ON THE STRUCTURE OF CERTAIN BASIC SEQUENCES ASSOCIATED WITH AN ARITHMETIC FUNCTION

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1. Introduction

We have previously studied in some detail the multiplicative properties of a given arithmetic function f with respect to a fixed basic sequence $\mathscr{B}$ (see, for example, (**1**), (**2**)). We investigate here the structure of $M(f)$, the collection of all basic sequences $\mathscr{B}$ such that f is multiplicative with respect to $\mathscr{B}$, and in particular we focus our attention on the maximal members of $M(f)$. Our principal result will be a proof that each maximal member of $M(f)$ contains the same set of type II primitive pairs. Moreover, we will give a simple criterion for determining, in terms of the behaviour of f, whether or not a particular primitive pair (p, p) is in any (and therefore every) maximal member of $M(f)$.

A *basic sequence* $\mathscr{B}$ is a set of pairs (a, b) of natural numbers for which

(1) $(1, k) \in \mathscr{B}, \quad k = 1, 2, \ldots;$

(2) if $(a, b) \in \mathscr{B}$, then $(b, a) \in \mathscr{B}$;

(3) $(a, bc) \in \mathscr{B}$ if and only if $(a, b) \in \mathscr{B}$ and $(a, c) \in \mathscr{B}$.

If Φ is any collection of pairs of natural numbers, we set $\Gamma[\Phi] = \bigcap \mathscr{E}$, where the intersection is taken over all basic sequences $\mathscr{E}$ which contain Φ. If $\Phi = \varnothing$, then $\Gamma[\Phi] = \mathscr{S}$, where $\mathscr{S}$ is the basic sequence consisting only of all pairs of the form $(1, k)$ and $(k, 1)$ $(k = 1, 2, \ldots)$. A pair (a, b) of natural numbers is called a *primitive pair* if both a and b are primes. It is of *type I* if $a \neq b$, *type II* if $a = b$.

We assume, in order to avoid trivial situations, that no arithmetic function is eventually zero. An arithmetic function f is said to be *multiplicative* with respect to a basic sequence $\mathscr{B}$ if $f(m)f(n) = f(mn)$ for all $(m, n) \in \mathscr{B}$. The set of all arithmetic functions which are multiplicative with respect to $\mathscr{B}$ is denoted by $M(\mathscr{B})$, and for a given arithmetic function f, $M(f)$ represents the set of all basic sequences $\mathscr{B}$ for which $f \in M(\mathscr{B})$.

A basic sequence $\mathscr{B}$ is a *maximal* member of $M(f)$ if $f \in M(\mathscr{B})$, but $f \notin M(\mathscr{B}')$ for any basic sequence $\mathscr{B}'$ which properly contains $\mathscr{B}$. The set of maximal members of $M(f)$ is denoted by $M^*(f)$. We prove in Lemma 2.1 that every member of $M(f)$ is contained in a member of $M^*(f)$, hence for the study of $M(f)$ it is sufficient to confine our attention to $M^*(f)$.

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2. The structure of $M^*(f)$

We will show first that for $M^*(f)$ to be non-empty it is both necessary and sufficient that $f(1) = 1$. This result is an easy consequence of

Lemma 2.1. *If $\mathscr{B}_0 \in M(f)$, then there is a basic sequence $\mathscr{B}'$ in $M^*(f)$ such that $\mathscr{B}_0 \subset \mathscr{B}'$.*

Proof. The proof of the lemma will depend on Tukey's Lemma: Every non-empty family of sets of finite character has a maximal member. (A family $\mathfrak{A}$ of sets is of *finite character* provided $A \in \mathfrak{A}$ if and only if every finite subset of A is in $\mathfrak{A}$. A is a *maximal* member of $\mathfrak{A}$ if $A \in \mathfrak{A}$ and if there is no member A' of $\mathfrak{A}$ such that A' properly contains A.)

Let Φ_0 be the set of primitive pairs in $\mathscr{B}_0$ (take $\Phi_0 = \varnothing$ if $\mathscr{B}_0 = \mathscr{S}$) and define $\mathfrak{A}$ to be the family of all sets Φ of primitive pairs for which

$$f \in M(\Gamma[\Phi \cup \Phi_0]).$$

We will show that $\mathfrak{A}$ is non-empty and of finite character. By Tukey's Lemma, $\mathfrak{A}$ will then contain a maximal member, say Φ', and it is clear that the basic sequence $\mathscr{B}' = \Gamma[\Phi']$ will satisfy the requirements of Lemma 2.1.

