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The connectivity of total graphs

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We associate with a graph (finite, undirected, without loops and multiple lines) a graph $T(G)$, called the *total graph* of G . This new graph has the property that a one-to-one correspondence can be established between its points and the elements (points and lines) of G such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident. The object of this article is to prove the following theorem: If $\kappa(G_1) = n$, $n \geqq 1$, and $\lambda(G_2) = m$, $m \geqq 1$, then $\kappa(T(G_1)) \geqq n + 2 + [(n - 2)/3]$, $\lambda(T(G_1)) \geqq 2n$, $\kappa(T(G_2)) \geqq m + 1$, and $\lambda(T(G_2)) \geqq 2m$, where $\kappa(G)$ and $\lambda(G)$ denote the connectivity and line-connectivity of the graph G .

1. Introduction

The (point) *connectivity* $\kappa(G)$ of a graph (finite, undirected, with no loops and multiple lines) G is the least number of points whose removal disconnects G or reduces it to K_1 . The *line-connectivity* $\lambda(G)$ of a nontrivial graph G is the minimum number of lines whose removal results in a disconnected graph. (For completeness, $\lambda(K_1)$ is defined to be zero.)

We associate with a graph G another graph $T(G)$, called the *total graph* of G . This new graph has the property that a one-to-one correspondence can be established between its points and the elements (the set of points and lines) of G such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are points or both are lines) or they are indicent (if one element

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is a point and the other a line).

In this note we investigate the connectivity relationships between a graph and its total graph. In particular, we show that if $\kappa(G) = n$, $n \geqq 1$, and $\lambda(G) = m$, then $\lambda(T(G)) \geqq 2m$, and $\kappa(T(G)) \geqq n + 2 + [(n - 2)/3]$, where $[x]$ is the greatest integer less than or equal to x.

2. Preliminaries

In this section we review some useful terminologies and results dealing with the problem.

The point set of a graph G will be denoted by $V(G)$ and its line set by $X(G)$. The *degree*, $\deg_G a$, of a point a of G is the number of lines incident with a. If $\deg_G a = d$ is constant on $V(G)$, then G is called *regular of degree* d. A regular graph of order (the number of elements of $V(G)$) p and degree $p - 1$ is denoted by K_p. A connected regular graph of degree 2 is called a *cycle*.

The *line-graph*, $L(G)$, of G is that graph whose point set is $X(G)$, and in which two points are adjacent if and only if they are adjacent in G.

Following these definitions we observe that both G and $L(G)$ are (disjoint) subgraphs of $T(G)$. (See [1], [2].) Moreover, for a point a of $T(G)$ belonging to $V(G)$ we have $\deg_{T(G)} a = 2 \deg_G a$, and for a point b of $T(G)$ belonging to $X(G) = V(L(G))$ we have $\deg_{T(G)} b = \deg_G u + \deg_G v$, where u and v are the points of G incident with b. (For an illustration, a graph G is given in Fig. 1 together with $T(G)$. In $T(G)$ the "dark" points correspond to the points of G while the "light" points correspond to the lines of G; $L(G)$ consists of the "light" points and the lines of $T(G)$ joining two such points. These lines are drawn in Fig. 1 with dashed lines.)

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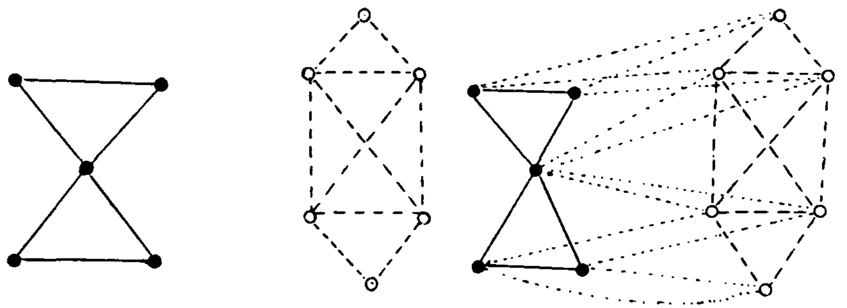

Figure 1

A graph G is said to be *n-connected* if $\kappa(G) \geq n$ and *m-line connected* if $\lambda(G) \geq m$. Characterizations of n-connected graphs and m-line connected graphs presented next are due to Whitney [4], [5].

