J. Aust. Math. Soc. **99** (2015), 175–185
doi:10.1017/S1446788715000026

BOUNDED MULTIPLE SOLUTIONS FOR p-LAPLACIAN PROBLEMS WITH ARBITRARY PERTURBATIONS

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(Received 22 October 2014; accepted 17 November 2014; first published online 26 February 2015)

Communicated by J. M. Borwein

Abstract

In the present paper we deal with the existence of multiple solutions for a quasilinear elliptic problem involving an arbitrary perturbation. Our approach, based on an abstract result of Ricceri, combines truncation arguments with Moser-type iteration technique.

2010 *Mathematics subject classification*: primary 35J20; secondary 35J62, 35J92.

Keywords and phrases: variational methods, bounded multiple solutions, quasilinear elliptic equations, Moser iteration technique, arbitrary perturbations.

1. Introduction

In the present paper, we deal with the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,\mu})$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$, $p > 1$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and λ, μ nonnegative parameters.

The existence of three solutions for λ and μ in convenient intervals has been established by Ricceri in [7] when f is superlinear at zero and sublinear at infinity, and, when $p < N$, g has a subcritical growth at infinity, that is,

$$|g(x,u)| \le C(1 + |u|^{q-1})$$

for some $1 < q < p^* = Np/(N-p)$, $C > 0$.

In [8], Zhao and Zhao proved the existence of two nonzero bounded weak solutions for the elliptic problem $(P_{\lambda,\mu})$ when $p < N$ and $g(x,u) = |u|^{r-2}u$ with $r \ge p^*$. When the perturbation has a supercritical growth, variational techniques do not apply as the energy functional related to the problem is not well defined on the Sobolev space

$W_0^{1,p}(\Omega)$. This difficulty is overcome in [8] by truncating the power term and by proving that the critical points of the truncated functional are actually bounded. More precisely, combining the abstract three critical points theorem of Ricceri [7] with a Moser iteration technique, they obtain two nonzero, bounded, weak solutions of the above problem for suitable λs and μs.

In the present note, we will prove that the method of [8] can be adapted to arbitrary perturbations (not only pure power terms) and also when $p \geq N$. Indeed, because of the particular nature of the conclusion of [7], the match with a Moser-type iteration argument turns out to be successful also in the present framework (see Remark 2.2).

In connection with multiplicity results for nonlinear problems with arbitrary perturbations, we mention the papers of Anello [1], Chen and Li [2], Iturriaga *et al.* [3] and Lorca and Ubilla [5]. In [2, 3, 5] the authors deal with a nonlinear problem of the following type:

$$\begin{cases} -\Delta_p u = \lambda h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where h is a continuous function, λ a positive parameter. In [5] the authors, among other applications, prove the existence of a nonnegative solution for problem (P_λ) when $p = 2$ and λ is big enough, with suitable monotonicity and growth assumptions on the nonlinearity near zero. The approach consists in modifying h outside a neighborhood of the origin in order to ensure a subcritical behavior of the associated Euler Lagrange functional so that variational methods apply. Suitable *a priori* estimates allow a solution of the original problem to be obtained. Multiple solutions for (P_λ) are obtained in [3] when h has a positive zero and it satisfies a p-linear condition only at zero. It is proved, for λ big enough, that there exist a first solution via sub-supersolution techniques and a second one by topological degree arguments. No assumptions at infinity are required. A superlinear behavior at zero is assumed in [2] where truncation methods and minimax theorems are employed to study the multiplicity of solutions for problem (P_λ) when $p = 2$.

A closer comparison is possible with the main result of [1]. The author investigates the existence and multiplicity of solutions of the problem

$$\begin{cases} -\Delta u = f(x, u) + \mu g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where both f and g have arbitrary growth, f is a changing sign nonlinearity, sublinear at zero. The existence of two solutions, one positive and the other negative, is obtained for small values of the parameter μ by employing direct methods and truncation arguments.

