W(K,E)/A_E - b$ with $b \geq 0$. From (3.2), it follows that $b \geq \delta/A_E \geq 0$. By the convexity of the function $F(x)$ (see [8, Lemma 2.9]),

$$\begin{aligned} F(\rho_1) + F(\rho_2) &= F\Big(\frac{W(K,E)}{A_E} + b\Big) + F\Big(\frac{W(K,E)}{A_E} - b\Big) \\ &\geq F\Big(\frac{W(K,E)}{A_E} + \frac{\delta}{A_E}\Big) + F\Big(\frac{W(K,E)}{A_E} - \frac{\delta}{A_E}\Big) \\ &= F(-t_1) + F(-t_2). \end{aligned}$$

Hence,

$$\frac{1}{2A_E}\int_0^{2\pi} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta \geq \frac{1}{2}(F(-t_1) + F(-t_2)).$$

On the one hand, if K and E are homothetic, it is clear that equality holds in (3.7), since $-t_1 = -t_2 = \rho(\theta)$. On the other hand, to prove that K and E are homothetic when equality holds in (3.7), it is enough to show that, when K and E are not homothetic,

$$\frac{1}{2A_E}\int_0^{2\pi} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta > \frac{1}{2}(F(-t_1) + F(-t_2)).$$

If K and E are not homothetic, $\delta = \sqrt{W(K,E)^2 - A_K A_E} > 0$. By Lemma 3.4, there exists $b > 0$ such that $\rho_1 = W(K,E)/A_E + b$ and $\rho_2 = W(K,E)/A_E - b$. Furthermore, it follows from (3.3) that $b > \delta/A_E > 0$. Again, by the strict convexity of $F(x)$,

$$\begin{aligned} F(\rho_1) + F(\rho_2) &= F\Big(\frac{W(K,E)}{A_E} + b\Big) + F\Big(\frac{W(K,E)}{A_E} - b\Big) \\ &> F\Big(\frac{W(K,E)}{A_E} + \frac{\delta}{A_E}\Big) + F\Big(\frac{W(K,E)}{A_E} - \frac{\delta}{A_E}\Big) \\ &= F(-t_1) + F(-t_2). \end{aligned}$$

Therefore,

$$\frac{1}{2A_E}\int_0^{2\pi} F(\rho(\theta))\gamma(\theta)(\gamma(\theta) + \gamma''(\theta))\,d\theta \geq \frac{1}{2}(F(\rho_1) + F(\rho_2)) > \frac{1}{2}(F(-t_1) + F(-t_2)),$$

which completes the proof. □

Remark 3.6. If $\mathbb{R}^2$ is equipped with a suitable Minkowski metric such that the boundary of E becomes the isoperimetrix of the Minkowski plane, then the Minkowski perimeter, $\mathcal{L}(K)$, of K is given by (cf. [9, page 310])

$$\mathcal{L}(K) = 2W(K,E).$$

Following the notation of [6, (2.6)], set

$$\mathcal{A}(K) = 2A_K \quad \text{and} \quad \alpha = 2A_E.$$

The Minkowski element of arc length $d\sigma$ at a point on the curve ∂K with Minkowski unit tangent can be written as (cf. [6, (2.7)])

$$d\sigma = \gamma(\theta)(p(\theta) + p''(\theta))\,d\theta.$$

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press

With these observations, (3.7) turns into

$$\frac{1}{\alpha}\int_0^{\mathcal{L}} F(\rho(\sigma))\frac{1}{\rho(\sigma)}d\sigma \geq \frac{1}{2}(F(-t_1)+F(-t_2)),$$

which is an inequality in Minkowski geometry.

Acknowledgements

We are grateful to the anonymous referee for a careful reading of the original manuscript of this paper and giving us some invaluable comments. We would also like to thank Professor Shengliang Pan for many suggestions.

References

[1] T. Bonnesen, *Les problèmes des isopérimètres et des isépiphanes* (Gauthier-Villars, Paris, 1929).
[2] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies* (BCS Associates, Moscow, 1987).
[3] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, 'The log-Brunn–Minkowski inequality', *Adv. Math.* **231** (2012), 1974–1997.
[4] M. E. Gage, 'An isoperimetric inequality with applications to curve shortening', *Duke Math. J.* **50** (1983), 1225–1229.
[5] M. E. Gage, 'Curve shortening makes convex curves circular', *Invent. Math.* **76** (1984), 357–364.
[6] M. E. Gage, 'Evolving plane curves by curvature in relative geometries', *Duke Math. J.* **72** (1993), 441–466.
[7] M. E. Gage and Y. Li, 'Evolving plane curves by curvature in relative geometries II', *Duke Math. J.* **75** (1994), 79–98.
[8] M. Green and S. Osher, 'Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves', *Asian J. Math.* **3** (1999), 659–676.
[9] H. Guggenheimer, 'Pseudo-Minkowski differential geometry', *Ann. Mat. Pura Appl.* **70** (1965), 305–370.
[10] M. Henk, M. A. Hernández Cifre and E. Saorín, 'Steiner polynomials via ultra-logconcave sequences', *Commun. Contemp. Math.* **14** (2012), 1250040, 16 pp.
[11] M. A. Hernández Cifre and E. Saorín, 'On the roots of the Steiner polynomial of a 3-dimensional convex body', *Adv. Geom.* **7** (2007), 275–294.
[12] M. Jetter, 'Bounds on the roots of the Steiner polynomial', *Adv. Geom.* **11** (2011), 313–317.
[13] L. S. Jiang and S. L. Pan, 'On a non-local curve evolution problem in the plane', *Comm. Anal. Geom.* **16** (2008), 1–26.
[14] R. Osserman, 'Bonnesen-style isoperimetric inequalities', *Amer. Math. Monthly* **86** (1979), 1–29.
[15] C. Peri, J. M. Wills and A. Zucco, 'On Blaschke's extension of Bonnesen's inequality', *Geom. Dedicata* **4**(8) (1993), 349–357.
[16] C. Peri and A. Zucco, 'On the minimal convex annulus of a planar convex body', *Monatsh. Math.* **114** (1992), 125–133.
[17] J. R. Sangwine-Yager, 'Bonnesen-style inequalities for Minkowski relative geometry', *Trans. Amer. Math. Soc.* **307** (1988), 373–382.

YUNLONG YANG, Department of Mathematics,
Tongji University, Shanghai 200092, PR China
e-mail: 88ylyang@tongji.edu.cn

DEYAN ZHANG, School of Mathematical Sciences,
Huaibei Normal University, Huaibei 235000, PR China
e-mail: zhangdy8005@126.com

https://doi.org/10.1017/S0004972715001859 Published online by Cambridge University Press