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THE ODD-LOCAL SUBGROUPS OF THE MONSTER

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Abstract

We determine all conjugacy classes of maximal p-local subgroups of the Monster for $p \neq 2$.

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1. Introduction

In this paper we find all the maximal p-local subgroups of the Fischer-Griess Monster group, for each odd prime p. The results for $p \geqslant 5$ are not new—they were first obtained by S. P. Norton, but his proofs have not been published. The hardest case, however, is the case $p = 3$, and our main result is

THEOREM 3. *Any 3-local subgroup of the Monster group M is contained in a conjugate of one of the seven groups*

$$\begin{aligned} N(3A) &\simeq 3 \cdot Fi_{24}, \\ N(3A^2) &\simeq \left(3^2{:}2 \times O_8^+(3)\right) \cdot S_4, \\ N(3B) &\simeq 3_+^{1+12} \cdot 2 \cdot Suz{:}2, \\ N(3B^2) &\simeq 3^2 . 3^5 . 3^{10}{:}(M_{11} \times 2S_4), \\ N(3B^3) &\simeq 3^3 . 3^6 . [3^8] . (D_8{:}2 \times L_3(3)), \\ N(3^8) &\simeq 3^8 \cdot O_8^-(3) \cdot 2, \\ N(3C) &\simeq S_3 \times Th. \end{aligned}$$

Conversely, none of these groups is contained in a conjugate of any other.

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The proof is based largely on calculations inside the group $3^{1+12} \cdot 2Suz$, and therefore depends heavily on the simple notation for this group introduced in Section 3 below. This notation can be restricted to various subgroups of $3^{1+12} \cdot 2Suz$, and enables one to find the maximal 3-local subgroups of B [10], Th [11], Fi_{24} and Fi_{23} [12]. It is really an extension and modification of the notation used in [9] to study the 3-local subgroups of Fi_{22}.

Other notation is taken from the ATLAS [3].

The results for $p = 5$ and 7 are

THEOREM 5. *Let Y be any non-trivial 5-subgroup of the Monster. Then $N(Y)$ is contained in a conjugate of one of the 6 groups*:

$$N(5A) \simeq (D_{10} \times HN) \cdot 2,$$
$$N(5A^2) \simeq (5^2{:}4 \cdot 2^2 \times U_3(5)){:}S_3,$$
$$N(5B) \simeq 5_+^{1+6}{:}2 \cdot J_2{:}4,$$
$$N(5B^2) \simeq 5^2.5^2.5^4{:}(S_3 \times GL_2(5)),$$
$$N(5B^3) \simeq 5^{3+3} \cdot (2 \times L_3(5)),$$
$$N(5B^4) \simeq 5^4{:}(3 \times 2 \cdot L_2(25)){:}2.$$

Conversely, none of these groups is contained in a conjugate of any other.

THEOREM 7. *Let Y be any non-trivial 7-subgroup of the Monster. Then $N(Y)$ is contained in a conjugate of one of the 5 groups*

$$N(7A) \simeq (7{:}3 \times He){:}2,$$
$$N(7A^2) \simeq (7^2{:}(3 \times 2A_4) \times L_2(7)){:}2,$$
$$N(7B) \simeq 7_+^{1+4}{:}(3 \times 2S_7),$$
$$N(7B^2) \simeq 7^2 \cdot 7_+^{1+2}{:}GL_2(7),$$
$$N(7B^2) \simeq 7^2{:}SL_2(7).$$

Conversely, none of these groups is contained in a conjugate of any other.

Note. This last group is missing from the list given in [3].

The results for $p > 7$ are given in the final section of the paper.

2. Basic 3-local structure

There are three classes of elements of order 3 in the Monster, with normalizers

$$N(3A) \simeq 3 \cdot Fi_{24} \simeq 3 \cdot Fi'_{24}{:}2,$$
$$N(3B) \simeq 3_+^{1+12} \cdot 2Suz{:}2,$$
$$N(3C) \simeq S_3 \times Th.$$

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Of these three classes, just the $3B$-elements are 3-central. In particular, the group $3^{1+12} \cdot 2Suz$ contains a Sylow 3-subgroup of M, and hence contains a representative of every conjugacy class of elementary Abelian subgroups. Throughout the rest of Sections 2–8, Y will denote an elementary Abelian 3-group of order at least 3^2.

PROPOSITION 2.1. *If Y contains $3C$-elements, then $N(Y)$ is contained in $N(Y_0)$ where Y_0 is a non-trivial elementary Abelian 3-group containing no $3C$-elements.*

PROOF. The subgroup Th contains no elements of M-class $3C$, while all diagonal 3-elements in $3 \times Th$ are of M-class $3C$. It follows that Y has a unique subgroup Y_0 of index 3 containing no $3C$-elements, so $N(Y) \leqslant N(Y_0)$, and the proposition is proved.

From now on, therefore, we concern ourselves only with the $3A$- and $3B$-elements, and we begin by studying their centralizers.

