Bull. Austral. Math. Soc. 46B20
Vol. 49 (1994) [21–24]

TWO COUNTER-EXAMPLES IN NONSEPARABLE BANACH SPACES

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It is shown that the well-known theorem of Kadec for the H_Γ renorming of separable Banach spaces, when Γ is a norming subspace in the dual, cannot be extended to the class of nonseparable Banach spaces.

1. Introduction

Let X be a Banach space and let Γ be a total subspace in the dual space X^*.

The norm $\|.\|$ on a Banach space X is said to have the H_Γ-*property*, if for sequences on the unit sphere, $\sigma(X, \Gamma)$ and norm convergence coincide, that is whenever $x_0, x_n \in X$ $(n < \infty)$, $\lim_n \|x_n\| = \|x_0\|$ and $\lim_n f(x_n) = f(x_0)$ for all $f \in \Gamma$, then $\lim_n \|x_n - x_0\| = 0$.

The norm $\|.\|$ on a Banach space X is said to have the K_Γ-*property*, if the $\sigma(X, \Gamma)$ and norm topologies coincide on the unit sphere.

Obviously, if the norm has the K_Γ-property, then it has the H_Γ-property. The converse is not true.

When $\Gamma = X^*$, then the H_{X^*}-property is known as the *H-property* or the *Kadec-Klee property* and the K_{X^*}-property is known as the *Kadec property*.

If the Banach space X admits an equivalent norm with the H_Γ or K_Γ property, then we write $X \in (H_\Gamma)$ or $X \in (K_\Gamma)$.

It is easy to see that, if Γ is a total separable subspace in X^*, then $X \in (K_\Gamma)$ if and only if $X \in (H_\Gamma)$.

The following result of Kadec [5] is well-known: Let X be a separable Banach space and Γ is a norming subspace in X^*. Then $X \in (H_\Gamma)$.

Naturally, the question arises: Can we extend this result to the class of nonseparable Banach spaces?

The answer to this question is negative.

Plicko proved in [7] that, if Γ is a total subspace in X^* such that $\operatorname{dens}(\Gamma) < \operatorname{dens}(X)$, then $X \notin (K_\Gamma)$. In particular, it follows that, if X is a separable Banach space with a nonseparable dual space X^*, then $X^* \notin (H_X)$.

Received 20 January 1993

Here, we give two examples of total subspaces Γ in X^*, for some concrete Banach spaces X, with $\operatorname{dens}(\Gamma) = \operatorname{dens}(X)$, such that $X \notin (H_\Gamma)$.

We denote by $\overline{\operatorname{lin}}(A)$ the closed linear hull of the set $A \subset X$; $\operatorname{dens}(X)$ is the density character of X, that is, the smallest cardinal for which X has a dense subset of the same cardinality.

Let Γ be a subspace of X^*. We say that Γ is *norming* if its Dixmier characteristic

$$r(\Gamma) = \inf_{\|x\|=1} \sup_{f\in\Gamma} \frac{|f(x)|}{\|f\|} > 0.$$

Lemma. *Let (X, ρ_1) be an uncountable separable metric space, (Y, ρ_2) be a separable metric space and $T\colon X \to Y$ be an arbitrary map. Then there exist point $x_0 \in X$ and sequence $\{x_n\}_{n<\infty}$ in X, $x_n \neq x_0$, $\forall n < \infty$, such that $\lim_n \rho_1(x_n, x_0) = 0$ and $\lim_n \rho_2(Tx_n, Tx_0) = 0$.*

In this case, we say that x_0 is a *point of partial continuity* for the map T.

2. First example

Let AP be the Banach space of all almost periodic functions defined on the real line $\mathbb{R}$ with the supremum norm $\|.\|_\infty$.

We define the linear functionals $\delta_t \in AP^*$ for every $t \in \mathbb{R}$ by the equality $\delta_t(f) = f(t)$, $f \in AP$, and define the subspace $\Gamma = \overline{\operatorname{lin}}(\delta_t)_{t\in\mathbb{R}}$ in AP^*.

The subspace Γ is norming.

Really, if $f \in AP$, $\|f\|_\infty = 1$, then there exists a sequence $\{t_n\}_{n<\infty} \subset R$ such that $\lim_n |f(t_n)| = 1$, that is $\lim_n |\delta_{t_n}(f)| = 1$. Since $\sup_{\delta\in\Gamma}(|\delta(f)| / \|\delta\|) \geqslant |\delta_{t_n}(f)|$, $\forall n < \infty$, then $r(\Gamma) = 1$.

Proposition 1. *The space $AP \notin (H_\Gamma)$.*

Proof: Let $f_\lambda(t) = e^{i\lambda t}$, $\lambda \in \mathbb{R}$, and let $\|.\|$ be an equivalent norm on Banach space AP.

We examine the function $\lambda \mapsto \|f_\lambda\|$, $\lambda \in \mathbb{R}$.

