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MINIMAL GENERATING SYSTEMS OF A SUBGROUP OF $SL(2, \mathbb{C})$

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Introduction

Let H be any group. We call a cardinal number r the rank $r(H)$ of H if H can be generated by a generating system X with cardinal number r but not by a generating system Y with cardinal number s less than r. Let $r(H)$ be the rank of H.

We call a generating system X of H a minimal generating system (M.G.S.) of H if X has the cardinal number $r(H)$.

In this note we prove the following.

Theorem. *Let G be a non-elementary and non-elliptic subgroup of $SL(2,\mathbb{C})$. Then G can be generated by a minimal generating system which contains either only hyperbolic matrices or only loxodromic matrices.*

Preliminary remarks

We write $[A,B]=ABA^{-1}B^{-1}$ for the commutator of $A,B\in SL(2,\mathbb{C})$ and $\operatorname{tr} A$ for the trace of $A\in SL(2,\mathbb{C})$; also E denotes the unit matrix in $SL(2,\mathbb{C})$.

We call an element $A\in SL(2,\mathbb{C})$, $A\neq \pm E$,
hyperbolic if $\operatorname{tr} A\in\mathbb{R}$, $|\operatorname{tr} A|>2$,
parabolic if $\operatorname{tr} A\in\mathbb{R}$, $|\operatorname{tr} A|=2$,
elliptic if $\operatorname{tr} A\in\mathbb{R}$, $|\operatorname{tr} A|<2$, and loxodromic if $\operatorname{tr} A\notin\mathbb{R}$.

Now let G be a subgroup of $SL(2,\mathbb{C})$. We say G is elementary if the commutator of any two elements of infinite order has trace 2; equivalently, G is elementary if any two elements of infinite order (regarded as linear fractional transformations) have at least one common fixed point.

The elementary subgroups of $SL(2,\mathbb{C})$ are well known and easily dealt with (cf. [**1**, pp. 117–147]).

We say that G is elliptic if each of its elements $A\neq\pm E$ is elliptic.

We often use the identity

$$\operatorname{tr} AB^{-1}=\operatorname{tr} A\cdot\operatorname{tr} B-\operatorname{tr} AB$$

for two elements A,B in $SL(2,\mathbb{C})$.

Proof of the Theorem

Throughout this section, let G be a non-elementary and non-elliptic subgroup of $SL(2,\mathbb{C})$.

Proposition 1. *Let* $\operatorname{tr} A \in \mathbb{R}$ *for all* $A \in X$ *and for all M.G.S.'s* X *of* G. *Then* G *can be generated by a M.G.S. which contains only hyperbolic matrices.*

Proof. Let X be a M.G.S. of G and $\operatorname{tr} A \in \mathbb{R}$ for all $A \in X$. We assume that $\operatorname{tr} B \notin \mathbb{R}$ for some $B \in G$ and we will show that this assumption leads to a contradiction.

Then there are $A_1, \ldots, A_n \in X$, different in pairs, with $B \in \langle A_1, \ldots, A_n \rangle$, the subgroup generated by $A_1, \ldots, A_n$, and $\operatorname{tr} B$ can be expressed as a polynomial with integral coefficients in the $2^n - 1$ traces $\operatorname{tr} A_{i_1} \ldots A_{i_\nu}$, $1 \leqq \nu \leqq n$, $1 \leqq i_1 < i_2 < \cdots < i_\nu \leqq n$ (cf. e.g. [**2**, p. 220]).

Now we have $\operatorname{tr} A_{i_1} \ldots A_{i_\nu} \notin \mathbb{R}$ for some $A_{i_1} \ldots A_{i_\nu}$ because $\operatorname{tr} B \notin \mathbb{R}$. If we replace A_{i_1} by $A_{i_1} \ldots A_{i_\nu}$ we obtain a M.G.S. Y of G which contains an element with non-real trace. This gives a contradiction.

Therefore we have $\operatorname{tr} B \in \mathbb{R}$ for all $B \in G$.

After a suitable conjugation we may assume that $G \subset SL(2,\mathbb{R})$ because G is non-elementary and non-elliptic (cf. [**3**, p. 42]). The result then follows from [**4**, p. 350]. □

From now on we assume that G has a M.G.S. X with $\operatorname{tr} A \notin \mathbb{R}$ for some $A \in X$ and we show that G then can be generated by a M.G.S. which contains only loxodromic elements.