Restricting the 196883-character of M to $3Fi_{24}'$ we obtain $1a + 8671a + 57477a + 783ab + 64584ab$, where the last four characters are faithful and the first three characters are not. Calculating the character values on the various elements of order 3 we deduce the class fusion:

Fi_{24}'-class	M-classes	3^2-centralizer in M
$3A$	$3A_3$	$3^2 \times O_8^+(3)$
$3B$	$3B_1A_2$	$(3 \times 3^{1+10}):U_5(2)$
$3C$	$3A_1B_2$	$3^8 \cdot 2U_4(3)$
$3D$	$3B_3$	$[2^5 \cdot 3^{14}]$
$3E$	$3C_3$	$3^2 \times G_2(3)$

PROPOSITION 2.2. *If Y contains $3A$-elements but no $3B$-elements, then $N(Y)$ is contained in*

$$N(3A^2) \simeq \left(3^2:2 \times O_8^+(3)\right) \cdot S_4.$$

PROOF. From Proposition 2.1 and the above table it suffices to show that Fi_{24} contains no 3^2-group whose non-trivial elements are all of Fi_{24}-class $3A$. This is proved in [12] by restricting the characters of degrees 783 and 2808.

From now on, we can assume that Y is an elementary Abelian group containing $3B$-elements but no $3C$-elements. For the rest of the paper we fix a $3B$-element x contained in Y, and let $X \simeq 3^{1+12}$ be normal in $N(x)$.

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Our next task is to investigate the structure of the $3B$-centralizer in detail. Now the values of the 196883-character on the elements of classes $2B$, $3B$ and $6B$ show that it restricts to $3^{1+12} \cdot 2Suz$ as

$$143 \oplus 65520 \oplus \{(729 \oplus 729') \otimes (12 \oplus 78)\},$$

where characters are denoted by their degrees, and 12, 78, 143 are faithful irreducible characters of $6Suz$, $3Suz$ and Suz respectively, 729 and 729′ are extensions of the faithful irreducible characters of 3^{1+12} to $3^{1+12} \cdot 2Suz$, and 65520 is a faithful monomial representation of $3^{12}:2Suz$. Again we calculate the character values on the various elements of order 3, and deduce that the fusion from $6Suz$ to M is given by

Suz-class	M-classes	3^2-centralizer in M
$3A$	$3A_2B_1$	$3^8 \cdot 2U_4(3)$
$3B$	$3A_1B_2$	$[2^5 \cdot 3^{14}]$
$3C$	$3C_3$	$3^6 . 2A_6$

3. A notation for $3^{1+12} \cdot 2Suz$

In this section we introduce a notation for the group $3^{12}:2Suz = C(x)/\langle x\rangle$. Since the normal 3^{12}-group is really a reduction of the complex Leech lattice modulo $\theta = \sqrt{-3}$, it is useful to describe this lattice first. More details can be found in [7], though for present purposes it is best to use the "quaternionic" version given in [6], with slight modifications. It must be borne in mind however that the group $6Suz$ does not preserve the quaternionic structure.

We write vectors of the Leech lattice in the 6 by 4 *MOG* array in the usual way, and associate the rows to the quaternions 1, i, j and k respectively. By interpreting each column of the *MOG* as a single quaternion, and dividing by 2, we identify the Leech lattice with a certain set of vectors in a quaternionic 6-space. There is a "monomial" subgroup $2^{5+12}:3A_6$ of the automorphism group of the lattice which preserves the associated decomposition into 6 quaternionic 1-spaces (real 4-spaces). This may be generated by left and right multiplications by diagonal matrices corresponding to "hexacode" words, together with the hexacode automorphism group $3A_6$. This multiplicative hexacode may be generated by the elements $\langle j, k, k, j, k, j\rangle$, $\langle k, j, j, k, k, j\rangle$, $\langle k, j, k, j, j, k\rangle$ and their images under ω. Here ω denotes the map defined by $1 \to 1$, $i \to j \to k \to i$: this may be realized as conjugation by $\frac{1}{2}(-1 + i + j + k)$. Full details are given in [8].

We want generators for $6Suz$ as a subgroup of the lattice automorphism group $2Co_1$, so we take the group 2^{5+6} generated by the right multiplications by hexacode words, and extend by the hexacode automorphism group $3A_6$, which

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may be generated by $\langle \omega, \bar{\omega}, +,$  $\rangle$ and $\langle$  $\rangle$. Then we need one extra generator, which can be taken to be right multiplication by the matrix

$$\frac{j-i}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Note that since ω always denotes conjugation by $\omega = \frac{1}{2}(-1 + i + j + k)$, this group is not a group of right multiplications by quaternionic matrices.

Now let $\theta = i + j + k$, and consider the endomorphism Θ of L defined by multiplying each quaternionic coordinate of a vector on the left by θ. Then $L/\Theta L$ is an Abelian group of exponent 3 and order 3^{12}. Now left multiplication by ω acts trivially on $L/\Theta L$, so conjugation by ω is the same as right multiplication by ω. Hence the group $6Suz$ can be re-interpreted as a group of right multiplications on $L/\Theta L$, where the central element of order 3 acts trivially. The group $C(x)/\langle x \rangle$ is then isomorphic to the split extension of $L/\Theta L$ by this group of matrices.