THEOREM A *A graph G is n-connected (m-line connected) if and only if between every pair of distinct points there exist at least n disjoint (m line-disjoint) paths.*

The next theorem is due to Chartrand and Stewart [3].

THEOREM B *If $\kappa(G_1) = n$ and $\lambda(G_2) = m$, then $\kappa(L(G_1)) \geq n$ and $\lambda(L(G_1)) \geq 2n - 2$ while $\kappa(L(G_2)) \geq m$ and $\lambda(L(G_2)) \geq 2m - 2$.*

In conclusion of this section we state an observation due to Whitney [6]. We write *min deg G* to denote the smallest degree among the points in G.

THEOREM C *For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \min \deg G .$$

3. Main results

Before we prove our first theorem we observe that G is connected if and only if $T(G)$ is connected; and that in $T(G)$ a point of G is adjacent to at least min deg G points of $L(G)$.

THEOREM 1 *If G is m-line connected, then $T(G)$ is 2m-line*

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connected.

Proof If $m = 0$, then the theorem is clearly true. So assume $m \geqq 1$. First we show between each pair u and v of distinct points of $T(G)$ belonging to $L(G)$ there exist at least $2m$ line-disjoint paths. By Theorem B, there exist at least $2m - 2$ line-disjoint paths in $L(G)$. Let u and v correspond to the lines $x = ab$ and $y = cd$, respectively. If x and y have a point in common, that is, if for example $d = b$, then the paths (u, b, v) and (u, a, b, c, v) are two line-disjoint $u - v$ paths, and no line of these paths belongs to $L(G)$. In case x and y have no points in common, $m \geqq 1$ implies that there exists at least one $b - d$ path, say $(b = b_0, b_1, b_2, \ldots, b_n = d)$ in G, where n is a positive integer. The $u - v$ paths $(u, b, b_1, b_2, \ldots, b_{n-1}, d, v)$ and $(u, a, b, b_1, \ldots, b_n, c, v)$ are line-disjoint. Again no line of these paths is in $L(G)$. Hence the assertion follows.

Next suppose a set S, $|S| \leqq 2m - 1$, of lines disconnects $T(G)$. Remove S and denote the resulting graph by H. In H all points of $L(G)$ must be in one of its components, say H_1. Let H_2 be another component of H. All points of H_2 are points of G, moreover, the number of points of H_2 is at least 2. This contradicts the inequality $|S| \leqq 2m - 1$, since in $T(G)$ there are at least $2 \min \deg G$ lines joining points of H_1 to points of H_2, and by Theorem C $2m \leqq 2 \min \deg G$.

COROLLARY 1.1 *If G is m-connected, then $T(G)$ is $2m$-line connected.*

Proof $\kappa(G) \leqq \lambda(G)$ implies that G is m-line connected.

The equalities $\kappa(K_{m+1}) = \lambda(K_{m+1}) = m$ and $\min \deg T(K_{m+1}) = 2m$ show that the results of Theorem 1 and Corollary 1.1 are the best.

THEOREM 2 *If G is m-line connected, $m \geqq 1$, then $T(G)$ is $(m + 1)$-connected.*

Proof Suppose a set S consisting of s points of $T(G)$, $s \leqq m$, disconnects $T(G)$. Let $S = S_1 \cup S_2$, where S_1 is the set of all elements of S which are points of $L(G)$, and $S_2 = S - S_1$. If

$|S_1| < m$, then the removal of S from $L(G)$ results in a connected graph. This and the fact that a point of G in $T(G)$ is adjacent to at least m points of $L(G)$ give rise to a contradiction. So $|S_1| = m$ and $|S_2| = 0$. But then every point of $L(G)$ being adjacent to two points of G in $T(G)$ gives rise to a contradiction again. This completes the proof of the theorem.

The result of Theorem 2 is best possible, too. Identify two copies of K_{m+1} at one point v and denote the resulting graph by G. The point v is a cut-point of G and $\lambda(G) = m$. The subgraph $L(G)$ of $T(G)$ has point connectivity m. The m points which disconnect $L(G)$ together with the point v, disconnect $T(G)$. Hence $\kappa(T(G)) = m + 1$. The graph G in Fig. 1 illustrates this for $m = 2$.

Next, we note that a point of $L(G)$ in $T(G)$ is adjacent with at least 2 (min deg $G - 1$) other points of $L(G)$.

