J. Austral. Math. Soc. (*Series A*) **50** (1991), 391–408

BEST SIMULTANEOUS APPROXIMATION OF QUASI-CONTINUOUS FUNCTIONS BY MONOTONE FUNCTIONS

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(Received 28 June 1989)

Communicated by E. N. Dancer

Abstract

Let $\mathscr{Q}$ denote the Banach space (under the sup norm) of quasi-continuous functions on the unit interval $[0, 1]$. Let $\mathscr{M}$ denote the closed convex cone comprised of monotone nondecreasing functions on $[0, 1]$. For f and g in $\mathscr{Q}$ and $1 < p < \infty$, let h_p denote the best L_p-simultaneous approximant of f and g by elements of $\mathscr{M}$. It is shown that h_p converges uniformly as $p \to \infty$ to a best L_∞-simultaneous approximant of f and g by elements of $\mathscr{M}$. However, this convergence is not true in general for any pair of bounded Lebesgue measurable functions. If f and g are continuous, then each h_p is continuous; so is $\lim_{p\to\infty} h_p = h_\infty$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): primary 41 A 28; secondary 41 A 30, 41 A 65.

0. Introduction

Let Ω be the real unit interval $[0, 1]$. Let μ be the Lebesgue measure on Ω and let $\mathscr{A}$ be the collection of all measurable subsets of Ω. Let $L_p(\Omega, \mathscr{A}, \mu)$, $1 < p < \infty$, be the well known L_p Banach spaces and let $\mathscr{Q} \subseteq L_p(\Omega, \mathscr{A}, \mu)$ be the Banach space (under the sup norm) comprised of all quasi-continuous functions defined on Ω, that is, functions having at most discontinuities of the first kind only. Let $\mathscr{C} \subseteq \mathscr{Q}$ be the subspace of continuous functions on Ω, and let $\mathscr{M}$ be the closed convex cone in $\mathscr{Q}$ consisting of all monotone nondecreasing functions on Ω. Let f and g be two bounded Lebesgue measurable functions on Ω. It was shown in [3] that

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https://doi.org/10.1017/S1446788700032997 Published online by Cambridge University Press

if $f \notin \mathscr{M}$ or $g \notin \mathscr{M}$, then there exists a unique $h_p \in \mathscr{M}$, $p \in (1, \infty)$, such that

$$(0.1) \qquad [\|f - h_p\|_p^p + \|g - h_p\|_p^p]^{1/p} = \inf_{h \in \mathscr{M}} [\|f - h\|_p^p + \|g - h\|_p^p]^{1/p}.$$

We call h_p the best L_p-simultaneous approximant of f and g by elements of $\mathscr{M}$. Unless indicated otherwise h_p will be referred to as the b.s.a. of f and g. In general we say that f and g have the simultaneous Polya property if $h_\infty = \lim_{p\to\infty} h_p$ is well defined as a bounded Lebesgue measurable function on Ω.

When $f = g$ in (0.1), we have the usual L_p-approximation of a single function f by elements of $\mathscr{M}$. If f_p is its best L_p-approximant, then $\lim_{p\to\infty} f_p = f_\infty$ exists provided f is quasi-continuous, that is, f has the Polya property in this case (see [2]). In this paper, we try primarily to generalize the results discussed in [2]. As for now, there has been no similar work concerning the convergence of h_p as $p \to \infty$.

We devote the next section to studying the case when both f and g are real-valued functions defined on a finite point set X. We state formulas for computing h_p and h_∞ in this case, and we establish the basic convergence results needed later.

In Section 2, we utilize the results of Section 1 to establish convergence results in the space of step functions defined on Ω.

The fact that the step functions are dense in the space of quasi-continuous functions together with the results of Section 2, enable us to obtain the simultaneous Polya property. This is done in Section 3 where we establish as well the continuity of h_p, $p > 1$, whenever f ang g are continuous.

In Section 4, we show by an example that the simultaneous Polya property does not hold in general for any pair of bounded Lebesgue measurable functions. In particular, we consider the case when f is approximately continuous on Ω.

Throughout this report we may assume (unless otherwise indicated) that either f and/or g does not belong to $\mathscr{M}$.

1. Best L_p-simultaneous approximation on a finite set

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite subset of $\mathscr{R}$ with $x_1 < x_2 < \cdots < x_n$. Let $B = B(X)$ be the linear space of real functions on X and $\mathscr{M} = \mathscr{M}(X)$ the closed convex cone of monotone nondecreasing functions in B, that is, functions h satisfying $h(x) \le h(y)$ whenever $x, y \in X$ and

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$x \leq y$. For each $p \in [1, \infty)$, define a weighted L_p-norm ${}_w\|.\|_p$ by

$$(1.1) \qquad {}_w\|f\|_p = \left(\sum_{i=1}^{n} w_i |f_i|^p\right)^{1/p},$$

where $f = \{f_i\}_{i=1}^n = \{f(x_i)\}_{i=1}^n \in B$, and $w = \{w_i\}_{i=1}^n > 0$ is a weight function satisfying $\sum_{i=1}^n w_i = 1$.

Let $f = \{f_i\}_{i=1}^n$ and $g = \{g_i\}_{i=1}^n$ in B be fixed. For each $p \in [1, \infty)$, a function $h_p = \{h_{p,i}\}_{i=1}^n \in \mathscr{M}$ is called a best weighted L_p-simultaneous approximant of f and g if

$$(1.2) \qquad \begin{aligned} &({}_w\|f - h_p\|_p^p + {}_w\|g - h_p\|_p^p)^{1/p} \\ &\quad = \inf\{({}_w\|f - h\|_p^p + {}_w\|g - h\|_p^p)^{1/p} : h \in \mathscr{M}\}, \end{aligned}$$

or,

$$(1.2') \qquad \begin{aligned} &\left[\sum_{i=1}^{n} w_i (|f_i - h_{p,i}|^p + |g_i - h_{p,i}|^p)\right]^{1/p} \\ &\quad \leq \left[\sum_{i=1}^{n} w_i (|f_i - h_i|^p + |g_i - h_i|^p)\right]^{1/p}, \end{aligned}$$

for all $h = \{h_i : i = 1, \dots, n\} \in \mathscr{M}$.