Now the vector $(8,0,0,0,0,0)$ is in L, and we can divide by 2 in $L/\Theta L$, so we can take these 6 vectors as a basis. Then the image in $L/\Theta L$ of a vector in L is obtained simply by reducing its coordinate modulo θ. If we take the imaginary part of the quaternionic inner product (that is, the sum of the coefficients of i, j and k), and reduce modulo 3, then we obtain the symplectic form corresponding to the commutator map on 3^{1+12}.

4. The 3-elements in $3^{1+12} \cdot 2Suz$

In this section we aim to classify the conjugacy classes of elements of order 3 in $3^{12}{:}2Suz$, and to identify the corresponding groups of order 9 in $3^{1+12} \cdot 2Suz$. First, there are two classes in 3^{12}, as follows:

Representative	M-centralizer	M-type
$(1,0,0,0,0,0)$	$(3 \times 3^{1+10}){:}\,U_5(2)$	$3A_3B_1$
$(1,1,1,0,0,0)$	$(3 \times 3^{1+10}).3^5.M_{11}$	$3B_4(\mathrm{i})$

The minimal vectors of the Leech lattice correspond to subgroups of type $3B_1A_3$ in 3^{1+12}. There are five orbits under the monomial group $2^{5+6}:3A_6$, as

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follows

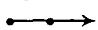
$(1,0,0,0,0,0)$	24
$(1,1,0,0,0,0)$	480
$(1,1,1,1,0,0)$	7680
$(1,1,1,1,1,0)$	6144
$(1,1,1,1,1,i)$	18432
	Total = 32760

The type 3 vectors in the Leech lattice correspond to subgroups of type $3B_4$ in 3^{1+12}. There are 8 orbits under the monomial group, as follows:

$(1,1,1,0,0,0)$	5120
$(1,1,1,i,0,0)$	23040
$(1,1,1,1,i,0)$	92160
$(1,1,1,1,1,1)$	1024
$(1,1,1,1,1,-1)$	1024
$(1,1,1,1,1,-i)$	18432
$(1,1,1,1,i,i)$	46080
$(1,1,1,1,i,-i)$	46080
	Total = 232960

Next, for each class of 3-elements in *Suz* we must find the classes they lift to in $3^{12}:2Suz$. Now each element t is conjugate to its multiples by vectors in the image of $1-t$, so we need to find the orbits of $C(t)$ on the vectors in $3^{12}/\mathrm{im}(1-t)$. There are 3 classes of elements of order 3 in $2Suz$, with centralizers:

$$C(3a) \simeq 6U_4(3),$$
$$C(3b) \simeq 2\,.\,3^{2+4}(2A_4 \times 2),$$
$$C(3c) \simeq 3^2 \times 2A_6$$

and these three classes may be represented respectively by the elements $\langle \omega, \omega, \omega, \omega, \omega, \omega \rangle$, $\langle \omega, \overline{\omega}, +,$  $\rangle$, and $\langle$  $\rangle$.

Case a: $t = \langle \omega, \omega, \omega, \omega, \omega, \omega \rangle$. Here $\mathrm{im}(1-t)$ is 6-dimensional, spanned by $(1,0,0,0,0,0)$, $(0,1,0,0,0,0)$, $(0,0,1,0,0,0)$, $(0,0,0,1,0,0)$, $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$. The centralizer $6U_4(3)$ acts on the quotient 6-space as $2 \cdot O_6^-(3)$, and so there are four orbits on vectors, of which two are interchanged by the outer automorphism

Name	Number	Representative	M-centralizer	M-type
$3a$	1	$(0,0,0,0,0,0)$	$3^8 \cdot 2U_4(3)$	$3A_2B_2$
$3a'$	224	$(i,i,i,0,0,0)$	$3^6\,.\,3^5\,.\,A_6$	$3B_4(\mathrm{ii})$
–	252	$(i,0,0,0,0,0)$	$3^7\,.\,U_4(2)$	$9A$
–	252	$(i,i,0,0,0,0)$	$3^7\,.\,U_4(2)$	$9A$

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Case b: $t = \langle \omega, \bar{\omega}, +, \bullet\!\!-\!\!\bullet\!\!\rightarrow \rangle$. Here $\mathrm{im}(1-t)$ is again 6-dimensional, and is spanned by $(1,0,0,0,0,0)$, $(0,1,0,0,0,0)$, $(0,0,0,1,-1,0)$, $(0,0,0,i,-i,0)$, $(0,0,0,1,1,1)$ and $(0,0,0,i,i,i)$. This time it is not totally isotropic, but lifts to $3^4 \times 3^{1+2}$ in 3^{1+12}. The quotient $3^{12}/\mathrm{im}(1-t)$ may be generated by $(i,0,0,0,0,0)$, $(0,i,0,0,0,0)$, $(0,0,1,0,0,0)$, $(0,0,i,0,0,0)$, $(0,0,0,1,0,0)$ and $(0,0,0,i,0,0)$. If we multiply t by any vector involving either of the last two basis elements, then we obtain an element of order 9 in $3^{12}:2Suz$. On the remaining 4-space $C(t)$ has an invariant 2-space, and some calculation or other easy arguments show that it has 5 orbits on vectors, two of which are interchanged by an outer automorphism