THEOREM 3 *If G is m-connected, $m \geqq 1$, then $T(G)$ is $(m + 2 + [(m - 2)/3])$-connected.*

Proof Since S is m-line connected, $T(G)$ is $(m + 1)$-connected. Hence for $m = 1$, the theorem is true. So assume $m \geqq 2$. Suppose there exists a set S having $s = m + 1 + [(m - 2)/3]$ or less points of $T(G)$ whose removal from $T(G)$ results in a disconnected graph H. Suppose $S_1 \subset S$ consists of those points of S belonging to $L(G)$ and $S_2 = S - S_1$.

If $|S_1| \leqq m - 1$, then the removal of S_1 from $L(G)$ results in a connected graph. This together with the fact that in $T(G)$ each point of G is adjacent to m points of $L(G)$ contradicts the fact that H is a disconnected graph. Thus $|S_1| \geqq m \geqq 2$. From this we conclude that

(1) $$|S_2| = |S| - |S_1| \leqq s - m = 1 + [(m - 2)/3] \leqq m - 1 .$$

Since H is disconnected, $|S_2| \geqq 2$. Hence:

(2) $$2 \leqq |S_2| \leqq m - 1 .$$

Therefore, the removal of S_2 from G results in a connected graph.

Now remove S from $T(G)$ and denote the connected subgraph containing all remaining points of G (and possibly some points of $L(G)$) by H_1 and let H_2 denote the rest of the resulting graph H. The graph H_2 contains at least one point, say u. The first inequality in (2) implies that

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(3) $$|S_1| \leqq m - 1 + [(m - 2)/3] .$$

From (3) and the note preceding Theorem 3 we get:

(4) $$2m - 2 - m + 1 - [(m - 2)/3] \geqq 1 .$$

Hence u is adjacent to another point v of $L(G)$ in H_2 . The points u and v correspond to two adjacent lines in G . These two lines are incident with 3 points in G which must belong to S_2 . Hence:

(5) $$|S_1| \leqq s - 3 = m - 2 + [(m - 2)/3] .$$

Again, from (5) and the note preceding the theorem, we obtain:

(6) $$2m - 2 - m + 2 - [(m - 2)/3] \geqq 2 .$$

Therefore, besides v , the point u is adjacent to another point w of $L(G)$ in H_2 . The points u , v , and w correspond to three lines U , V , and W , respectively, of G . Since the line U is adjacent to both V and W , one of the graphs in Fig. 2 must be a subgraph of G .

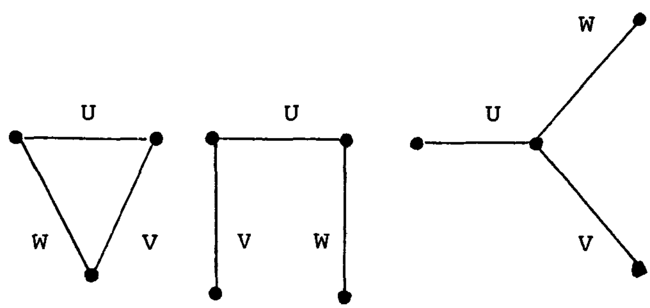

Figure 2

In each case there are at least $3m - 6$ lines in G , different from U , V , and W , which are adjacent to U , V , or W . Hence, in addition to u , v , and w , there are at least $3m - 6$ points in $L(G)$ which are adjacent to the points u , v , or w . Therefore, we have:

(7) $$3m - 6 - (s - 3) = 2m - 4 - [(m - 2)/3] \geqq m - 2 .$$

Now (7) implies that at least $m - 2$ points of $L(G)$ are left which are adjacent to u , v , or w in H_2 . These points correspond to $m - 2$ lines of G adjacent to U , V , or W . These $m - 2$ lines together with the lines U , V , and W are adjacent with at least $[(m - 2)/3]$ points of G which must belong to S_2 . Hence the set S contains at least $m + 3 + [(m - 2)/3]$ points. Since this number is greater than s , the theorem must hold.

Now we summarize our main results in the following

THEOREM 4 *If* $\kappa(G_1) = n$, $n \geqq 1$, *and* $\lambda(G_2) = m$, $m \geqq 1$, *then*

$$\kappa(T(G_1)) \geqq n + 2 + [(n - 2)/3] ,$$

$$\lambda(T(G_1)) \geqq 2n ,$$

$$\kappa(T(G_2)) \geqq m + 1 ,$$

and

$$\lambda(T(G_2)) \geqq 2m .$$

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