ON A METRIC THAT CHARACTERIZES DIMENSION

J. DE GROOT

1. Introduction. Sometimes it is possible to characterize topological properties of a metrizable space M by claiming that a certain (topology-preserving) metric ρ can be introduced in M. For example:

(α) A metrizable space C is compact, that is, is a compactum, if and only if C is totally bounded[1] in every metric.

(β) A metrizable space M is separable, if and only if there exists a totally bounded metric in M.

(γ) A (non-empty) metrizable space M is 0-dimensional (dim $M = 0$), if and only if there exists a metric ρ in M which satisfies—instead of the triangle axiom—the stronger axiom

1.1 $$\rho\,(y, z) \leqslant \max\,[\rho(x, y),\ \rho(x, z)],$$

(that is, every "triangle" in this metric has two equal "sides" and the third "side" is smaller than or equal to the other ones) (see **2**, **3**).

Nagata **(7)** gave a characterization of a metrizable space M of dim $\leqslant n$ (for every non-negative integer n) by means of a certain metric, which he showed to be equivalent with (γ) in the case $n = 0$. However, this characterization (see §**2**) is rather complicated. In this note we give another generalization of (γ) which gives a simplification of Nagata's result for arbitrary dimension n, but only for the case of *separable* metrizable spaces, i.e., metrizable spaces with a countable base.

THEOREM. *A topological space M is a separable metrizable space of dimension $\leqslant n$ if and only if one can introduce a totally bounded metric ρ in M satisfying the following condition: for every $n + 3$ points*

$$x, y_1, y_2, y_3, \ldots y_k, \ldots y_{n+2}$$

in M there is a triplet of indices i, j, k, such that

1.2 $$\rho(y_i, y_j) \leqslant \rho(x, y_k), \qquad (i \neq j).$$

COROLLARY. *A compactum has dimension $\leqslant n$, if and only if one can introduce a metric ρ, such that for every $n + 3$ points $x, y_k (k = 1,2, \ldots, n + 2)$ the relation 1.2 holds for suitable i, j, k.*

Received May 28, 1957.

[1] ϵ-net: A finite number of points p such that the system of ϵ-neighbourhoods cover the space. Totally bounded: there is an ϵ-net for every $\epsilon > 0$. See **(1)** in general for our terminology. See **(4)** for dimension theory in separable metrizable spaces and **(5; 6)** for dimension theory in metrizable spaces.

It has to be observed that condition 1.2 is essentially weaker than the condition which is satisfied by Nagata's metric **(7)** (see also § 2). Indeed, the ordinary metric of a segment of real numbers is a metric ρ with 1.2 (for the case $n = 2$), but does not satisfy Nagata's condition.

2. Proof of Theorem. Suppose M is a separable metric space with $\dim M \leqslant n$. Since M is separable, we can embed M, according to a theorem of Hurewicz, in a compactum $\bar{M}$, such that M is dense in $\bar{M}$, and

$$\dim M = \dim \bar{M} \leqslant n.$$

We introduce in $\bar{M}$ the metric ρ of Nagata **(7)**, which has the following characterizing property: for every $\epsilon > 0$ and for every point $x \in \bar{M}$ the relations[2]

2.1 $$\rho(U_{\frac{1}{2}\epsilon}(x), y_k) < \epsilon \qquad (k = 1, 2, \ldots n + 2),$$

where $U_\delta(x)$ is the set of all points p with $\rho(x, p) < \delta$, imply

2.2 $$\min_{i \neq j} \rho(y_i, y_j) < \epsilon.$$

It is easy to see that this metric ρ in particular satisfies our condition 1.2. Indeed, being given the points x, y_k ($k = 1,2, \ldots n + 2$), consider all ϵ with

$$\epsilon > \mu = \max_k \rho(x, y_k).$$

For these ϵ, 2.1 obviously holds, so 2.2 holds.
Since $\inf \epsilon = \mu$, we have

$$\min_{i \neq j} \rho(y_i, y_j) \leqslant \mu \qquad \text{q.e.d.}$$

Moreover, the metric ρ in the compact space $\bar{M}$ is necessarily totally bounded. Hence the metric ρ of $M \subset \bar{M}$ is also totally bounded and satisfies 1.2, which we had to prove.