Name	Number	Representative	M-centralizer	M-type
$3b$	1	$(0,0,0,0,0,0)$	$[2^5 \cdot 3^{14}]$	$3A_1B_3$
$3b'$	8	$(0,0,1,0,0,0)$	$[2^2 \cdot 3^{13}]$	$3B_4$(iii)
–	18	$(i,0,0,0,0,0)$	$[2^4 \cdot 3^{11}]$	$9B$
–	18	$(-i,0,0,0,0,0)$	$[2^4 \cdot 3^{11}]$	$9B$
$3b''$	36	$(i,i,0,0,0,0)$	$[2^3 \cdot 3^{11}]$	$3B_1C_3$

Case c: $t = \langle$  $\rangle$. Here $\mathrm{im}(1-t)$ is 8-dimensional, and $3^{12}/\mathrm{im}(1-t)$ may be spanned by $(1,0,1,0,1,0)$, $(0,1,0,1,0,1)$, $(i,0,i,0,i,0)$ and $(0,i,0,i,0,i)$. Then $C(t)$ acts on this space as $SL_2(9)$, so there are just two orbits on vectors. But it is clear that the non-zero vectors give rise to elements of order 9 in $3^{12}:2Suz$, so we have only one case:

Name	Number	Representative	M-centralizer	M-type
$3c$	1	$(0,0,0,0,0,0)$	$3^6.2A_6$	$3B_1C_3$

5. The 3^2-normalizers

PROPOSITION 5.1. *Let Y be an elementary Abelian subgroup of 3^{1+12} of order 3^2, not containing the centre. Then Y corresponds to a totally isotropic 2-space in the symplectic 12-space. There are 7 conjugacy classes of such groups Y, as follows*

Type	Centralizer in $N(3B)$	Centralizer in M	Alias
$3A_4$	$(3^2 \times 3_+^{1+8}).2^{1+6}:3^3$	$3^2 \times O_8^+(3)$	
$3A_3B_1$	$(3^2 \times 3_+^{1+8}).3^4A_5$	$(3 \times 3_+^{1+10}):U_5(2)$	
$3A_2B_2$	$(3^2 \times 3_+^{1+8}).(3 \times A_6)$	$(3^2 \times 3^6)\cdot 2U_4(3)$	$3a$
$3A_1B_3$	$(3^2 \times 3_+^{1+8}).[2^2 \cdot 3^3]$	$[2^5 \cdot 3^{14}]$	$3b$
$3B_4$	$(3^2 \times 3_+^{1+8}).3^4.3^2:D_8$	$3^2.3^5.3^{10}:M_{11}$	
$3B_4$	$(3^2 \times 3_+^{1+8}).A_6$	$(3^2 \times 3_+^{1+8}).A_6$	$3a'$
$3B_4$	$(3^2 \times 3_+^{1+8}).[2^2 \cdot 3^2]$	$(3^2 \times 3_+^{1+8}).[2^2 \cdot 3^2]$	$3b'$

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Proof. To find the conjugacy classes and centralizers in $3^{1+12} \cdot 2Suz$ requires only some elementary geometry of the complex Leech lattice. (Note, however, that the classification of the 2-spaces spanned by minimal vectors given (but not used) in [7] is wrong, in that the orbit sizes given as 2970 and 5346 should be 1980 and 6336.) We have also already classified 3^2-groups containing $3A$-elements, so all that remains is to identify the pure $3B$-type groups. To do this we must first describe the structure of the normalizer of a 3^2-group of type $3B_4$(i), which has the shape $3^2 . 3^5 . 3^{10}:(M_{11} \times 2S_4)$. We do this by analogy with the description of $2^2 \cdot 2^{11} \cdot 2^{22} \cdot (M_{24} \times S_3)$ given in [2]. The analogy is not complete, however, and there does not seem to be any underlying loop corresponding to Parker's loop. On the other hand, the fact that the group splits over its O_3-subgroup means that we can manage without such a loop.

The normal subgroup $3^2 . 3^5 \simeq 3^7$ is the intersection of the groups 3^{1+12} corresponding to the $3B$-elements in the original 3^2-group. The action of the next factor 3^{10} on the 3^7 is the natural one, that is, it consists of all elements which fix both the 3^2 and the quotient 3^5 pointwise. As a module for $M_{11} \times 2S_4$ it is therefore isomorphic to $\mathrm{Hom}(5,2) \simeq 5^* \otimes 2$, where modules are denoted by their degrees, and * denotes the dual.