According to the Lemma, there exists a point of partial continuity for this function, that is, there exist $\lambda_0, \lambda_n \in \mathbb{R}$, $\lambda_n \neq \lambda_0$ $(n < \infty)$ such that

$$\lim_n \lambda_n = \lambda_0, \tag{1}$$

$$\lim_n \|f_{\lambda_n}\| = \|f_{\lambda_0}\|. \tag{2}$$

From (1) and the definition of the functions f_{λ_n} we get $\lim_n f_{\lambda_n}(t) = f_{\lambda_0}(t)$, $\forall t \in \mathbb{R}$, which is equivalent to

$$\lim_n \delta_t(f_{\lambda_n}) = \delta_t(f_{\lambda_0}), \quad \forall t \in \mathbb{R}. \tag{3}$$

Consequently, from (3) for all $\delta \in \Gamma$ we have

(4) $$\lim_n \delta(f_{\lambda_n}) = \delta(f_{\lambda_0}).$$

Now, if we suppose that the space $AP \in (H_\Gamma)$ then from (2) and (4) it follows that $\lim_n \|f_{\lambda_n} - f_{\lambda_0}\| = 0$ which is, obviously, impossible, since the system $\{f_\lambda\}_{\lambda \in \mathbb{R}}$ is minimal (see [6]). The proposition is proved. ∎

3. Second example

Let $QC[0,1]$ be the Banach space of all real-valued functions defined on $[0,1]$ for which $f(t+0) = f(t)$ for every t, that is, f is continuous from the right and $f(t-0)$ exists for every t with the supremum norm $\|.\|_\infty$.

Let E be a dense subset in $[0,1]$ such that the set $E_1 = [0,1] \setminus E$ is uncountable.

We define the linear functionals $\delta_t \in QC[0,1]^*$ for every $t \in E$ by the equality $\delta_t(f) = f(t)$, $f \in QC[0,1]$, and define the subspace $\Gamma = \overline{\text{lin}}(\delta_t)_{t \in E}$ in $QC[0,1]^*$.

The subspace Γ is norming.

Really, if $f \in QC[0,1]$, $\|f\|_\infty = 1$, then there exists a sequence $\{t_n\}_{n<\infty} \subset E$ such that $\lim_n |f(t_n)| = 1$, that is, $\lim_n |\delta_{t_n}(f)| = 1$. Since $\sup_{\delta \in \Gamma}(|\delta(f)| / \|\delta\|) \geqslant |\delta_{t_n}(f)|$, $\forall n < \infty$, then $r(\Gamma) = 1$.

Proposition 2. *The space* $QC[0,1] \notin (H_\Gamma)$.

Proof: For every $s \in E_1$ we define the functions $f_s(t) = 0$, if $0 \leqslant t < s$ and $f_s(t) = 1$, if $s \leqslant t \leqslant 1$. Obviously, $f_s \in QC[0,1]$ for all $s \in E_1$. Let $\|.\|$ be an equivalent norm on the Banach space $QC[0,1]$.

We examine the function $s \mapsto \|f_s\|$, $s \in E_1$.

According to the Lemma, there exists a point of partial continuity for this function, that is, there exist $s_0, s_n \in E_1$, $s_n \neq s_0$ $(n < \infty)$ such that

(5) $$\lim_n s_n = s_0,$$

(6) $$\lim_n \|f_{s_n}\| = \|f_{s_0}\|.$$

From (5) and definition of the functions f_{s_n} we get $\lim_n f_{s_n}(t) = f_{s_0}(t)$, $\forall t \in E$, which is equivalent to

(7) $$\lim_n \delta_t(f_{s_n}) = \delta_t(f_{s_0}), \quad \forall t \in E.$$

Consequently, from (3) for all $\delta \in \Gamma$ we have

(8) $$\lim_n \delta(f_{s_n}) = \delta(f_{s_0}).$$

Now, if we suppose that the space $QC[0,1] \in (H_\Gamma)$ then from (6) and (8) it follows that $\lim_n \|f_{s_n} - f_{s_0}\| = 0$ which is, obviously, impossible, since $\|f_{s_n} - f_{s_0}\| = 1$ for all $n < \infty$. The proposition is proved. ∎

https://doi.org/10.1017/S0004972700016038 Published online by Cambridge University Press

REMARKS. (i) The spaces AP and $QC[0,1]$ possess an equivalent locally uniformly convex norm and, consequently, they have the Kadec property (see [**1**, **2**, **3**]).

(ii) Godun proved in [4] that, if $(x_i, f_i)_{i \in I}$ is M-basis in the Banach space X and $\Gamma = \overline{\text{lin}}(f_i)_{i \in I}$, then $X \in (H_\Gamma)$ if and only if the subspace Γ is norming.

This gives rise to the following.

QUESTION. *Let X be a nonseparable Banach space and let Γ be a norming subspace in dual the space X^*. What sufficient conditions must Γ satisfy so that X admits the H_Γ-property?*

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