In the present note, as mentioned above, we focus on the arbitrary perturbation. We follow the idea of [8] consisting in combining the three critical points theorem of Ricceri [7] and a truncation technique. It is worth mentioning that the Moser iteration methods employed in [8] does not work in the presence of an arbitrary perturbation and a different approach is employed here (see [6]).

Denote by $\mathcal{A}$ the class of Carathéodory functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following assumption:

$$\text{for every } M > 0, \ \sup_{|s| \le M} |h(x, s)| \in L^\infty(\Omega).$$

We say that u is a solution of problem ($P_{\lambda,\mu}$) if $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_\Omega [\lambda f(x, u) + \mu g(x, u)] v \, dx$$

for every $v \in W_0^{1,p}(\Omega)$.

Our result is as follows.

Theorem 1.1. *Assume that* $f \in \mathcal{A}$ *and*

(i) $\lim_{s \to 0} (f(x, s))/|s|^{p-1} = 0$ *uniformly with respect to* $x \in \Omega$;
(ii) $\lim_{|s| \to \infty} (f(x, s))/|s|^{p-1} = 0$ *uniformly with respect to* $x \in \Omega$;
(iii) $\sup_{s>0} \inf_{x \in \Omega} f(x, s) > 0$.

Set

$$\lambda^* = \inf \left\{ \frac{\int_\Omega |\nabla u(x)|^p \, dx}{p \int_\Omega \int_0^{u(x)} f(x, s) \, ds \, dx} : \int_\Omega \int_0^{u(x)} f(x, s) \, ds \, dx > 0 \right\}.$$

Then, for every compact interval $[a, b] \subseteq \,]\lambda^*, +\infty[$, *there exists* $\rho > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and for every* $g \in \mathcal{A}$, *there exists* $\mu^* > 0$ *such that for every* $\mu \in [0, \mu^*]$, *problem* ($P_{\lambda,\mu}$) *has at least three solutions in* $C_0^1(\bar{\Omega})$ *whose norms in* $W_0^{1,p}(\Omega)$ *are less than* ρ.

Remark 1.2. As will be clear from the proof of Theorem 1.1, λ^* and ρ, given by Theorem 2.1 below, depend on f but not on the particular choice of g. This is the key to our arguments as it allows the perturbation g to be truncated in a suitable way (via a constant depending on λ, f and ρ) and the existence of multiple solutions for the truncated problem to be proved. A technical lemma (see below) provides a very precise estimate of the L^∞-norm of the solutions of the auxiliary problem, which turn out to be solutions of ($P_{\lambda,\mu}$).

Remark 1.3. A comparison with the results of [1–3, 5] is in order. We point out that our assumptions are quite different than those of the papers mentioned. Besides the different structure of the nonlinearity, in [3], h is required to be p-linear at the origin (that is, $\liminf_{s \to 0} h(s)/|s|^{p-2} s \ge 1$) and to have a positive zero, while in [5], besides the superlinear behavior at zero, the function $s \to h(s)/s$ is increasing in a neighborhood of zero. In [2], only the superlinear growth at zero is required, while in [1], the key assumption is that f changes sign.

If, for instance, $p = 2$ and f and g are defined as

$$f(s) = \begin{cases} 0 & \text{if } s \le 0, \\ \min\{s^2, \sqrt{s}\} & \text{if } s > 0, \end{cases} \qquad g(s) = e^s,$$

then Theorem 1.1 applies. Notice that f is a nonnegative function; if $h_\lambda = f + (\mu/\lambda)g$ for every positive λ and μ, then $s \to h_\lambda(s)/s$ is decreasing in a neighborhood of zero, has no positive zeros, and $\lim_{s\to 0^+} h_\lambda(s)/s^{q-1} = +\infty$ for any $q \geq 2$.

2. Proofs

Let X be a Banach space. Denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ with the following property: if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Our main abstract tool is the following result of Ricceri.

THEOREM 2.1 [7, Theorem 2]. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X, and whose derivative admits a continuous inverse on X^*; and $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\alpha = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\beta = \sup_{u\in\Phi^{-1}(]0,+\infty[)} \frac{J(u)}{\Phi(u)},$$

assume that $\alpha < \beta$.