Take a copy of the ternary Golay code containing the vector (1^{12}), and let M_{11} act by fixing this vector. The resulting 5-dimensional module is the one we have called 5^*, so the module denoted 5 is obtained by taking all words in the co-code orthogonal to (1^{12}). Now M_{11} has orbits $55 + 66$ on the 1-spaces in 5, and from what we have already proved of Proposition 5.1 these give rise to 3^3-groups of types $3B_{13}$ and $3B_4A_9$ respectively.

We are now in a position to finish the proof of Proposition 5.1. First note that the structure of $3^2 . 3^5 . 3^{10}:(M_{11} \times 2S_4)$ implies that the first 2-space of type $3B_4$ in the Leech lattice is of class $3B_4$(i), but that neither of the others is of this class. Hence the second 2-space of type $3B_4$ is of class $3B_4$(ii), since it is centralized by elements of order 5. It follows that the third 2-space of type $3B_4$ cannot be conjugate to the second, so must be of class $3B_4$(iii). This concludes the proof.

Theorem 5.2. *If Y is an elementary Abelian group of order 9 then $N(Y)$ is contained in one of the groups $N(3A)$, $N(3B)$, $N(3A^2)$ (whose structures are given above) or in one of the groups*

$$N(3B^2) \simeq 3^2 . 3^5 . 3^{10}:(M_{11} \times 2S_4),$$
$$N(3^8) \simeq 3^8 \cdot O_8^-(3) \cdot 2.$$

Proof. We have seen that the normalizer of a 3^2-group of type $3B_4$(ii) or $3B_4$(iii) is contained in $N(3B)$, and that the normalizer of a 3^2-group of type $3B_4$(i) has the shape given above. We have also dealt with the $3A$-pure groups, and it is clear that $N(3A_1B_3) \leqslant N(3A)$ and $N(3A_3B_1) \leqslant N(3B)$. Finally, $C(3A_2B_2) \simeq 3^8 \cdot 2U_4(3)$, so $N(3A_2B_2) \leqslant N(3^8)$. But now $N(3^8)$ is transitive on the $3A_2B_2$-subgroups, so has the indicated structure.

From now on we may assume that Y is an elementary Abelian 3-group of order at least 3^3, containing a $3B$-element but no $3C$-element. The group 3^8 with normalizer $3^8 \cdot O_8^-(3) \cdot 2$ will be denoted by E, and the group 3^7 with normalizer $3^2 . 3^5 . 3^{10} : (M_{11} \times 2S_4)$ will be denoted by F.

6. Subgroups of X containing x

Next we show that if we have a 3^2-group of type $3B_4$(ii) or $3B_4$(iii), embedded in 3^{1+12}, then all the $3A$- or $3B$-elements which centralize it are also contained in the same group 3^{1+12}. We do this by showing that their centralizers in $2Suz$ contain only elements of type $3c$.

PROPOSITION 6.1. *The centralizer of a 2-space of type* $3B_4$(ii) *contains no elements of type* $3a$ *or* $3b$ *in* $2Suz$.

PROOF. We have seen that the full 3^2-normalizer is in $3^{1+12} \cdot 2Suz:2$, and that it has the shape $(3^2 \times 3^{1+8}) . (A_6 \times 2A_4) . 2$. Now embedding $(2 \cdot A_4 \times A_6) . 2$ in $2Suz:2$ we see that the A_6 centralizes $2B$-elements of Suz, since these are the ones which lift to elements of order 4 in $2Suz$. Hence all the 3-elements in this A_6 are of Suz-class $3C$ (that is, type $3c$), and we have seen that there are no elements of M-class $3A$ or $3B$ in such a coset of 3^{1+12}.

PROPOSITION 6.2. *There is no* 3^2-*group of type* $3B_4$(iii) *in the fixed point set in* 3^{1+12} *of an element of type* $3a$ *or* $3b$.

PROOF. In the $3a$-case, the fixed point set is a subgroup of $E \simeq 3^8$. But all $3B_4$-groups in E are conjugate, so are of type $3B_4$(i).

In the $3b$-case, the fixed point set is the group $3^4 \times 3^{1+2}$ given in Section 4 above. But all the maximal Abelian subgroups of this are conjugate to a subgroup of F, so we must show that F contains no 3^2-groups of type $3B_4$(iii). Such a subgroup would obviously have to have trivial intersection with the invariant 3^2-subgroup, so the problem reduces to showing that there is no 3^2 in the 3^5 consisting entirely of elements from the 55-orbit under M_{11}. The proof can now be completed by easy calculations with the Golay code.

COROLLARY 6.3. *If we have an elementary Abelian group containing a* 3^2*-group of type* $3B_4$(ii) *or* $3B_4$(iii), *then we can embed it in a unique* 3^{1+12}, *and can adjoin the centre of this* 3^{1+12}, *since it is central in all Sylow 3-subgroups of the* 3^2*-centralizer.*

In general, we need only consider elementary Abelian groups Y whose normalizer acts irreducibly on Y. The following special case of a result of Piper [5] is very useful in this connection.