It was shown in [3] that h_p is unique (up to equivalence) when $p \in (1, \infty)$.

To this end we shall discuss briefly the computation of the values of h_p explicitly, we start with the following definitions.

Definition. A subset $L \subseteq X$ is said to be a lower set if $x_i \in L$ and $x_j \in X$, $x_j \leq x_i$, imply that $x_j \in L$. Similarly, $U \subseteq X$ is an upper set if $x_i \in U$ and $x_j \in X$, $x_j \geq x_i$, imply that $x_j \in X$. For simplicity we will write $i \in Y \subseteq X$ instead of $x_i \in Y$.

Fix $p \in (1, \infty)$. If $L \cap U$ is nonempty, define $\mu_p(L \cap U)$ to be the unique real number minimizing

$$\left\{\sum_j w_j [|f_j - u|^p + |g_j - u|^p] : j \in L \cap U\right\}.$$

Let $h_p = \{h_{p,i} : i = 1, 2, \dots, n\}$ be the function defined on X by

$$(1.3) \qquad \begin{aligned} h_{p,i} &= \max_{\{U : i \in U\}} \min_{\{L : i \in L\}} \mu_p(L \cap U), \\ &= \min_{\{L : i \in L\}} \max_{\{U : i \in U\}} \mu_p(L \cap U). \end{aligned}$$

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It is shown in [8, pages 21–38], that h_p is the unique solution satisfying (1.2). Before we proceed we remind the reader of the sup norm defined by

$$\|f\|_\infty = \max\{|f_i| : i = 1, \dots, n\}.$$

DEFINITION. Let $a = \min\{-\|f\|_\infty, -\|g\|_\infty\}$ and $b = \max\{\|f\|_\infty, \|g\|_\infty\}$. For fixed $f = \{f_i\}_{i=1}^n$ and $g = \{g_i\}_{i=1}^n$, we define functions $\tau_p : [a, b]^n \to \mathscr{R}$ and $\kappa_p : [a, b] \to \mathscr{R}$ for $1 \le p \le \infty$ by

$$\tau_p(\overline{u}) = \sum_{i=1}^n w_{p,i}(|f_i - u_i|^p + |g_i - u_i|^p), \qquad p < \infty, \tag{1.4}$$

$$\tau_\infty(\overline{u}) = \max_{1 \le i \le n}(|f_i - u_i|, |g_i - u_i|), \tag{1.5}$$

$$\kappa_p(u) = \sum_{i=1}^n w_{p,i}(|f_i - u|^p + |g_i - u|^p), \tag{1.6}$$

$$\kappa_\infty(u) = \max_{1 \le i \le n}(|f_i - u|, |g_i - u|), \tag{1.7}$$

where $\overline{u} = (u_1, \dots, u_n) \in [a, b]^n$ and $u \in [a, b]$.

LEMMA 1. *Suppose* $w_p = \{w_{p,i}\}_{i=1}^n > 0$ *with* $\sum_{i=1}^n w_{p,i} = 1$, *and*

$$\liminf_{p \to \infty} w_{p,i} > 0. \tag{1.8}$$

Then

$$\lim_{p \to \infty}(\tau_p(\overline{u}))^{1/p} = \tau_\infty(\overline{u}), \tag{1.9}$$

and

$$\lim_{p \to \infty}(\kappa_p(u))^{1/p} = \kappa_\infty(u), \tag{1.10}$$

and the convergence is uniform on the compact sets $[a, b]^n$ *and* $[a, b]$ *respectively.*

PROOF. From (1.8) we conclude that there exist real numbers $\delta_1, \delta_2 \in (0, 1]$ and $p_0 \ge 1$ such that $w_{p,i} \in [\delta_1, \delta_2]$ for all i and all $p \ge p_0$. Hence

$$\begin{aligned} \tau_p(\overline{u}) &= \sum_{i=1}^n w_{p,i}(|f_i - u_i|^p + |g_i - u_i|^p), \\ &\le \sum_{i=1}^n \delta_2[(\tau_\infty(\overline{u}))^p + (\tau_\infty(\overline{u}))^p], \\ &= \sum_{i=1}^n 2\delta_2(\tau_\infty(\overline{u}))^p = 2n\delta_2(\tau_\infty(\overline{u}))^p, \end{aligned}$$

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for all $p \geq p_0$. In other words,

$$(1.11) \qquad (\tau(\overline{u}))^{1/p} - \tau_\infty(\overline{u}) \leq [(2n\delta_2)^{1/p} - 1]\tau_\infty(\overline{u}).$$

Since X is finite, we may assume without loss of generality the existence of an integer k, $1 \leq k \leq n$, such that

$$\tau_\infty(\overline{u}) = |f_k - u_k|.$$

Hence,

$$\tau_p(\overline{u}) \geq w_{p,k}[|f_k - u_k|]^p \geq \delta_1(\tau_\infty(\overline{u}))^p,$$

which implies that

$$(1.12) \qquad (\delta_1^{1/p} - 1)\tau_\infty(\overline{u}) \leq (\tau_p(\overline{u}))^{1/p} - \tau_\infty(\overline{u}).$$

Combining (1.11) and (1.12) it follows that

$$|\tau_p(\overline{u})^{1/p} - \tau_\infty(\overline{u})| \leq \max\{|(2n\delta_2)^{1/p} - 1|, |\delta_1^{1/p} - 1|\}\tau_\infty(\overline{u}).$$

Now $\tau_\infty(\overline{u})$ is continuous in $\overline{u}$, so it is bounded on the compact domain $[a, b]^n$. Hence the uniform convergence of $(\tau_p(\overline{u}))^{1/p}$ to $\tau_\infty(\overline{u})$ follows. This establishes (1.9). To obtain (1.10) take $\overline{u} = (u, \dots, u)$.