Then, for each compact interval $[a, b] \subset]1/\beta, 1/\alpha[$ (with the conventions $1/0 = +\infty$, $1/\infty = 0$), there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$, and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu\Psi'(u)$$

has at least three solutions in X whose norms are less than ρ.

REMARK 2.2. Notice that in [7], the above result is applied to a quasilinear elliptic problem of ($P_{\lambda,\mu}$) type with $p > 1$. When $p \leq N$, the perturbation $g \in \mathcal{A}$ has growth

$$|g(x, s)| \leq c(1 + |s|^{q-1}), \quad \text{for a.e. } x \in \Omega, \text{ for every } s \in \mathbb{R},$$

with $1 < q < p^*$ if $p < N$, with any $q > 1$ if $p = N$. The novelty of [8] when $p < N$ consists in dealing with critical or even supercritical perturbations, that is, $q \geq p^*$. In our result, the only assumption on g is that $g \in \mathcal{A}$.

The following technical lemma provides an estimate of the L^∞-norm of the solution which turns out to be crucial in our proof. The proof employs a variant of the Moser iteration argument inspired by Theorem C of [6].

2.1. A technical lemma.

Lemma 2.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$), $p > 1$, $\bar{p}^* = p^*$ if $p < N$, $\bar{p}^*$ any fixed number greater than p if $p \geq N$. Assume that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for some $a, b \in \mathbb{R}^+$,*

$$|h(x,s)| \leq a|s|^{p-1} + b \quad \text{for a.e. } x \in \Omega, \text{every } s \in \mathbb{R}.$$

If $u \in W_0^{1,p}(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta_p u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_h}$$

then $u \in L^\infty(\Omega)$ and there exists a constant $K_1 = K_1(h)$ (not depending on u) such that

$$\|u\|_\infty \leq K_1 \max\{1, \|u\|_{\bar{p}^*}\}$$

(for an explicit formula for K_1 see the end of the proof).

Proof. Let u be a weak solution of (P_h). Hence, for any $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega h(x,u)\varphi. \tag{2.1}$$

Denote $u^+ = \max\{u, 0\}$ and put, for each $L > 0$,

$$u_L(x) = \begin{cases} u^+(x) & \text{if } u^+(x) \leq L, \\ L & \text{if } u^+(x) > L. \end{cases}$$

It is well known that $u_L \in W_0^{1,p}(\Omega)$. For any $q > 0$, plug $\varphi = u_L^{q+1}$ into (2.1) as a test function. Provided that $u \in L^{p+q}(\Omega)$, we get

$$\begin{aligned} (q+1)\int_\Omega |\nabla u_L|^p u_L^q &\leq \int_\Omega (a|u|^{p-1}u_L^{q+1} + bu_L^{q+1}) \\ (\text{since } u_L \leq u_+) &\leq \int_\Omega (au_+^{p+q} + bu_+^{q+1}) \\ (\text{by Hölder inequality}) &\leq a\|u_+\|_{p+q}^{p+q} + b|\Omega|^{p-1/p+q}\|u_+\|_{p+q}^{q+1} \\ &\leq (a + b\max\{1, |\Omega|^{p-1/p}\})(1 + \|u_+\|_{p+q}^{p+q}). \end{aligned} \tag{2.2}$$

If $q' = 1 + q/p$,

$$\begin{aligned} \|u_L\|_{\bar{p}^* q'}^{p+q} = \|u_L^{q'}\|_{\bar{p}^*}^p &\leq S\|u_L^{q'}\|^p \\ &= S q'^p \int_\Omega |\nabla u_L|^p u_L^q, \end{aligned} \tag{2.3}$$

where S is the embedding constant of $W_0^{1,p}(\Omega)$ into $L^{\bar{p}^*}(\Omega)$, that is,

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{(\|u\|_{\bar{p}^*})^p}.$$

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From (2.2) and (2.3) (using also the fact that $q > 0$),

$$\|u_L\|_{\bar{p}^* q'}^{p+q} \le S q'^p (a + b \max\{1, |\Omega|^{p-1/p}\})(1 + \|u_+\|_{p+q}^{p+q}).$$