Lemma. *Let* $\operatorname{tr} A \notin \mathbb{R}$ *for some* $A \in X$ *and some M.G.S.* X *of* G.
Then G *can be generated by a M.G.S.* Y *which contains two elements* A *and* B *with* $\operatorname{tr} A \notin \mathbb{R}$, $\operatorname{tr} B \notin \mathbb{R}$ *and* $\operatorname{tr}[A,B] \neq 2$.

Proof. If X is a M.G.S. of G then we may assume without any loss of generality that no pair of different elements generates a cyclic group (this is trivial if G is finitely generated).

Case 1. G has a M.G.S. X with $\operatorname{tr} A \notin \mathbb{R}$ for some $A \in X$ and $\operatorname{tr} B = 0$ for all $B \in X$, $B \neq A$.

Let X be a M.G.S. of G, $A \in X$ with $\operatorname{tr} A \notin \mathbb{R}$ and $\operatorname{tr} B = 0$ for all $B \in X$ with $B \neq A$. Because G is non-elementary there exists an element B in X such that $\operatorname{tr} B = 0$ and $\operatorname{tr}[A,B] \neq 2$.

If $\operatorname{tr} AB \notin \mathbb{R}$ then we replace B by AB and the lemma is proved because $\operatorname{tr}[A,AB] = \operatorname{tr}[A,B] \neq 2$.

If $\operatorname{tr} AB \in \mathbb{R}$, $\operatorname{tr} AB \neq 0$, then $\operatorname{tr} A^2B \notin \mathbb{R}$. Now we replace B by A^2B and the lemma is proved because $\operatorname{tr}[A, A^2B] = \operatorname{tr}[A,B]$. Now let $\operatorname{tr} AB = 0$. We may assume (after a suitable conjugation) that

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} 0 & -\rho \\ 1/\rho & 0 \end{pmatrix}, \ |\rho| \neq 1.$$

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Because G is non-elementary the rank $r(G)$ is greater than 2 and there exists an element

$$C=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

of X with $a\neq 0$ and $c\neq 0$ or $b\neq 0$.

Now let

$$C_1 .= BC=\begin{pmatrix} -c & a \\ a & b \end{pmatrix} \text{ if } c\neq b \text{ and } C_1 .= ABC=\begin{pmatrix} -\rho c & \rho a \\ a/\rho & b/\rho \end{pmatrix} \text{ if } c=b.$$

Then we have $\operatorname{tr} C_1\neq 0$ and $\operatorname{tr}[A,C_1]\neq 2$ because $|\rho|\neq 1$, $a\neq 0$ and $c\neq 0$ or $b\neq 0$.

Now we replace B by C_1 and consider A and C_1.

If $\operatorname{tr} C_1\notin\mathbb{R}$ then the lemma is proved. If $\operatorname{tr} C_1\in\mathbb{R}$ and $\operatorname{tr} AC_1\notin\mathbb{R}$ then we replace C_1 by AC_1 and the lemma is proved because $\operatorname{tr}[A,AC_1]=\operatorname{tr}[A,C_1]$. If $\operatorname{tr} C_1\in\mathbb{R}$ and $\operatorname{tr} AC_1\in\mathbb{R}$ then $\operatorname{tr} A^{-1}C_1\notin\mathbb{R}$. Now we replace C_1 by $A^{-1}C_1$ and the lemma is proved because $\operatorname{tr}[A,A^{-1}C_1]=\operatorname{tr}[A,C_1]$.

Case 2. G has no M.G.S. X with $\operatorname{tr} A\notin\mathbb{R}$ for some $A\in X$ and $\operatorname{tr} B=0$ for all $B\in X$, $B\neq A$.

Let X be a M.G.S. of G and $A\in X$ with $\operatorname{tr} A\notin\mathbb{R}$. Because G is non-elementary there exists an element B in X such that $\operatorname{tr}[A,B]\neq 2$.

If $\operatorname{tr} B\notin\mathbb{R}$ then the lemma is proved. If $\operatorname{tr} B\in\mathbb{R}$ and $\operatorname{tr} AB\notin\mathbb{R}$ then we replace B by AB and the lemma is proved because $\operatorname{tr}[A,AB]=\operatorname{tr}[A,B]$. If $\operatorname{tr} B\in\mathbb{R}$, $\operatorname{tr} B\neq 0$ and $\operatorname{tr} AB\in\mathbb{R}$ then $\operatorname{tr} AB^{-1}\notin\mathbb{R}$. Now we replace B by AB^{-1} and the lemma is proved because $\operatorname{tr}[A,AB^{-1}]=\operatorname{tr}[A,B]$.