LEMMA 6.4. *If* p *is an odd prime,* $n \geqslant 3$, *and* H *is an irreducible subgroup of* $GL_n(p)$ *containing all the transvections fixing a given point, then* $H \geqslant SL_n(p)$. *In particular,* H *acts transitively on the points and lines of the associated projective space.*

THEOREM 6.5. *If* Y *is any subgroup of* 3^{1+12} *containing the centre, then its normalizer is in one of*

$$N(3B) \simeq 3^{1+12} \cdot 2Suz:2,$$
$$N(3B^2) \simeq 3^2 . 3^5 . 3^{10} : (M_{11} \times 2S_4),$$
$$N(3B^3) \simeq 3^3 . 3^6 . [3^8] . (L_3(3) \times D_8 : 2).$$

PROOF. We assume that $N(Y)$ acts irreducibly on Y. But $N(Y)$ contains all the transvections fixing x, induced by elements of 3^{1+12}, so by Lemma 6.4 $N(Y)$ is transitive on the non-trivial elements of Y, and on the 3^2-subgroups of Y. Hence Y is $3B$-pure, and all the 3^2-groups in Y are of class $3B_4$(i). But the calculation in Proposition 6. 2 shows that the 3^5 chief factor contains no $3B$-pure 3^2-group, and this concludes the proof.

7. Subgroups of E

We have dealt now with any elementary Abelian 3-group containing x which is either in 3^{1+12} or contains an element of class $3a'$ or $3b'$. It remains therefore to consider those groups containing only elements of class $3a$ or $3b$, which give rise to 3^2-groups of types $3A_2B_2$ and $3A_1B_3$ respectively. Now Suz contains no 3^2-groups of type $3a_4$, so we can divide the problem into two cases:

(1) $Y \leqslant 3^{1+12}:3$, of type $3a$,

(2) Y contains an element of type $3b$.

In the first case Y is a subgroup of the group $E \simeq 3^8$. There is a quadratic form on this 3^8-group, which is invariant up to sign. This quadratic form is inherited by subgroups, and we want to see to what extent it remains invariant. Note first that the $3A$-elements are non-isotropic, while the $3B$-elements are isotropic, so that the kernel of the quadratic form on a subgroup Y is well defined: it is the set of all $3B$-elements in Y not contained in any $3A_2B_2$-subgroup of Y. Hence we may assume that the quadratic form on Y is either zero, or non-singular. But if it is zero then Y is a subgroup of X, and its normalizer is given by Theorem 6.5.

PROPOSITION 7.1. *If the quadratic form is non-singular, then $N(Y)$ is contained in* $3^8 \cdot O_8^-(3) \cdot 2$.

PROOF. Since $|Y| \geqslant 3^3$, Y contains a $3A_2B_2$-subgroup, so $C(Y) \leqslant 3^8 \cdot O_8(3) \cdot 2$. Furthermore, the normalizer of Y in $3^8 \cdot O_8(3) \cdot 2$ is transitive on the $3A_2B_2$-subgroups, and $N(3A_2B_2) \leqslant 3^8 \cdot O_8(3) \cdot 2$. Hence the result.

REMARK. This is a variant of Lemma 5.1 of [9], using subgroups rather than elements.

8. Groups containing a $3A_1B_3$-subgroup

First note that $C(3A_1B_3)$ is contained in $3^2 \,.\, 3^5 \,.\, 3^{10}{:}(M_{11} \times 2S_4)$. For the 3^2-group of type $3A_1B_3$ can be embedded in the 3^5 consisting of all Golay co-code words orthogonal to (1^{12}), so that its centralizer in $N(3B^2)$ is $3^2 \,.\, 3^5 \,.\, 3^6 \,.\, (2^2 \times 2A_4)$, which has the same order as its centralizer in M.

Now let $Y_1 = Z(O_3(C(Y)))$, so that $N(Y) \leqslant N(Y_1)$. Then Y_1 contains Y and at least one $3B$-element from the normal 3^2 in $3^2 \,.\, 3^5 \,.\, 3^{10}{:}(M_{11} \times 2S_4)$. From the point of view of such a $3B$-element, therefore, it contains a 2-space of type $3A_1B_3$ in the 3^{1+12}. The centralizer of this 3^3-group is $(3^2 \times 3^{1+8}) \,.\, [2^2 \cdot 3^2]$. If our elementary Abelian 3-group is contained in the 3^{1+12}, then its normalizer is given by Theorem 6.5. From the results of Sections 5 and 6, it therefore suffices to prove:

PROPOSITION 8.1. *If Y is an elementary Abelian group in $3^{1+12} \cdot 2Suz$, containing the centre, an outer element of class $3b$, and a subspace of type $3A_1B_3$ in the 3^{1+12}, then it contains an element of class $3b'$, and hence a 3^2-group of class* $3B_4$(iii).

PROOF. Taking the $3b$ element given above, its centralizer in 3^{1+12} is a group $3^4 \times 3^{1+2}$. Now the only $3B$-elements in the centre of this group are those in the group generated by $(0,0,0,1,1,1)$ and $(0,0,0,i,i,i)$, which does not contain a

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subgroup of type $3A_1B_3$. Hence Y must contain a non-central element of $3^4 \times 3^{1+2}$, and multiplying this by our $3b$-element we obtain an element of type $3b'$.