Remark. In the above lemma, we may waive the requirement that

$$\sum_{i=1}^{n} w_{p,i} = 1,$$

if we require instead that

$$0 < \liminf_{p\to\infty} w_{p,i} \leq \limsup_{p\to\infty} w_{p,i} < \infty.$$

The proof is essentially the same. See [7, Theorem 1] for a similar argument.

Lemma 2. *For $1 < p \leq \infty$, $\kappa_p(u)$ has a unique minimizer u_p. Moreover,*

$$\lim_{p\to\infty} u_p = u_\infty,$$

and u_∞ is the minimizer of $\kappa_\infty(u)$.

Proof. The proof of [7, Lemma 2] can be modified to obtain the desired result.

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THEOREM 3. *Assume that* $\omega_p = \{\omega_{p,i}\}_{i=1}^n > 0$ *satisfies* (1.8) *and*

$$\sum_{i=1}^n \omega_{p,i} = 1 \quad \text{for all } p,\ 1 < p < \infty.$$

Then the solution $h_p = \{h_{p,i}: i = 1, \dots, n\}$ *given by* (1.3) *and satisfying* (1.2) *converges as* $p \to \infty$ *to a solution* $h_\infty = \{h_{\infty,i}: i = 1, \dots, n\}$ *satisfying*

$$(1.13)\qquad \begin{aligned} &\max(\|f - h_\infty\|_\infty, \|g - h_\infty\|_\infty) \\ &\quad = \inf\{\max(\|f - h\|_\infty, \|g - h\|_\infty): h \in \mathscr{M}\}, \end{aligned}$$

or,

$$\max_{1\le i\le n}(|f_i - h_{\infty,i}|, |g - h_{\infty,i}|) \le \max_{1\le i\le n}(|f_i - h_i|, |g_i - h_i|), \qquad h = \{h_i\}_{i=1}^n \in \mathscr{M}.$$

Moreover

$$(1.14)\qquad \begin{aligned} h_{\infty,i} = \lim_{p\to\infty} h_{p,i} &= \max_{\{U : i\in U\}} \min_{\{L : i\in L\}} \mu_\infty(L \cap U) \\ &= \min_{\{L : i\in L\}} \max_{\{U : i\in U\}} \mu_\infty(L \cap U), \end{aligned}$$

for every i, *where* L *and* U *are lower and upper sets respectively and* $\mu_\infty(L \cap U)$ *is the unique real number minimizing*

$$\max_{j\in L\cap U}(|f_j - u|, |g_j - u|),$$

for all real u.

PROOF. Putting $L \cap U$ instead of X in Lemma 2 above we conclude that

$$\lim_{p\to\infty} \mu_p(L \cap U) = \mu_\infty(L \cap U)$$

exists and $\mu_\infty(L\cap U)$ is the minimizer of $\max\{|f_j - u|, |g_j - u|: j \in L \cap U\}$. Since X is finite, the number of lower and upper sets is finite, so (1.3) implies that the limit of $h_{p,i}$ exists as $p \to \infty$ for all i, and therefore (1.14) holds and it has a real value, say $h_{\infty,i}$, $i = 1, \dots, n$.

It remains to show that (1.13) holds, that is, h_∞ is indeed a best L_∞-simultaneous approximant of f and g. Since $\{h_{p,i}\}_{i=1}^n \in \mathscr{M}$ for every $p < \infty$, $\{h_{\infty,i}\}_{i=1}^n \in \mathscr{M}$. Clearly

$$\min(f_i, g_i) \le h_{p,i} \le \max(f_i, g_i),$$

for all p and i. By definition of $\{h_{p,i}\}_{i=1}^n$, we have $(\tau_p(h_p))^{1/p} \le (\tau_p(\overline{u}))^{1/p}$

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for all $\overline{u} \in \mathscr{R}^n$. Now, let $p \to \infty$ to conclude from (1.9) that $\tau_\infty(h_\infty) \le \tau_\infty(\overline{u})$ for all $\overline{u} \in \mathscr{R}^n$. Hence h_∞ satisfies (1.13). This completes the proof.

2. B.s.a. to step functions

DEFINITION. Let π be a finite partition Ω with points $\{t_i : i = 0, 1, \dots, n\}$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$. Let I_E denote the indicator function of a subset E of Ω. Let S_π be the linear space comprised of all step functions of the form

$$f = \sum_{i=1}^{n} f_i I_{(t_i-1, t_i]}, \tag{2.1}$$

where $f_i \in \mathscr{R}$ for every i, and for $i = 1$, we include the point $t_0 = 0$.

LEMMA 4. *Let f and g be in S_π and let h_p, $1 < p < \infty$, be the b.s.a. of f and g by elements of $\mathscr{M}$. Then $h_p \in S_\pi$.*

PROOF. Suppose h_p is not constant on some sub-interval $(t_j - 1, t_j]$. Let

$$l = \operatorname{essinf}\{h_p(t) : t_{j-1} < t \le t_j\},$$

and

$$u = \operatorname{essup}\{h_p(t) : t_{j-1} < t \le t_j\}.$$

Clearly $l < u$. Choose $\zeta \in [l, u]$ such that

$$|f_j - \zeta|^p + |g_j - \zeta|^p = \inf\{|f_j - r|^p + |g_j - r|^p : r \in [l, u]\},$$

where f_j and g_j are respectively the values of f and g on $(t_{j-1}, t_j]$. Now, let h_p^* be the element of $\mathscr{M}$ defined by

$$h_p^*(t) = \begin{cases} \zeta, & t_{j-1} < t \le t_j, \\ h_p(t), & \text{otherwise.} \end{cases}$$