So, provided that $u \in L^{p+q}(\Omega)$, by Fatou's lemma, as $L \to +\infty$,

$$\|u_+\|_{\bar{p}^* q'}^{p+q} \le S q'^p (a + b \max\{1, |\Omega|^{p-1/p}\})(1 + \|u_+\|_{p+q}^{p+q}). \tag{2.4}$$

In particular, if $u_+ \in L^{p+q}(\Omega)$, then $u_+ \in L^{\bar{p}^* q'}(\Omega)$. Let us choose a sequence of positive numbers $\{q_m\}_m$ in the following way:

$$\begin{cases} q_0 = \bar{p}^* - p, \\ q_{m+1} = \dfrac{\bar{p}^*(p + q_m)}{p} - p. \end{cases}$$

It is easy to prove the following relations for any $m \ge 0$:

(i) $\bar{p}^* q'_m = q_{m+1} + p$;
(ii) $q_m \ge (\bar{p}^*/p)^m q_0$ (in particular, $q_m \to \infty$);
(iii) $q_m \le (m+1)(\bar{p}^*/p)^m q_0$;
(iv) $(\bar{p}^*/p - 1) q_m = [(\bar{p}^*/p)^{m+1} - 1] q_0$.

Since $u \in L^{\bar{p}^*}(\Omega)$, from (i) and (ii) we deduce that $u_+ \in L^\infty(\Omega)$. Let us now obtain the estimate in the thesis which is the most delicate part of the proof. In view of (i), we can rephrase (2.4) as

$$\begin{aligned} \|u_+\|_{p+q_m}^{p+q_{m-1}} &\le S q'^p_{m-1} (a + b \max\{1, |\Omega|^{(p-1)/p}\})(1 + \|u_+\|_{p+q_{m-1}}^{p+q_{m-1}}) \\ &\le C_0 q'^p_{m-1} (1 + \|u_+\|_{p+q_{m-1}}^{p+q_{m-1}}), \end{aligned} \tag{2.5}$$

where we are putting

$$C_0 = \max\{1, S(a + b \max\{1, |\Omega|^{(p-1)/p}\})\}.$$

Denote

$$b_m = \max\{1, \|u_+\|_{p+q_m}^{p+q_m}\}.$$

From (2.5), for any $m \ge 1$,

$$\begin{aligned} \log b_m &\le \frac{p + q_m}{p + q_{m-1}} [p \log(C_0^{1/p} q'_{m-1}) + \log b_{m-1}] \\ (\text{def. of } q_m) &= \frac{\bar{p}^*}{p} [p \log(C_0^{1/p} q'_{m-1}) + \log b_{m-1}] \\ &\le p \sum_{i=1}^m \Big(\frac{\bar{p}^*}{p}\Big)^i \log(C_0^{1/p} q'_{m-i}) + \Big(\frac{\bar{p}^*}{p}\Big)^m \log b_0 \\ (\text{def. of } q'_{m-i}) &= p \sum_{i=1}^m \Big(\frac{\bar{p}^*}{p}\Big)^i \log\Big(C_0^{1/p}\Big(1 + \frac{q_{m-i}}{p}\Big)\Big) + \Big(\frac{\bar{p}^*}{p}\Big)^m \log b_0 \\ (\text{for (iii)}) &\le p \sum_{i=1}^m \Big(\frac{\bar{p}^*}{p}\Big)^i \log\Big[C_0^{1/p}\Big(1 + \frac{q_0}{p}(m - i + 1)\Big(\frac{\bar{p}^*}{p}\Big)^{m-i}\Big)\Big] + \Big(\frac{\bar{p}^*}{p}\Big)^m \log b_0. \end{aligned}$$