This concludes the proof of Theorem 3.

9. The 5-local subgroups

In this section we shall prove Theorem 5. Our proof follows in outline the same method as for the 3-local subgroups, so we content ourselves with giving fewer details. Many of the results stated here without proof can either be found in [4] or else readily deduced from results of that paper.

First we consider the various classes of subgroups of order 5 and 25. There are two classes of elements of order 5, with normalizers

$$N(5A) \simeq (D_{10} \times HN) \cdot 2,$$
$$N(5B) \simeq 5_+^{1+6}:2 \cdot J_2:4.$$

There is also one class of elements of order 25, which have centralizer $25 \times D_{10}$ and 5th-power to $5B$-elements. We have the following class fusion from $5 \times HN$ to M:

HN-class	$5A$	$5B$	$5CD$	$5E$
M-class	$5A$	$5B$	$5B$	$5A$
diagonal elements	$5A$	$5A$	$5A/B$	$5B$
5^2-type	$5A_6$	$5A_5B_1$	$5A_3B_3$	$5A_2B_4$
5^2-centralizer	$5^2 \times U_3(5)$	$5 \times 5_+^{1+4}:2_-^{1+4}:5$	$5^2 \times 5^2:2 \cdot A_5$	$5^2 \times 5_+^{1+2}:2^2$

Next we wish to find the conjugacy classes of elements of order 5 in $N(5B)$. First note that there are two orbits of $2J_2$ on the 3906 1-spaces in the 5^6, as follows

Number	Class	5^2-type	5^2-centralizer
1890	$5A$	$5A_5B_1$	$5 \times 5_+^{1+4}:2^{1+4}:5$
2016	$5B$	$5B_6$(i)	$(5 \times 5^{1+4}):5^2:S_3$

The other classes of 5-elements in $5^6:2 \cdot J_2:4$ are given names corresponding to their images in J_2, and are as follows:

Name	5^2-type	5^2-centralizer
$5AB$	$5A_3B_3$	$5^2 \times 5^2:2 \cdot A_5$
$5AB'$	$5B_6$(ii)	$5^4:2$
$5CD$	$5A_2B_4$	$5^4:(2 \times D_{10}) \simeq 5^2 \times 5_+^{1+2}:2^2$
$5CD'$	$5B_6$(iii)	$5^4:2$
$5CD''$	$25A, 5B$	$25 \times D_{10}$

Now consider the various 2-spaces in the 5^6 on which $2J_2$ acts, and the normalizers of the corresponding 5^2-subgroups of M. By calculation we obtain

Number	Type	Stabilizer
6300	$5A_6$	$(5^2 \times 5_+^{1+2}):[2^7].S_3$
37800	$5A_2B_4$	$(5^2 \times 5_+^{1+2}):[2^7]$
50400	$5A_3B_3$	$(5^2 \times 5_+^{1+2}):[2^4].S_3$
6048	$5A_5B_1$	$(5^2 \times 5_+^{1+2}):[2^5 \cdot 5^2]$
1008	$5B_6$(i)	$(5^2 \times 5_+^{1+2}):(5 \times 2A_5).[2^3]$

Proof of Theorem 5. Let Y be an elementary Abelian 5-subgroup of M. Consider first the case when Y is $5A$-pure, that is, Y contains no $5B$-elements. Then it has order at most 5^2 and its normalizer is either in $N(5A)$ or in $N(5A^2) \simeq (5^2{:}4 \cdot 2^2 \times U_3(5)){:}S_3$.

Next, suppose Y contains a $5B$-element x, and consider Y as a subgroup of $C(x) \simeq 5^{1+6}{:}2J_2$. If Y is contained in 5^{1+6} then by Lemma 6.4 we may assume it is $5B$-pure. Hence Y has order at most 5^3 and its normalizer is in one of $N(5B)$, $N(5B^2) \simeq 5^2 \cdot 5^2 \cdot 5^4{:}(S_3 \times GL_2(5))$, or $N(5B^3) \simeq 5^{3+3} \cdot (2 \times L_3(5))$.

Finally, suppose Y contains elements outside 5^{1+6}. If Y contains an element of type $5AB'$ or $5CD'$ then its centralizer contains a unique 5^4-subgroup, which is the same in both cases, and $N(Y)$ is contained in $N(5^4) \simeq 5^4{:}(3 \times 2 \cdot L_2(25)){:}2$. If Y contains an element of type $5CD$, then it is conjugate to a subgroup of 5^{1+6}, and we can adjoin the centre of this since it is central in the unique Sylow 5-subgroup of $C(Y)$. Hence this reduces to an earlier case. The only remaining case is when Y contains only elements of type $5AB$. If Y has order 5^2 then its normalizer is in $N(5B^2)$. Otherwise, it is conjugate to a subgroup of 5^{1+6}, containing the centre, and again this reduces to an earlier case.

This concludes the proof of Theorem 5.