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Then h_p^* is a better b.s.a. of f and g since

$$\begin{aligned}\|f-h_p^*\|_p^p+\|g-h_p^*\|_p^p &= \sum_{\substack{i=1\\ i\neq j}}^{n}\int_{t_{i-1}}^{t_i}(|f_i-h_p(t)|^p+|g_i-h_p(t)|^p)dt\\ &\quad+\int_{t_{j-1}}^{t_j}(|f_j-\zeta|^p+|g_j-\zeta|^p)dt\\ &<\sum_{\substack{i=1\\ i\neq j}}^{n}\int_{t_{i-1}}^{t_i}(|f_i-h_p(t)|^p+|g_i-h_p(t)|^p)dt\\ &\quad+\int_{t_{j-1}}^{t_j}(|f_j-h_p(t)|^p+|g_j-h_p(t)|^p)dt,\end{aligned}$$

or,

$$\|f-h_p^*\|_p^p+\|g-h_p^*\|_p^p<\|f-h_p\|_p^p+\|g-h_p\|_p^p,$$

a contradiction. Therefore h_p must be constant on $(t_j-1, t_j]$, or $h_p\in S_\pi$.

LEMMA 5. *Let $p\in(1,\infty)$ be fixed. Let f_1, f_2, g_1 and g_2 be elements of S_π. Let h_1 and h_2 be the b.s.a. of f_1, g_1 and f_2, g_2 respectively. If $f_1\le f_2$ and $g_1\le g_2$, then $h_1\le h_2$.*

PROOF. It was shown in [5, Lemma 2] that for $1<p<\infty$ and for all real numbers a, b, c, d with $a\ge c$,

(2.2) $$|a-b|^p+|c-d|^p\ge|a-\max(b,d)|^p+|c-\min(b,d)|^p.$$

Define functions T_1 and T_2 by $T_1(x)=\min(h_1(x), h_2(x))$ and $T_2(x)=\max(h_1(x), h_2(x))$. Applying (2.2) at every $x\in[0,1]$ with $a=f_2(x)$, $b=h_2(x)$, $c=f_1(x)$ and $d=h_1(x)$, we obtain

$$|f_2(x)-h_2(x)|^p+|f_1(x)-h_1(x)|^p\ge|f_2(x)-T_2(x)|^p+|f_1(x)-T_1(x)|^p,$$

and hence by integrating over $[0,1]$ we get

(2.3) $$\|f_2-h_2\|_p^p+\|f_1-h_1\|_p^p\ge\|f_2-T_2\|_p^p+\|f_1-T_1\|_p^p.$$

Similarly, we obtain

(2.4) $$\|g_2-h_2\|_p^p+\|g_1-h_1\|_p^p\ge\|g_2-T_2\|_p^p+\|g_1-T_1\|_p^p.$$

Adding (2.3) to (2.4), we conclude that either

(2.5) $$\|f_2-h_2\|_p^p+\|g_2-h_2\|_p^p\ge\|f_2-T_2\|_p^p+\|g_2-T_2\|_p^p,$$

or

(2.6) $$\|f_1-h_1\|_p^p+\|g_1-h_1\|_p^p\ge\|f_1-T_1\|_p^p+\|_p^p+\|g_1-T_1\|_p^p,$$

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or both of them. If (2.5) holds, then by definition of h_2 we must have

$$h_2 = T_2 = \max(h_1, h_2) \geq h_1.$$

If (2.6) holds, then we end up with $h_1 = T_1 \leq h_2$. This completes the proof.

LEMMA 6. *Let f and g be elements of S_π, and let h_p be their b.s.a. Then $h_p + c$ is the b.s.a. of $f + c$ and $g + c$ where $c \in \mathscr{R}$.*

PROOF. This is clear, since $h_p + c \in \mathscr{M}$ for all c.

REMARK. The last two lemmas are true in general for all bounded Lebesgue measurable functions on [0,1]. The proofs are essentially the same.

THEOREM 7. *Let f and g be elements of S_π given by*

$$f = \sum_{i=1}^{n} f_i I_{(t_{i-1}, t_i]}, \tag{2.7}$$

and

$$g = \sum_{i=1}^{n} g_i I_{(t_{i-1}, t_i]}. \tag{2.8}$$

For every p, $1 < p < \infty$, let $\omega_p = \{\omega_{p,i}\}_{i=1}^n$ be defined by $\omega_{p,i} = t_i - t_{i-1}$ for all i, and let $h_p = \{h_{p,i}\}_{i=1}^n$ be given by (1.3). Then, the b.s.a. of f and g is given by

$$h_p^* = \sum_{i=1}^{n} h_{p,i} I_{(t_{i-1}, t_i]}. \tag{2.9}$$

PROOF. By Lemma 4, we know that $h_p^* \in S_\pi$. Let $X = \{x_1, \dots, x_n\}$, where $x_i = (t_i + t_{i-1})/2$, $i = 1, \dots, n$. Consider $\{f_i : i = 1, \dots, n\}$ and $\{g_i : i = 1, \dots, n\}$ as two finite real valued functions defined on X and let $\{h_i : i = 1, \dots, n\}$, $h_i \leq h_j$ for all $i < j$, be a monotone nondecreasing function on X. Then by substituting the values of $\omega_{p,i}$ in equation (1.2′) we obtain

$$\left[\sum_{i=1}^{n} (t_i - t_{i-1})(|f_i - h_{p,i}|^p + |g_i - h_{p,i}|^p)\right]^{1/p}$$
$$\leq \left[\sum_{i=1}^{n} (t_i - t_{i-1})(|f_i - h_i|^p + |g_i - h_i|^p)\right]^{1/p}$$

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or,

$$\left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(|f_i-h_{p,i}|^p+|g_i-h_{p,i}|^p)\right]^{1/p} \le \left[\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(|f_i-h_i|^p+|g_i-h_i|^p)\right]^{1/p},$$

which implies that

$$\|f-h_p^*\|_p^p+\|g-h_p^*\|_p^p \le \|f-h\|_p^p+\|g-h\|_p^p,$$

for any monotone nondecreasing function $h=\sum_{i=1}^n h_i I_{(t_{i-1},t_i]}$ belonging to S_π.