$\mathfrak{A}$ is non-empty, since $f \in M(\mathscr{B}_0) = M(\Gamma[\phi \cup \Phi_0])$, so $\phi \in \mathfrak{A}$.

Suppose that $\Phi \in \mathfrak{A}$ and that Ψ is any finite subset of Φ. Since

$$f \in M(\Gamma[\Phi \cup \Phi_0])$$

and since $\Gamma[\Psi \cup \Phi_0] \subset \Gamma[\Phi \cup \Phi_0]$, it follows that $f \in M(\Gamma[\Psi \cup \Phi_0])$; hence $\Psi \in \mathfrak{A}$.

Conversely, let Φ be a set of primitive pairs and suppose $\Psi \in \mathfrak{A}$ for every finite subset Ψ of Φ. Let (a, b) be any pair in $\Gamma[\Phi \cup \Phi_0]$ and let Ψ_1 be the set of all primitive pairs (p, q) for which $p \mid a$ and $q \mid b$. Clearly $\Psi_1 \subset \Phi \cup \Phi_0$. Now set

$$\Psi = \Psi_1 \cap \Phi.$$

Then Ψ is a finite subset of Φ, so by assumption $\Psi \in \mathfrak{A}$. Therefore

$$f \in M(\Gamma[\Psi \cup \Phi_0]).$$

But

$$(a, b) \in \Gamma[\Psi_1] = \Gamma[\Psi_1 \cap (\Phi \cup \Phi_0)]$$
$$= \Gamma[\Psi \cup (\Psi_1 \cap \Phi_0)] \subset \Gamma[\Psi \cup \Phi_0],$$

so $f(ab) = f(a)f(b)$. It follows that $f \in M(\Gamma[\Phi \cup \Phi_0])$ and so $\Phi \in \mathfrak{A}$.

Thus $\mathfrak{A}$ is of finite character and the proof is complete.

The result of Lemma 2.1 provides the basis for an easy proof of

Theorem 2.2. *$M^*(f) \neq \varnothing$ if and only if $f(1) = 1$.*

Proof. If $f(1) = 1$, then $f \in M(\mathscr{S})$, so $\mathscr{S} \in M(f)$. By Lemma 2.1 there is a basic sequence $\mathscr{B}'$ in $M^*(f)$, therefore $M^*(f) \neq \varnothing$.

On the other hand, if $f(1) \neq 1$, then $f \notin M(\mathscr{B})$ for any basic sequence $\mathscr{B}$; that is, $M(f) = \varnothing$. But since $M^*(f) \subset M(f)$, we have $M^*(f) = \varnothing$.

The next theorem will yield not only the previously mentioned result about the type II primitive pairs in the members of $M^*(f)$, but also will provide information about the distribution of the type I primitive pairs. We will use the following notation: For a given basic sequence $\mathscr{B}$ and a given prime p, we define

$$C_{\mathscr{B}}(p) = \{q \mid q \text{ prime}, (p, q) \in \mathscr{B}\}.$$

Theorem 2.3. *If*

$$f(p^a q^b) = f(p^a) f(q^b) \tag{2.1}$$

for all natural numbers a and b, then the primitive pair (p, q) is contained in every basic sequence $\mathscr{B}$ in $M^*(f)$ for which $C_{\mathscr{B}}(p) = C_{\mathscr{B}}(q)$.

Proof. Suppose that $\mathscr{B} \in M^*(f)$ and $C_{\mathscr{B}}(p) = C_{\mathscr{B}}(q)$, but that the primitive pair $(p, q) \notin \mathscr{B}$. Define the basic sequence $\mathscr{B}'$ by

$$\mathscr{B}' = \Gamma[\mathscr{B} \cup (p, q)].$$

Since $\mathscr{B}'$ properly contains $\mathscr{B}$, $f \notin M(\mathscr{B}')$.