If $\operatorname{tr} B\in\mathbb{R}$, $\operatorname{tr} AB\in\mathbb{R}$ and $\operatorname{tr} AB\neq 0$ then $\operatorname{tr} A^2B\notin\mathbb{R}$. Now we replace B by A^2B and the lemma is proved because $\operatorname{tr}[A,A^2B]=\operatorname{tr}[A,B]$.

From now on let $\operatorname{tr} B=\operatorname{tr} AB=0$. Because G is non-elementary and by our assumptions about the M.G.S's there exists an element C in X, $A\neq C\neq B$, such that $\operatorname{tr} C\neq 0$, $\operatorname{tr} AC\neq 0$, $\operatorname{tr} A^{-1}C\neq 0$, $\operatorname{tr} BC\neq 0$, $\operatorname{tr} BAC\neq 0$ and $\operatorname{tr} BA^{-1}C\neq 0$ (if for instance $\operatorname{tr} BA^{-1}D=0$ for $D\in X$, $D\neq A$, with $\operatorname{tr} D\neq 0$ then we replace D by $D' .= BA^{-1}D$ to get a new M.G.S. Y with $D'\in Y$ and $\operatorname{tr} D'=0$).

If $\operatorname{tr} C\in\mathbb{R}$ and $\operatorname{tr} AC\notin\mathbb{R}$ then we replace C by AC.

If $\operatorname{tr} C\in\mathbb{R}$, $\operatorname{tr} AC\in\mathbb{R}$ then $\operatorname{tr} A^{-1}C\notin\mathbb{R}$ and we replace C by $A^{-1}C$.

Then (after the suitable replacement) $\operatorname{tr} C\notin\mathbb{R}$ and $\operatorname{tr} BC\neq 0$.

If $\operatorname{tr}[A,C]\neq 2$ then the lemma is proved.

Now let $\operatorname{tr}[A,C]=2$.

If $\operatorname{tr} BC\notin\mathbb{R}$ and $\operatorname{tr}[B,C]\neq 2$ then we replace B by BC and the lemma is proved because $\operatorname{tr}[BC,C]=\operatorname{tr}[B,C]$. If $\operatorname{tr} BC\in\mathbb{R}$ then $\operatorname{tr} BC^2\notin\mathbb{R}$.

Therefore, if $\operatorname{tr}[B,C]\neq 2$ and $\operatorname{tr} BC\in\mathbb{R}$ we replace B by BC^2 and the lemma is proved because $\operatorname{tr}[BC^2,C]=\operatorname{tr}[B,C]$.

Now let $\operatorname{tr}[B,C]=2$. Then, altogether, we have the situation $\operatorname{tr} A\notin\mathbb{R}$, $\operatorname{tr} C\notin\mathbb{R}$, $\operatorname{tr} B=\operatorname{tr} AB=0$, $\operatorname{tr} BC\neq 0$, $\operatorname{tr}[A,C]=\operatorname{tr}[B,C]=2$ and $\operatorname{tr}[A,B]\neq 2$.

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Therefore we may assume (after a suitable conjugation) that

$$C=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix},\ B=\begin{pmatrix}a & 0\\ c & -a\end{pmatrix},\ a^2=-1,\quad \text{and } A=\begin{pmatrix}\alpha & \gamma\\ 0 & \alpha^{-1}\end{pmatrix},$$

where $\gamma\neq 0\neq c$ because $\operatorname{tr}[A,B]\neq 2$. Then $\operatorname{tr} AB=0$ implies

$$c\gamma+a\alpha-a\alpha^{-1}=0.$$

Therefore we obtain

$$\begin{aligned}\operatorname{tr}[A,CB]&=2+c\gamma(a\alpha-a\alpha^{-1}+c\gamma\beta^{-2}+a\alpha\beta^{-2}-a\alpha^{-1}\beta^{-2})\\&=2+ac\gamma(\alpha-\alpha^{-1})\neq 2\end{aligned}$$

because $a\neq 0$, $c\neq 0$, $\gamma\neq 0$ and $\operatorname{tr} A\notin\mathbb{R}$.