10. The 7-local subgroups

In this section we prove Theorem 7. As with Theorem 5, we do not give full details, but much of the information can be derived from [1].

There are two classes of elements of order 7 in M, with normalizers

$$N(7A) \simeq (7:3 \times He):2,$$
$$N(7B) \simeq 7_+^{1+4}:(3 \times 2S_7).$$

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There are 3 classes of 7-elements in $He:2$, which give rise to 3 classes of 7^2-group in M, with the following class fusion:

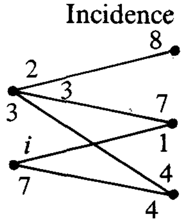
He-class	$7AB$	$7C$	$7DE$
M-class	$7A$	$7B$	$7B$
diagonal elements	$7A$	$7A$	$7A/B$
7^2-type	$7A_8$	$7A_7B_1$	$7A_4B_4$
7^2-centralizer	$7^2 \times L_2(7)$	$7 \times 7_+^{1+2}:3$	$7^2 \times D_{14}$

Now consider the group $7^{1+4}:(3 \times 2S_7)$. In the 7^4-factor the isotropic subspaces (corresponding to the elementary Abelian subgroups of 7^{1+4}) are as follows

Points | Lines

Class	Number
$7A$	280
$7B$	120

Incidence



Number	Class
70	$7A_8$
120	$7A_7B_1$
210	$7A_4B_4$

Next let us consider the conjugacy classes of elements of order 7 in $7^4:(3 \times 2S_7)$ outside the 7^4-subgroup. The 7-elements in $2S_7$ act on the 7^4 with one Jordan block of size 4, since we already know that they centralize just $7^2 \times D_{14}$ in $7^{1+4}:(3 \times 2S_7)$. Fixing a particular such 7-element x, we see that $7^4/\mathrm{Im}(1-x)$ has dimension 1, and the normalizer of x permutes the non-trivial vectors transitively. Hence there are two classes of these outer 7-elements, one of which corresponds to the 7^2-group of type $7A_4B_4$ which we have already found, the other of which is a new 7^2-group of type $7B_8$.

PROOF OF THEOREM 7. Let Y be any elementary Abelian 7-group in M. If Y is of pure $7A$-type, then it has order at most 7^2, and its normalizer is contained in one of the groups

$$N(7A) \simeq (7:3 \times He):2,$$
$$N(7A^2) \simeq \left(7^2:(3 \times 2A_4) \times L_2(7)\right):2.$$

Otherwise, Y contains a $7B$-element x. If it is in the corresponding 7^{1+4}, then its normalizer is in one of the groups

$$N(7B) \simeq 7_+^{1+4}:(3 \times 2S_7),$$
$$N(7B^2) \simeq 7^2 \cdot 7^{1+2}:GL_2(7).$$

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Otherwise, it contains outer elements of $7^{1+4}:2S_7$. If these are of the first type, then Y may be conjugated into 7^{1+4}, and its normalizer is then in $N(7B)$, as we have already seen. If they are of the second type, however, then Y has order 7^2, is self-centralizing, and has normalizer $7^2:SL_2(7)$.

11. The p-local subgroups, $p \geqslant 11$

The p-local subgroups for $p \geqslant 11$ can more or less be read off from the character table. For $p = 11$ we have

$$N(11A) \simeq (11:5 \times M_{12}):2 < (L_2(11) \times M_{12}):2,$$
$$N(11A^2) \simeq 11^2:(5 \times 2 \cdot A_5).$$

For $p = 13$, there are two classes of elements of order p, with normalizers

$$N(13A) \simeq (13:6 \times L_3(3)) \cdot 2,$$
$$N(13B) \simeq 13_+^{1+2}:(3 \times 4 \cdot S_4).$$

The 14 groups of shape 13^2 in 13^{1+2} are 8 of type $13A_{13}B_1$ and 6 of type $13B_{14}$, so we get just one more maximal 13-local subgroup

$$N(13B^2) \simeq 13^2:4 \cdot L_2(13) \cdot 2.$$

For the remaining primes, the Sylow p-subgroup has order p, and we have

$$N(17A) \simeq (17:8 \times L_3(2)) \cdot 2 < (S_4(4):2 \times L_3(2)) \cdot 2,$$
$$N(19A) \simeq (19:9 \times A_5):2 < (U_3(8):3 \times A_5):2,$$
$$N(23AB) \simeq 23:11 \times S_4 < 2^2 \cdot 2^{11} \cdot 2^{22} \cdot (M_{24} \times S_3),$$
$$N(29A) \simeq (29:14 \times 3) \cdot 2 < 3 \cdot Fi_{24},$$
$$N(31AB) \simeq 31:15 \times S_3 < S_3 \times Th,$$
$$N(41A) \simeq 41:40,$$
$$N(47AB) \simeq 47:23 \times 2 < 2 \cdot B,$$
$$N(59AB) \simeq 59:29,$$
$$N(71AB) \simeq 71:35.$$

In all cases where the group is known not to be maximal in M, we give a containing group.

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