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Put

$$T = \frac{\bar{p}^*}{p} > 1.$$

By the previous inequality we get

$$\begin{aligned}
\log \max\{1, \|u_+\|_{p+q_m}\} &= \frac{\log b_m}{p+q_m} \\
(\text{for (iv)}) &= \frac{(T-1)\log b_m}{(T-1)p + (T^{m+1}-1)q_0} \\
&\leq \frac{(T-1)p}{(T-1)p + (T^{m+1}-1)q_0} \\
&\quad \times \sum_{i=1}^{m} T^i \log\Big[C_0^{1/p}\Big(1 + \frac{q_0}{p}(m-i+1)T^{m-i}\Big)\Big] \\
&\quad + \frac{(T-1)T^m \log b_0}{(T-1)p + (T^{m+1}-1)q_0}.
\end{aligned}$$

Since

$$\frac{(T-1)}{(T-1)p + (T^{m+1}-1)q_0} < \frac{T^{-m}}{q_0},$$

we can continue with the above estimate and obtain

$$\begin{aligned}
\log \max\{1, \|u_+\|_{p+q_m}\} &\leq \frac{p}{q_0} \sum_{i=1}^{m} T^{-m+i} \log\Big[C_0^{1/p}\Big(1 + \frac{q_0}{p}(m-i+1)T^{m-i}\Big)\Big] \\
&\quad + \frac{(T-1)T^m \log b_0}{(T-1)p + (T^{m+1}-1)q_0} \\
&= \frac{p}{q_0} \sum_{k=0}^{m-1} T^{-k} \log\Big[C_0^{1/p}\Big(1 + \frac{q_0}{p}(k+1)T^k\Big)\Big] \\
&\quad + \frac{(1-T^{-1})\log b_0}{(T^{-m}-T^{-m-1})p + (1-T^{-m-1})q_0} \\
(\text{since } T > 1) &\leq \frac{p}{q_0} \sum_{k=0}^{m-1} T^{-k}\Big[\log C_0^{1/p} + \log\Big(\Big(1 + \frac{q_0}{p}(k+1)\Big)T^k\Big)\Big] \\
&\quad + \frac{(1-T^{-1})\log b_0}{(T^{-m}-T^{-m-1})p + (1-T^{-m-1})q_0} \\
(\text{since } \log(1+t) \leq t) \quad &\leq \frac{\log C_0}{q_0} \sum_{k=0}^{m-1} T^{-k} + \sum_{k=0}^{m-1} T^{-k}(k+1) + \frac{p}{q_0} \log T \sum_{k=0}^{m-1} T^{-k} k \\
&\quad + \frac{(1-T^{-1})\log b_0}{(T^{-m}-T^{-m-1})p + (1-T^{-m-1})q_0}
\end{aligned}$$

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$$(\text{def. of } b_0) = \Big(\frac{\log C_0}{q_0} + 1\Big)\sum_{k=0}^{m-1} T^{-k} + \Big(\frac{p}{q_0}\log T + 1\Big)\sum_{k=0}^{m-1} T^{-k}k + \frac{(1-T^{-1})\bar{p}^*\log\max\{1, \|u\|_{\bar{p}^*}\}}{(T^{-m}-T^{-m-1})p + (1-T^{-m-1})q_0}.$$

Notice that the right-hand side of the above inequality converges to

$$\Big(\frac{\log C_0}{q_0} + 1\Big)\frac{1}{1-T^{-1}} + \Big(\frac{p}{q_0}\log T + 1\Big)\sum_{k=0}^{\infty} T^{-k}k + \log\max\{1, \|u\|_{\bar{p}^*}\}.$$

So if we put

$$C_1 = \Big(\frac{\log C_0}{q_0} + 1\Big)\frac{1}{1-T^{-1}} + \Big(\frac{p}{q_0}\log T + 1\Big)\sum_{k=0}^{\infty} T^{-k}k, \tag{2.6}$$

we get that

$$\|u_+\|_\infty \le e^{C_1}\max\{1, \|u\|_{\bar{p}^*}\}. \tag{2.7}$$

Analogously, $\|u_-\|_\infty \le e^{C_1}\max\{1, \|u\|_{\bar{p}^*}\}$ and the thesis holds with $K_1 = e^{C_1}$. □

2.2. Proof of Theorem 1.1. Let $X = W_0^{1,p}(\Omega)$, endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$. According to the notation of Lemma 2.3, X is continuously embedded into $L^{\bar{p}^*}(\Omega)$.