Any element in $\mathscr{B}' - \mathscr{B}$ must be of the form $(p^a v, q^b w)$ or $(q^b w, p^a v)$ where $p \nmid v$ and $q \nmid w$, where $a \geqq 1$ and $b \geqq 1$, where v and w are divisible only by primes from $C_{\mathscr{B}}(p)$ $(= C_{\mathscr{B}}(q))$, and where $(v, w) \in \mathscr{B}$. Since $(p, v), (p, w), (q, v), (q, w)$ are all in $\mathscr{B}$, $(p^a q^b, vw)$ is also in $\mathscr{B}$. Therefore, for any pair $(p^a v, q^b w)$ in $\mathscr{B}' - \mathscr{B}$, we have

$$f(p^a q^b vw) = f(p^a q^b) f(vw) = f(p^a q^b) f(v) f(w), \tag{2.2}$$

$$f(p^a v) = f(p^a) f(v), \quad f(q^b w) = f(q^b) f(w). \tag{2.3}$$

On the other hand, since $f \notin M(\mathscr{B}')$ there is a pair (m, n) in $\mathscr{B}' - \mathscr{B}$ for which $f(mn) \neq f(m) f(n)$. So for some choice of a, b, v, w we have

$$f(p^a q^b vw) \neq f(p^a v) f(q^b w). \tag{2.4}$$

For this choice of a, b, v, w, relations (2.2), (2.3), (2.4) yield

$$f(p^a q^b) f(v) f(w) \neq f(p^a) f(q^b) f(v) f(w),$$

and so $f(p^a q^b) \neq f(p^a) f(q^b)$.

Corollary 2.4. *If $\mathscr{B}$ and $\mathscr{B}'$ are members of $M^*(f)$ and the primitive pair $(p, q) \in \mathscr{B}' - \mathscr{B}$, then $C_{\mathscr{B}}(p) \neq C_{\mathscr{B}}(q)$.*

A prime p is said to be *isolated* from a basic sequence $\mathscr{B}$ if $C_{\mathscr{B}}(p) = \varnothing$.

Corollary 2.5. *If $\mathscr{B}$ and $\mathscr{B}'$ are members of $M^*(f)$ and the primitive pair $(p, q) \in \mathscr{B}'$, then either p or q (or both) is not isolated from $\mathscr{B}$.*

If we set $p = q$ in Theorem 2.3 we get the desired characterization of the type II primitive pairs in the members of $M^*(f)$.

Theorem 2.6. *Every basic sequence in $M^*(f)$ contains the same set of type II primitive pairs, namely, those pairs (p, p) for which p satisfies*

$$f(p^n) = f^n(p) \quad (n = 1, 2, \dots). \tag{2.5}$$

Proof. If $f(1) \neq 1$, then $M^*(f)$ is empty and there is nothing to prove. Otherwise, suppose $f(1) = 1$ and $(p, p) \in \mathscr{B}_0$ for some basic sequence $\mathscr{B}_0$ in $M^*(f)$. Then $(p^a, p^b) \in \mathscr{B}_0$ and, since $f \in M(\mathscr{B}_0)$, (2.1) holds (with $p = q$). Since $p = q$ implies $C_{\mathscr{B}}(p) = C_{\mathscr{B}}(q)$ for *every* basic sequence $\mathscr{B}$, it follows from Theorem 2.3 that (p, p) is in every member of $M^*(f)$. Thus the members of $M^*(f)$ contain the same type II primitive pairs, and these are clearly just the pairs (p, p) such that p satisfies (2.5).

3. An example

In the previous section we investigated the structure of $M^*(f)$, the set of maximal members of $M(f)$. We may now ask the following question: Suppose the requirement that f be multiplicative with respect to $\mathscr{B}$ is replaced by the less stringent requirement that f be non-singular with respect to $\mathscr{B}$ (and, accordingly, $M(f)$ is replaced by the larger collection $N(f)$, consisting of those basic sequences $\mathscr{B}$ such that f is non-singular with respect to $\mathscr{B}$). What can be said about the structure of $N^*(f)$, the maximal members of $N(f)$? We will show here (in Example 3.1) that there are arithmetic functions f for which $N(f)$ has no maximal members, even though $f(1) = 1$ and $N(f)$ is not empty (compare this with Lemma 2.1 and Theorem 2.2). Thus while the requirement $f(1) = 1$ is enough to guarantee that $N(f)$ is not empty, it is not sufficient to ensure that $N^*(f)$ is not empty.

As a matter of convenience we repeat here the pertinent definitions (see (1) for a more complete exposition). For an arithmetic function f and a pair (m, n) of natural numbers we set

$$\alpha_f(m, n) = \begin{cases} \dfrac{f(m)f(n) - f(mn)}{|f(m)f(n)| + |f(mn)|} & \text{if } |f(m)f(n)| + |f(mn)| > 0, \\ 0 & \text{if } f(m)f(n) = f(mn) = 0. \end{cases}$$

We say that the *index of multiplicativity* of f with respect to the basic sequence $\mathscr{B}$ exists and has the value $I(f, \mathscr{B})$ provided

$$\lim_{k \to \infty} \alpha_f(m_k, n_k) = I(f, \mathscr{B})$$

for every sequence of pairs $\{(m_k, n_k)\}_{k=1}^{\infty}$ contained in $\mathscr{B}$ for which

$$\lim_{k \to \infty} m_k n_k = \infty.$$

We say f is *non-singular* with respect to $\mathscr{B}$ if $I(f, \mathscr{B})$ exists and has the value zero, and we denote the set of all functions which are non-singular with respect to $\mathscr{B}$ by $N(\mathscr{B})$. We denote by $N(f)$ the set of basic sequences $\mathscr{B}$ for which $f \in N(\mathscr{B})$, and by $N^*(f)$ the set of maximal members of $N(f)$.