Conversely, let M have a totally bounded metric satisfying 1.2. M is clearly separable. We shall now prove that $\dim M \leqslant n$.

M can be extended, just as every metric space, to a complete metric space $\bar{M}$ in which M is dense. Every sequence in M has a Cauchy sequence (fundamental sequence) as subsequence, since M is totally bounded under ρ. This Cauchy sequence converges in the complete $\bar{M}$. Hence $\bar{M}$ is compact and totally bounded under ρ, where ρ now denotes the natural extension of ρ (on M) to $\bar{M}$. Property 1.2 also holds in this extended metric ρ on $\bar{M}$. Indeed, suppose it does not hold for a set of certain points $\bar{x}$, $\bar{y}_k$. Then, since the distance function is continuous, we can determine small neighbourhoods of these points such that 1.2 does not hold for any set of points x, y_k chosen in these neighbourhoods respectively. We can, however, choose these points x, y_k from M, which leads to a contradiction. We shall now prove $\dim \bar{M} \leqslant n$, from which follows $\dim M \leqslant n$.

[2]The distance of the sets A and B is denoted by $\rho(A, B)$.

Consider an arbitrary finite open covering of $\bar{M}$. We have to find—according to the Lebesgue definition of dimension—a refinement of this covering of order $\leqslant n$ (i.e. each point of the refined covering is contained in at most $n+1$ elements of it).

Let $\sigma = 2\epsilon$ be a Lebesgue number of the given finite covering of $\bar{M}$. Choose a maximal set $p_1, p_2, \ldots, p_s$ in $\bar{M}$ such that $\rho(p_i, p_j) \geqslant \epsilon$ for all i, j with $i \neq j$. This set of points $\{p_i\}$ is an ϵ-net of $\bar{M}$ and the covering

2.3 $$\{U_\epsilon(p_i)\} \qquad (i = 1, 2, \ldots s)$$

is a refinement of the given covering. If a point $x \in \bar{M}$ belongs to at least $n+2$ elements of 2.3, we have $\rho(x, p_i) < \epsilon$ for $n+2$ different points p_i. Hence, using 1.2, $\rho(p_i, p_j) < \epsilon$ for suitable i, j with $i \neq j$, which is contradictory to the definition of $\{p_i\}$. Hence, the order of 2.3 is $\leqslant n$, so dim $\bar{M} \leqslant n$.

3. Questions. The corollary admits an immediate generalization to semi-compact[3] metrizable spaces, since we can apply in this case the sum theorem of dimension theory (a metric space which is the countable sum of closed subsets of dimension $\leqslant n$, has dimension $\leqslant n$), while the proof in the other direction is covered by Nagata's theorem, as mentioned in §2. So, our characterization by means of a metric satisfying 1.2 includes for example n-dimensional Euclidean spaces as well.

However, it remains uncertain whether in separable metric spaces M the property dim $\leqslant n$ can be characterized by a metric satisfying 1.2 only. There might be a possibility that the condition of total boundedness can be omitted in this case, if the condition 1.2 is strengthened in the following way: there is a metric ρ in M which satisfies 1.2 and also, if $\rho(x,y_1) = \rho(x,y_2) = \ldots = \rho(x,y_{n+2})$,

3.1 $$\rho(y_i, y_j) < \rho(x, y_k), \quad \text{for suitable } i, j, k \qquad (i \neq j).$$

However, does there exist such a metric? For $n = 0$, the answer is in the affirmative (**4**, §2).

The problem of generalizing the Theorem to metric spaces in general remains unanswered too.

[3]A space is semicompact if it is the sum of a countable number of compact spaces. Every locally compact, separable, metrizable space is semicompact, since such a space can be compactified by one point.

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Mathematisch Instituut
University of Amsterdam