Define in $W_0^{1,p}(\Omega)$ the functionals

$$\Phi(u) = \frac{1}{p}\|u\|^p \quad \text{and} \quad J_f(u) = \int_\Omega F(x, u(x))\,dx$$

where $F : \Omega\times\mathbb{R} \to \mathbb{R}$ is the primitive of f, that is,

$$F(x, s) = \int_0^s f(x, t)\,dt.$$

The above functionals comply with the hypotheses of Theorem 2.1. Indeed, due to the hypotheses on f, J_f is continuously Gâteaux differentiable with compact derivative. From assumptions (i) and (ii) it easily follows that $\alpha = 0$, while (iii) implies that $\beta > 0$ (for details, see [7]). Moreover, $\beta = 1/\lambda^*$. Hence, for any fixed compact interval $[a, b] \subseteq\,]\lambda^*, +\infty[$, there exists $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$, and every C^1 functional $\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) = \lambda J_f{}'(u) + \mu\Psi'(u)$$

has at least three solutions in $W_0^{1,p}(\Omega)$ whose norms are less than ϱ.

Fix $[a, b] \subseteq\,]\lambda^*, +\infty[$ and $\lambda \in [a, b]$. From $f \in \mathcal{A}$ and assumptions (i) and (ii), it follows that there exists some constant $C_f > 0$ such that

$$|f(x, s)| \le C_f|s|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ for every } s \in \mathbb{R}. \tag{2.8}$$

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Let $K = K(\lambda, f, \varrho) > 0$ be such that

$$\log \frac{K}{\max\{1, S^{1/p}\rho\}} > \Big[\Big(\frac{\log \max\{1, S\lambda C_f\}}{q_0} + 1\Big)\frac{1}{1 - T^{-1}} + \Big(\frac{p}{q_0}\log T + 1\Big)\sum_{k=0}^{\infty} T^{-k}k\Big]$$

(where S, T, q_0 are as in Lemma 2.3). We point out that the above constant depends only on λ and f. This allows us to apply in a very suitable way the abstract result of Ricceri. Indeed, let $g \in \mathcal{A}$ and define

$$g_K(x, s) = \begin{cases} g(x, -K) & \text{if } s < -K, \\ g(x, s) & \text{if } |s| \le K, \\ g(x, K) & \text{if } s > K, \end{cases}$$

$$G(x, s) = \int_0^s g_K(x, t)\, dt \quad \text{and} \quad J_g(u) = \int_\Omega G(x, u(x))\, dx.$$

Since $g \in \mathcal{A}$, J_g is continuously Gâteaux differentiable with compact derivative, and so there exists $\delta > 0$ (depending on λ, ρ, and K) such that, for any $\mu \in [0, \delta]$, the problem

$$\begin{cases} -\Delta_p u = \lambda f(x, u) + \mu g_K(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{P}_{\lambda,\mu}$}$$

has at least three weak solutions $u_i \in W_0^{1,p}(\Omega)$, $i = 1, 2, 3$, whose norms are less than ρ. Denote by u any such solutions. We claim that $\|u\|_\infty \le K$.

Again, from $g \in \mathcal{A}$, if $C_g = \| \sup_{|s| \le K} g(x, s)\|_\infty$, then

$$|g_K(x, s)| \le C_g \quad \text{for a.e. } x \in \Omega, \text{ for every } s \in \mathbb{R}. \tag{2.9}$$

Having in mind the definition of K, we can fix $0 < \mu^* \le \delta$ such that, for every $\mu \in [0, \mu^*]$,

$$\log \frac{K}{\max\{1, S^{1/p}\rho\}} > \Big[\Big(\frac{\log \max\{1, S(\lambda C_f + \mu C_g \max\{1, |\Omega|^{p-1/p}\})\}}{q_0} + 1\Big) \times \frac{1}{1 - T^{-1}} + \Big(\frac{p}{q_0}\log T + 1\Big)\sum_{k=0}^{\infty} T^{-k}k\Big].$$