Now we replace C by CB and consider A and CB.

If $\operatorname{tr} CB\notin\mathbb{R}$ then the lemma is proved. If $\operatorname{tr} CB\in\mathbb{R}$ and $\operatorname{tr} ACB\notin\mathbb{R}$ then we replace CB by ACB and the lemma is proved because $\operatorname{tr}[A,ACB]=\operatorname{tr}[A,CB]$.

If $\operatorname{tr} CB\in\mathbb{R}$ and $\operatorname{tr} ACB\in\mathbb{R}$ then $\operatorname{tr} A^{-1}CB\notin\mathbb{R}$. Now we replace CB by $A^{-1}CB$ and the lemma is proved because $\operatorname{tr}[A,A^{-1}CB]=\operatorname{tr}[A,CB]$. □

Proposition 2. *Let* tr $A\notin\mathbb{R}$ *for some* $A\in X$ *and some M.G.S.* X *of* G. *Then* G *can be generated by a M.G.S. which contains only loxodromic matrices.*

Proof. By the lemma, G has a M.G.S. X which contains two elements A and B with $\operatorname{tr} A\notin\mathbb{R}$, $\operatorname{tr} B\notin\mathbb{R}$ and $\operatorname{tr}[A,B]\neq 2$.

Now let

$$A=\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix},\ |\alpha|>1,\quad \text{and } B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

without any loss of generality. We have $c\neq 0$ and $b\neq 0$ because $\operatorname{tr}[A,B]\neq 2$.

Let

$$C=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$$

be any element of X with $A\neq C\neq B$. Then

$$BC=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}.$$

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Thus if $e=0$, $ae+bg\neq 0$ and if $h=0$ then $cf+dh\neq 0$. Thus replacing C by BC if necessary, we can assume that $e\neq 0$ or $h\neq 0$. Then there exists an integer n such that $|\operatorname{tr} A^nC|>2$ because $|\alpha|>1$. Therefore $\operatorname{tr} A^nC\notin\mathbb{R}$ or $\operatorname{tr} A^{n+1}C\notin\mathbb{R}$ or $\operatorname{tr} A^{n-1}C\notin\mathbb{R}$. Let $\operatorname{tr} A^nC\notin\mathbb{R}$ without any loss of generality. Then we replace C by A^nC.

This proves Proposition 2. □

Proposition 1 and Proposition 2, when taken together, prove the theorem. The following is a direct consequence of Proposition 1, Proposition 2 and their proofs.

Corollary. *Let G be a non-elementary and non-elliptic subgroup of $SL(2, \mathbb{C})$. If $\operatorname{tr} A\in\mathbb{R}$ for all $A\in G$ then G can be generated by a M.G.S. which contains only hyperbolic matrices. If $\operatorname{tr} A\notin\mathbb{R}$ for some $A\in G$ then G can be generated by a M.G.S. which contains only loxodromic matrices.*

Remark. Let P denote the natural map: $SL(2, \mathbb{C})\to PSL(2, \mathbb{C})$. We use, with slight ambiguity, the term $\operatorname{tr} A$ for the trace of $A, A\in PSL(2, \mathbb{C})$, and also the term E for the identity element in $PSL(2, \mathbb{C})$. We adapt the definitions accordingly for a subgroup G of $PSL(2, \mathbb{C})$. Then the theorem and the corollary naturally also hold for a non-elementary and non-elliptic subgroup G of $PSL(2, \mathbb{C})$.

We mention that our constructions lead to the following supplement. Let G be a finitely generated, non-elementary and non-elliptic subgroup of $SL(2, \mathbb{C})$ or $PSL(2, \mathbb{C})$ and let X be a M.G.S. of G. Then there exists a Nielsen-transformation from X to a M.G.S. Y of G which contains either only hyperbolic elements or only loxodromic elements.

Added in proof. Recently I learned that N. A. Isatschenko (preprint, Novosibirsk 1987) proved a partial version of the above theorem. He also remarked that the above theorem when taken together with the result of T. Jørgensen (On discrete groups of Möbius transformations, *Amer. J. Math.* **98** (1976), 739–749) leads to the following.

Let G be a non-elementary and non-elliptic subgroup of $SL(2, \mathbb{C})$. Then G is discrete if and only if each subgroup generated by two loxodromic or hyperbolic elements is discrete.

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