REMARK. Using Theorem 7, we are able to compute h_p, $p\in(1,\infty)$, when f and g are in S_π by applying (1.3). To compute h_∞, we may use (1.14) and the following theorem.

THEOREM 8. *Let f, g and h_p^* be as given above. Then h_p^* converges as $p\to\infty$ to the monotone nondecreasing function $h_\infty^*\in S_\pi$ given by*

$$h_\infty^*=\sum_{i=1}^n h_{\infty,i}I_{(t_{i-1},t_i]},$$

where $h_{\infty,i}=\lim_{p\to\infty}h_{p,i}$ is given by (1.14). *Furthermore, h_∞^* is a best L_∞-simultaneous approximant of f and g by elements of $\mathscr{M}$.*

PROOF. Let X and ω_p be as defined above. Then Theorem 3 implies the first part of the theorem. It also implies that h_∞^* is a best L_∞-simultaneous approximant of f and g by monontone nondecreasing functions in S_π.

Let h be any monotone nondecreasing function on [0,1]. We show that there is a monotone nondecreasing function $h^*\in S_\pi$ such that

$$\max(\|f-h^*\|_\infty,\|g-h^*\|_\infty)\le\max(\|f-h\|_\infty,\|g-h\|_\infty).$$

Indeed, for $i=1,2,\dots,n$, let

$$h_i^*=(1/2)[\mathrm{essup}(h(x))+\mathrm{essinf}(h(x))],\quad t_{i-1}<x\le t_i.$$

Then clearly

$$|f_i-h_i^*|\le \mathrm{essup}\,|f_i-h(x)|,\qquad t_{i-1}<x\le t_i,$$

and

$$|g_i-h_i^*|\le \mathrm{essup}\,|g_i-h(x)|,\qquad t_{i-1}<x\le t_i,$$

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for all i. Hence,

$$\max(|f_i - h_i^*|, |g_i - h_i^*|) \le \max(\text{essup } |f_i - h(x)|, \text{ essup } |g_i - h(x)|)$$

for all i. Now define $h^* \in S_\pi$ by

$$h^* = \sum_{i=1}^{n} h_i^* I_{(t_{i-1}, t_i]}.$$

Then surely we conclude from the last inequality that

$$\max(\|f - h^*\|_\infty, \|g - h^*\|_\infty) \le (\|f - h\|_\infty, \|f - g\|_\infty).$$

This completes the proof.

REMARK. We have seen that for $f, g \in S_\pi$, we also have $h_p, h_\infty \in S_\pi$. Let us define S to be the collection of all S_π, that is,

$$S = \left\{ \bigcup_\pi S_\pi : \pi \text{ is any finite partition of } [0, 1] \right\}.$$

If f and g are elements of a particular S_π, then we use the notation f_π, g_π. Let $h_{\pi,p}$ denote the best L_p-simultaneous approximant of f_π and g_π and let $h_{\pi,\infty} = \lim_{p\to\infty} h_{\pi,p}$ denote the limit of $h_{\pi,p}$ as p increases to infinity.

3. B.s.a. of quasi-continuous and continuous functions

DEFINITION. A function $f: [0, 1] \to \mathscr{R}$ is said to be quasi-continuous if it has at most discontinuities of the first kind only. Let $\mathscr{Q}$ be the Banach space (sup norm) consisting of all quasi-continuous functions on Ω.

Let $\mathscr{Q}^*$ denote the space of all elements of $\mathscr{Q}$ such that $f(0) = f(0^+)$ and $f(x) = f(x^-)$, $0 < x \le 1$. Then there is a linear isometry between $\mathscr{Q}^*$ and $\mathscr{Q}$, so that we may work with elements of $\mathscr{Q}^*$ only. For simplicity we denote $\mathscr{Q}^*$ by $\mathscr{Q}$. Hence S as defined in Section 1 is a proper subset of $\mathscr{Q}$. We also regard $\mathscr{M}$ as a proper subset of $\mathscr{Q}$, so in fact $\mathscr{M} = \mathscr{M} \cap \mathscr{Q}^*$.

DEFINITION. Let f be a bounded Lebesgue measurable function on $[0, 1]$, and let π be a partition of $[0, 1]$. Then $\overline{f}_\pi$ in S is defined by

$$\text{(3.1)} \qquad \overline{f}_\pi(x) = \sup\{f(y): y \in (t_{i-1}, t_i]\}, \quad x \in (t_{i-1}, t_i], \quad i \ge 1.$$

We define $\underline{f}_\pi$ similarly by replacing sup by inf.

A bounded function f is in $\mathscr{Q}$ if and only if, for any $\varepsilon > 0$, there exists a partition π of $[0, 1]$ such that $0 \le \overline{f}_\pi - \underline{f}_\pi < \varepsilon$. Thus, $\lim_\pi \overline{f}_\pi = \lim_\pi \underline{f}_\pi = f$. Moreover, if π' is a refinement partition of π, then we have

$$\text{(3.2)} \qquad \underline{f}_\pi \le \underline{f}_{\pi'} \le \overline{f}_{\pi'} \le \overline{f}_\pi.$$

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For more details see [2]. Thus, when $f, g \in \mathscr{Q}$, we are able to get as close as we like to f and g. Of course, we count here on the fact that the projection map $(f, g) \mapsto h_p$ is continuous.

LEMMA 9. *Let $f, g \in \mathscr{Q}$, and let $\varepsilon > 0$ be given. Let π be a partition such that $0 \le \overline{f}_\pi - \underline{f}_\pi < \varepsilon$ and $0 \le \overline{g}_\pi - \underline{g}_\pi < \varepsilon$. Then there exists a refinement π' of π such that*

$$\underline{h}_{\pi,p} \le \underline{h}_{\pi',p} \le \overline{h}_{\pi',p} \le \overline{h}_{\pi,p} \le \underline{h}_{\pi,p} + \varepsilon, \tag{3.3}$$

and

$$\underline{h}_{\pi,\infty} \le \underline{h}_{\pi',\infty} \le \overline{h}_{\pi',\infty} \le \overline{h}_{\pi,\infty} \le \underline{h}_{\pi,\infty} + \varepsilon, \tag{3.4}$$

where these h's are as defined earlier in the last remark at the end of Section 2.