For any arithmetic function f, if $f \in M(\mathscr{B})$ then clearly $f \in N(\mathscr{B})$. Therefore

$$M(f) \subset N(f).$$

The above inclusion may or may not be proper; it is easy to find functions satisfying either alternative.

Example 3.1. Define f by

$$f(1) = 1, f(p) = 1 \text{ (p prime)}, f(n) = 0 \text{ otherwise.}$$

We note that $M(f) = M^*(f) = \{\mathscr{S}\}$, for if (p, q) is any primitive pair, then $f(pq) = 0 \neq 1 = f(p)f(q)$.

The proof that $N^*(f)$ is empty will depend on the fact that $f \notin N(\mathscr{B})$ for any basic sequence $\mathscr{B}$ which contains infinitely many type II primitive pairs. For suppose the sequence of primitive pairs $\{(p_n, p_n)\}_{n=1}^{\infty}$ is in $\mathscr{B}$, where we may suppose that $p_1 < p_2 < \dots$. Then

$$\alpha_f(p_n, p_n) = \frac{f^2(p_n) - f(p_n^2)}{f^2(p_n) + f(p_n^2)} = 1.$$

Therefore $\lim_{n \to \infty} \alpha_f(p_n, p_n) \neq 0$ and so $f \notin N(\mathscr{B})$.

Suppose now that $\mathscr{B}$ is any member of $N(f)$ (these exist: $\mathscr{S}$, for example, or any basic sequence generated by a finite number of type II primitive pairs). By the remark above, $\mathscr{B}$ can contain only finitely many type II primitive pairs. Suppose then that $(q, q) \notin \mathscr{B}$ for some prime q and let

$$\mathscr{B}' = \Gamma[\mathscr{B} \cup (q, q)].$$

Since $\mathscr{B}'$ properly contains $\mathscr{B}$, it is sufficient to prove that $f \in N(\mathscr{B}')$.

Let $\{(m_\nu, n_\nu)\}_{\nu=1}^{\infty}$ be any sequence of pairs in $\mathscr{B}'$ for which $m_\nu > 1$, $n_\nu > 1$, $m_\nu n_\nu \to \infty$. Split the sequence $\{m_\nu, n_\nu\}$ into two parts: (1) those (m_ν, n_ν) in $\mathscr{B}$, (2) those (m_ν, n_ν) in $\mathscr{B}' - \mathscr{B}$.

For (1) we have immediately $\lim_{\nu \to \infty} \alpha_f(m_\nu, n_\nu) = 0$ since $f \in N(\mathscr{B})$.

Suppose then that $(m_\nu, n_\nu) \in \mathscr{B}' - \mathscr{B}$ and $m_\nu n_\nu > q^2$. Then

$$m_\nu = x_\nu q^{a_\nu}, \; n_\nu = y_\nu q^{b_\nu} \text{ with } a_\nu \geqq 1, \; b_\nu \geqq 1, \tag{3.1}$$

and either $m_\nu > q$ or $n_\nu > q$, say $m_\nu > q$. If m_ν were prime, then $m_\nu = q$ by (3.1). But $m_\nu > q$, so m_ν is not prime and $f(m_\nu) = 0$. Hence $f(m_\nu)f(n_\nu) = 0$. On the other hand, $m_\nu n_\nu$ is not prime since $m_\nu > 1$ and $n_\nu > 1$, and therefore $f(m_\nu n_\nu) = 0$. Thus $f(m_\nu n_\nu) = f(m_\nu)f(n_\nu)$ and $\lim_{\nu \to \infty} \alpha_f(m_\nu, n_\nu) = 0$ as $m_\nu n_\nu \to \infty$ with (m_ν, n_ν) in $\mathscr{B}' - \mathscr{B}$.

It follows that $f \in N(\mathscr{B}')$ and the proof that $N(f)$ has no maximal members is complete.

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