Fix $0 < \mu \le \mu^*$ (the case $\mu = 0$ is trivial). Put $h(x, s) = \lambda f(x, s) + \mu g_K(x, s)$. From (2.8) and (2.9),

$$|h(x, s)| \le \lambda C_f |s|^{p-1} + \mu C_g, \quad \text{for a.e. } x \in \Omega, \text{ for every } s \in \mathbb{R}.$$

From Lemma 2.3, $u \in L^\infty(\Omega)$ and there exists a constant $K_1 = K_1(\lambda f + \mu g_K)$ such that

$$\|u\|_\infty \le K_1 \max\{1, S^{1/p}\rho\}.$$

Recalling the definition of K and μ^*, (2.6) and (2.7), it follows that $K_1 \max\{1, S^{1/p}\rho\} \le K$, and so $\|u\|_\infty \le K$. Hence, u is a solution of the original problem ($P_{\lambda,\mu}$). From [4], one also has that $u \in C_0^1(\bar{\Omega})$, and this concludes the proof. □

REMARK 2.4.

(i) $\lambda^* \geq \lambda_1/C_f$, where C_f is from (2.8) and λ_1 is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$;
(ii) if $0 < \lambda < \lambda_1/C_f$, $(P_{\lambda,\mu})$ has no nontrivial solution if $g = 0$.

PROOF. (i) From (2.8), $J_f(u) \leq C_f/p\|u\|_p^p$. Recalling that

$$\lambda_1 = \min\left\{\frac{\|\nabla u\|_p}{\|u\|_p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\},$$

by the definition of λ^*,

$$\lambda^* \geq \frac{1}{C_f} \inf\left\{\frac{\|\nabla u\|_p}{\|u\|_p} : u \in W_0^{1,p}(\Omega),\ J_f(u) > 0\right\} \geq \frac{\lambda_1}{C_f}.$$

(ii) Notice first that since $f(x, 0) = 0$, if $g = 0$ then $(P_{\lambda,\mu})$ has the trivial solution. If $0 < \lambda < \lambda_1/C_f$, and u is a weak solution of $(P_{\lambda,\mu})$, then

$$\int_\Omega |\nabla u|^p\,dx = \lambda \int_\Omega f(x, u)u\,dx \leq \lambda C_f \int_\Omega |u|^p\,dx \leq \frac{\lambda C_f}{\lambda_1} \int_\Omega |\nabla u|^p\,dx,$$

a contradiction. □

Acknowledgement

The first author is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] G. Anello, 'Existence of solutions for a perturbed Dirichlet problem without growth conditions', *J. Math. Anal. Appl.* **330** (2007), 1169–1178.
[2] S. Chen and S. Li, 'On a nonlinear elliptic eigenvalue problem', *J. Math. Anal. Appl.* **307** (2005), 691–698.
[3] L. Iturriaga, S. Lorca and E. Massa, 'Positive solutions for the p-Laplacian involving critical and supercritical nonlinearities with zeros', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 763–771.
[4] G. M. Lieberman, 'Boundary regularity for solutions of degenerate elliptic equations', *Nonlinear Anal.* **12** (1988), 1203–1219.
[5] S. Lorca and P. Ubilla, 'Partial differential equations involving subcritical, critical and supercritical nonlinearities', *Nonlinear Anal.* **56** (2004), 119–131.
[6] S. Miyajima, D. Motreanu and M. Tanaka, 'Multiple existence results of solutions for the Neumann problems via super- and sub-solutions', *J. Funct. Anal.* **262** (2012), 1921–1953.
[7] B. Ricceri, 'A further three critical points theorem', *Nonlinear Anal.* **71** (2009), 4151–4157.
[8] L. Zhao and P. Zhao, 'The existence of three solutions for p-Laplacian problems with critical and supercritical growth', *Rocky Mountain J. Math.* **44** (2014), 1383–1397.

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