PROOF. The discussion preceding the statement of the lemma implies that (3.2) holds for both f and g with the addition of $\overline{f}_\pi < \underline{f}_\pi + \varepsilon$, and $\overline{g}_\pi < \underline{g}_\pi + \varepsilon$. Apply Lemmas 8 and 9 to conclude (3.3). Letting $p \to \infty$ gives us (3.4).

The proof of the next theorem can be obtained following the same line of proof as [2, Theorems 4 and 5], respectively, with the proper changes in the notations used.

THEOREM 10. *Let $f, g \in \mathscr{Q}$, and let h_p, $p \in (1, \infty)$, be their b.s.a.. Then*

$$\lim_\pi \overline{h}_{\pi,p} = \lim_\pi \underline{h}_{\pi,p} = h_p, \tag{3.5}$$

$$\lim_\pi \overline{h}_{\pi,\infty} = \lim_\pi \underline{h}_{\pi,\infty} = h_\infty = \lim_{p\to\infty} h_p. \tag{3.6}$$

The convergence being uniform in both cases.

REMARK. (a) Let f, g, h_p, h_∞ be as defined above. Let h_p^c be the best L_p-simultaneous approximant of $f + c$ and $g + c$. Then, $\lim_{p\to\infty} h_p^c = h_\infty + c$, where $c \in \mathscr{R}$.

(b) Let $f_1 \le f_2$, $g_1 \le g_2$, and let $h_{1,p}, h_{2,p}$ be the b.s.a. of f_1, g_1 and f_2, g_2 respectively, $1 < p < \infty$. If $h_{k,\infty} = \lim_{p\to\infty} h_{k,p}$, $k = 1, 2$, then $h_{1,\infty} \le h_{2,\infty}$.

Our next and final result in this section is a generalization of [2, Theorem 6].

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Theorem 11. *Suppose f and g are continuous. Then h_p is continuous, and so is h_∞.*

Proof. The second part of the conclusion is immediate once the first part is established. So let x be an arbitrary but fixed point in $(0, 1)$, and let $\varepsilon > 0$ be given. Then

$$|h_p(x) - h_p(y)| \le |h_p(x) - \overline{h}_{\pi,p}(x)| + |\overline{h}_{\pi,p}(x) - \overline{h}_{\pi,p}(y)| + |\overline{h}_{\pi,p}(y) - h_p(y)|. \tag{3.7}$$

Since

$$h_p(t) = \lim_\pi \overline{h}_{\pi,p}(t)$$

for all $t \in \Omega$, we may choose a partition $\pi = \{t_i : i = 0, 1, \dots, n\}$ such that

(i) each of the first and third term on the right-hand side of (3.7) is less than $\varepsilon/3$.

(ii) f_π and g_π can be written as

$$\overline{f}_\pi = \sum_{i=1}^{n} \overline{a}_i I_{(t_{i-1}, t_i]}, \tag{3.8}$$

and

$$\overline{g}_\pi = \sum_{i=1}^{n} \overline{b}_i I_{(t_{i-1}, t_i]}, \tag{3.9}$$

with

$$|\overline{a}_i - \overline{a}_{i-1}| < \varepsilon/9, \tag{3.10}$$

and

$$|\overline{b}_i - \overline{b}_{i-1}| < \varepsilon/9, \tag{3.11}$$

for all $i = 2, 3, \dots, n$.

Thus, (3.7) becomes

$$|h_p(x) - h_p(y)| < \varepsilon/3 + |\overline{h}_{\pi,p}(x) - \overline{h}_{\pi,p}(y)| + \varepsilon/3, \tag{3.12}$$

for all $y \in [0, 1]$. We still need to find $\delta > 0$ such that

$$|\overline{h}_{\pi,p}(x) - \overline{h}_{\pi,p}(y)| \le \varepsilon/3, \tag{3.13}$$

provided $y \in (x - \delta, x + \delta)$. We first observe that if $\overline{f}_\pi$ and $\overline{g}_\pi$ are given by (3.8) and (3.9) respectively, then Lemma 7 implies that $\overline{h}_{\pi,p}$ must have the form

$$\overline{h}_{\pi,p} = \sum_{i=1}^{n} c_i I_{(t_{i-1}, t_i]}, \tag{3.14}$$

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for some real numbers $c_1 \leq c_2 \leq \cdots \leq c_n$. We now have only a few cases to consider.

Case 1. If $t_{j-1} < x < t_j$ for some $j \leq n$, then it follows that

$$|\overline{h}_{\pi,p}(x) - \overline{h}_{\pi,p}(y)| = |c_j - c_j| = 0 < \varepsilon/3, \tag{3.15}$$

for all $y \in (t_{j-1}, t_j]$. Let $\delta = \min\{(x - t_{j-1}), (t_j - x)\} > 0$ so (3.13) holds for all $y \in (x - \delta, x + \delta)$ and the continuity of h_p at x is established.

Case 2. $x = t_j$ for some $j < n$. Then (3.15) holds for all $y \in (t_{j-1}, x]$. Thus suppose $y \in (x, t_{j+1}] = (t_j, t_{j+1}]$, and suppose (3.13) does not hold, that is,

$$|h_{\pi,p}(y) - h_{\pi,p}(x)| = h_{\pi,p}(y) - h_{\pi,p}(x) = c_{j+1} - c_j > \varepsilon/3. \tag{3.16}$$

In Figure 1 below we fix c_j, c_{j+1} and we may also without loss of generality fix a_j, a_{j+1} and then we discuss briefly the various possibilities for the values of $\overline{b}_j$, $\overline{b}_{j+1}$ and each time we end up with a contradiction.

