THE CENTRE OF A HEREDITARY LOCAL RING

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The purpose of this note is to establish the following

THEOREM. *The centre of a* (*left*) *hereditary local ring is either a field or a one-dimensional regular local ring.*

Before starting the proof, it is necessary to explain the terminology. A ring R with an identity element is called a *left local ring* if the elements of R which do not have left inverses form a left ideal I. In these circumstances (see [**1**, Proposition 2.1, p. 147]), I is necessarily a two-sided ideal and it consists precisely of all the elements of R which do not have right inverses. Furthermore, every element of R which is not in I possesses a two-sided inverse. Thus there is, in fact, no difference between a left local ring and a right local ring and therefore one speaks simply of a *local ring*. In addition, I contains every proper left ideal and every proper right ideal. We may therefore describe I simply as the *maximal ideal* of R.

A ring R is called *left hereditary* if every left ideal is a projective R-module. Now, by a theorem due to Kaplansky [**2**, Theorem 2, p. 374], every projective module with respect to a local ring is free. Accordingly, *if R is a left hereditary local ring, then every left ideal possesses a base.*

From now on let R be a (left) hereditary local ring and let I be its maximal ideal. Further, let Q be the centre of R and assume that Q is *not* a field. We contend that Q is a one-dimensional regular local ring. *This includes the assertion that Q is Noetherian.*

Put $J = Q \cap I$, so that J is a proper ideal of Q. Suppose that $q \in Q$, $q \notin J$. Then q has an inverse x in R and one verifies immediately that $x \in Q$. It follows that Q is a local ring and J is its maximal ideal. Since Q is not a field, $J \neq (0)$.

Now consider RJ. This is a left ideal of R and therefore it possesses an R-base. Furthermore, since $RJ \neq (0)$ and I is a two-sided ideal, RJ is not contained in $I(RJ)$; consequently we can find $\gamma \in J$ such that $\gamma \notin I(RJ)$. If now γ is expressed in terms of the base of RJ, at least one of the coefficients will be a unit in R. It follows that one of the base elements can be replaced by γ in such a way that we still get a base. But γ is now the *only* element in the new base. For suppose that u were a different base element. Then, since γ belongs to the centre of R, $u\gamma + (-\gamma)u = 0$, and this gives a contradiction. This shows that $RJ = R\gamma$. Furthermore, since γ is a base for this ideal, γ is not a zero-divisor in R.

Let $q \in R\gamma \cap Q$; then $q = z\gamma$ with $z \in R$. If now y is any element of R,

$$zy\gamma = z\gamma y = qy = yq = yz\gamma,$$

whence $zy = yz$ because γ is not a zero-divisor. This shows that $z \in Q$ and we conclude that $R\gamma \cap Q \subseteq Q\gamma$. But $R\gamma \cap Q = RJ \cap Q \supseteq J$ and therefore $J = Q$.

Next put

$$A = \bigcap_{=1}^{\infty} Q\gamma^{n}$$

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so that A is a Q-ideal. Since γ is not a zero-divisor, we see that $\gamma A = A$. Remembering that γ is in the centre of R, we obtain

$$RA = \gamma(RA) \subseteq I(RA).$$

But RA, being a left ideal of R, possesses an R-base and now the above relation shows that RA must be the zero ideal. Thus $A = (0)$ and hence

$$\bigcap_{n=1}^{\infty} Q\gamma^n = (0).$$

Finally, let B be any non-zero ideal of Q. If m is the largest integer such that $B \subseteq Q\gamma^m$, then we see easily that $B = Q\gamma^m$. Thus Q is a commutative local ring, which is not a field, in which every ideal is a principal ideal. Since the maximal ideal contains at least one element which is not a zero-divisor, this completes the proof.

REFERENCES

1. H. Cartan and S. Eilenberg, *Homological algebra* (Princeton, 1956).
2. I. Kaplansky, Projective modules, *Ann. of Math.* **68** (1958), 372–377.

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