P1. If $a_{j+1} < \overline{b}_{j+1} < c_{j+1}$, then we may replace c_{j+1} in (3.14) by $\max(c_j, \overline{b}_{j+1})$ to obtain a better b.s.a. of f and g, a contradiction. A similar conclusion holds if $c_j < \overline{b}_j < c_{j+1}$.

P2. If $\overline{b}_{j+1} < \overline{a}_{j+1} < c_{j+1}$, then we may replace c_{j+1} by $\max(c_j, \overline{b}_{j+1})$ to obtain a better b.s.a., a contradiction.

P3. If $\overline{b}_j > c_{j+1} > c_j$, then replace c_j by $\overline{a}_j$ to obtain a better b.s.a, a contradiction. The same argument is valid if in addition we assume that $\overline{b}_{j+1} > c_{j+1}$.

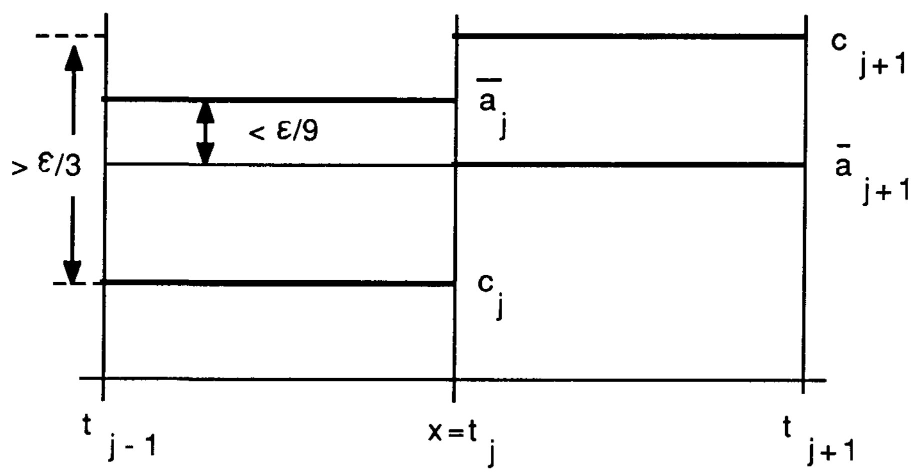

FIGURE 1

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P4. If $\bar{b}_j < c_j$ and $\bar{b}_{j+1} > c_{j+1}$, then $\bar{b}_{j+1} - \bar{b}_j > \varepsilon/9$, contradicting (3.11).

4. Examples

Example 1. Let f be the real-valued step function defined on $[0, 1]$ by

$$f = 3I_{[0,1/15]} + 5I_{(3/15,4/15]} + 7I_{(8/15,9/15]}, \tag{4.1}$$

and let $g \equiv 0$ on [0,1]. Then $h_2 \equiv 1/2$ on [0,1] which is the same as the best L_2-approximant of the single function $(f+g)/2$. This is always true, that is, the b.s.a. of f and g in the L_2-norm is equal to the best L_2-approximant of their mean [6, Theorem 3]. However, h_3 is not constant and it is given by

$$h_3 = (3/(\sqrt{5}+1))I_{[0,3/15]} + (5/4)I_{(3/15,8/15]} + (7/(\sqrt{13}+1))I_{(8/15,1]},$$

while

$$h_\infty = (3/2)I_{[0,3/15]} + (5/2)I_{(3/15,8/15]} + (7/2)I_{(8/15,1]}. \tag{4.2}$$

In general if $g \equiv 0$ and f is given by

$$f = k_1 I_{[0,t_1]} + k_2 I_{(t_2,t_3]} + \cdots + k_n I_{(t_{2(n-1)},t_{2n-1}]}, \tag{4.3}$$

where

$$2 < k_1 < k_2 < \cdots < k_n,$$

and

$$t_1 = \delta = \left(\sum_{j=1}^{n} k_j\right)^{-1},$$

$$t_{2i} = \left(\sum_{j=1}^{i} k_j\right)\delta, \qquad j \ge 1,$$

$$t_{2i+1} = t_{2i}\delta, \qquad i \ge 1,$$

then for every p, h_p must have the form

$$h_p = \zeta_1 I_{[0,t_2]} + \zeta_2 I_{(t_2,t_4]} + \cdots + \zeta_n I_{(t_{2(n-1)},1]}, \tag{4.4}$$

where $0 < \zeta_1 \le \zeta_2 \le \cdots \le \zeta_n$ and ζ_i depends on p for every i. Let us compute h_2 which has the form (4.4). Clearly ζ_1 is the unique real number minimizing the quadratic function

$$\begin{aligned} T_1(\zeta) &= \delta(k_1-\zeta)^2 + \delta(0-\zeta)^2 + (t_2-t_1)(0-\zeta)^2 + (t_2-t_1)(0-\zeta)^2 \\ &= \delta(k_1-\zeta)^2 + \delta\zeta^2 + 2\delta(k_1-1)\zeta^2. \end{aligned}$$

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Differentiating T_1 we get

$$T_1'(\zeta_1) = -2\delta(k_1 - \zeta_1) + 2\delta\zeta_1 + 4\delta(k_1 - 1)\zeta_1$$
$$= 2\delta k_1(2\zeta_1 - 1) = 0.$$

Thus, $\zeta_1 = 1/2$. Similarly, ζ_i is the unique real number minimizing the function

$$T_i(\zeta) = \delta(k_i - \zeta)^2 + \delta\zeta^2 + 2\delta(k_i - 1)\zeta^2,$$

which implies that $\zeta_i = 1/2$ for all $i \geq n$. Hence $h_2 \equiv 1/2$ on [0,1].

Suppose next, that we want to compute h_p for $p > 2$. Then in general ζ_i is the unique real number minimizing the function

$$T_i(\zeta) = \delta(k_i - \zeta)^p + \delta\zeta^p + 2\delta(k_i - 1)\zeta^p.$$

Differentiating T_i, we get

$$T_1'(\zeta_i) = -p(k_i - \zeta_i)^{p-1} + p(2k_i - 1)\zeta_i^{p-1} = 0,$$

which implies upon dividing by $(p\zeta_i^{p-1})$ that

$$2k_i - 1 = ((k_i/\zeta_i) - 1)^{p-1},$$

or

$$(2k_i - 1)^{1/\lambda} = (k_i/\zeta_i) - 1, \qquad \lambda = p - 1,$$

Hence

$$\zeta_i = k_i/((2k_i - 1)^{1/\lambda} + 1),$$

for $i = 1, 2, \ldots, n$.

Now observe that $\zeta_i \to k_i/2$ as $p = \lambda + 1 \to \infty$ which implies that h_p converges to a function

$$(4.6) \quad h_\infty = \lim_{p\to\infty} h_p = (k_1/2)I_{[t_0, t_2]} + (k_2/2)I_{(t_2, t_4]} + \cdots + (k_n/2)I_{(t_{2(n-1)}, 1]},$$

which would be identical with the value of h_∞ computed using (1.14).

It can also be shown that ζ_i increases as k_i increases by differentiating (4.5) with respect to k_i and observing that the derivative is always positive.

Notice that a function $h \in \mathscr{M}$ is a best simultaneous L_∞-approximant of f and g if and only if $\underline{h} \leq h \leq \overline{h}$, where $\underline{h}$, $\overline{h} \in \mathscr{M}$ are given by $\overline{h} \equiv k_n/2$ on $[0, 1]$, and

$$\underline{h} = ((2k_1 - k_n)/2)I_{[t_0, t_2]} + ((2k_2 - k_n)/2)I_{(t_2, t_4]} + \cdots + (k_n/2)I_{(t_{2(n-1)}, 1]}.$$

We show in Example 2 that the simultaneous Polya property does not hold in general for any two bounded Lebesgue measurable function f and g, that is, the statement of Theorem 14(b) is not true in general. But before we proceed, we prove a little lemma.

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LEMMA 12. *Let f and g be bounded Lebesgue measurable functions with $f \neq g$. Let f_p and g_p be the best L_p-approximant of f and g respectively by elements of $\mathscr{M}$, and let h_p be their b.s.a. by elements of $\mathscr{M}$. Then*

$$\|f - g_p\|_p \geq \|f - h_p\|_p, \tag{4.7}$$

and

$$\|g - f_p\|_p \geq \|g - h_p\|_p. \tag{4.8}$$

PROOF. By definition of f_p and g_p we have

$$\|f - f_p\|_p \leq \|f - h_p\|_p, \tag{4.9}$$

and

$$\|g - g_p\|_p \leq \|g - h_p\|_p. \tag{4.10}$$

If (4.7) does not hold, then

$$\|f - g_p\|_p < \|f - h_p\|_p.$$

Combining this inequality with (4.10) implies that

$$\|f - g_p\|_p^p + \|g - g_p\|_p^p < \|f - h_p\|_p^p + \|g - h_p\|_p^p,$$

or g_p is a better b.s.a. of f and g, a contradiction Therefore (4.7) must be true. Similarly we obtain (4.8). This completes the proof.

EXAMPLE 2. Next, let f be the bounded Lebesgue measurable function given in [1] and defined on $[0, 2]$. Let $g \equiv 7$ on $[0, 2]$. Then clearly $g \in \mathscr{M}$ which implies that $g = g_p \equiv 7$ on $[0, 2]$. Also f as given is continuous on $[0, 1) \cup (1, 2]$ and approximately continuous at 1. Let $\{p_k\}$ be as chosen in [1]. Then $7 < f_{p_{2n-1}} < 7.5$ on [1, 2] and $5 < f_{p_{2n}} < 6$ on [1,2], $n = 1, 2, \ldots$. Since $g_{p_{2n-1}} = g_{p_{2n}} \equiv 7$ on [0, 2]. Then, we clearly have by the last lemma that $h_{p_{2n-1}} > 7$, otherwise we get $\|f - f_p\|_p > \|f - h_p\|_p$. Similarly for p_{2n} we must have $h_{p_{2n}} < 7$, otherwise we get $\|f - g_p\|_p < \|f - h_p\|_p$, a contradiction. Therefore the sequence $\{h_{p_k}\}_{k=1}^{\infty}$ does not converge anywhere.

References

[1] R. B. Darst and R. Huotari, 'Monotone approximation on an interval', *Classical real analysis*, edited by D. Waterman, pp. 43–44 (Contemp. Math. 42, Amer. Math. Soc. Providence, R.I., 1985).

[2] R. B. Darst and S. Sahab, 'Approximation of continuous and quasicontinuous functions by monotone functions', *J. Approx. Theory* **38** (1983), 9–27.

https://doi.org/10.1017/S1446788700032997 Published online by Cambridge University Press

[3] D. S. Goel, A. S. B. Holland, C. Nasim and B. N. Sahney, 'Characterization of an element of best l^p-simultaneous approximation', *S. Ramanaujan Memorial Volume, Madras* (1984), 10–14.
[4] A. S. B. Holland and B. N. Sahney, 'Some remarks on best simultaneous approximation', *Theory of approximation with applications*, edited by A. G. Law and B. N. Sahney, pp. 332–337 (Academic Press, New York, 1976).
[5] D. Landers and L. Rogge, 'On projections and monotony in L_p-spaces', *Manuscripta Math* **26** (1979), 363–369.
[6] G. M. Phillips and B. N. Sahney, 'Best simultaneous approximation in the L_1 and L_2 norms', *Theory of approximation with applications*, edited by A. G. Law and B. N. Sahney, pp. 332–337 (Academic Press, New York, 1976).
[7] V. A. Ubhaya, 'Isotone optimization II', *J. Approx. Theory* **12** (1974), 315–331.
[8] C. Van Eeden, *Testing and estimating ordered parameters of probability distribution*, (Ph.D. dissertation, University of Amsterdam, Studentendrukkerij Poortpers N.V., Amsterdam, 1958).

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