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CHARACTERIZATIONS OF VITALI CONDITIONS WITH OVERLAP IN TERMS OF CONVERGENCE OF CLASSES OF AMARTS

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In a series of fundamental papers [**20**], [**21**], [**22**], [**23**], K. Krickeberg introduced 'Vitali' conditions on σ-algebras and showed that they are sufficient for convergence of properly bounded martingales, and supermartingales. It is now known that the conditions V_∞ $(= V)$, and V' are both sufficient and necessary for convergence of L^1-bounded amarts, and ordered amarts (Astbury [**1**]; [**24**], [**25**]); an *amart* (*ordered amart*) is a process (X_t) such that the net $(EX_\tau)_{\tau \in T^*}$ converges, where T^* is the net of simple (ordered) stopping times. We undertake here to similarly characterize the Vitali conditions V_p, $1 \leqq p < \infty$, in terms of convergence of properly defined classes of amarts. (In terms of convergence of L^∞-bounded martingales, Krickeberg himself [**22**] was able to characterize V_1.) It is easy to see that the condition V_∞ can be stated in terms of stopping times as follows: For any adapted family of sets (A_t), the set ess lim sup A_t can be covered up to ϵ by A_τ, where τ is a simple stopping time. To obtain an analogous formulation of V_p for $p \neq \infty$, we introduce multi-valued stopping times, with 'overlap' converging to zero in L^p. Essential convergence of L^1-bounded 'amarts for M_p' defined in terms of such stopping times, characterizes σ-algebras satisfying V_p. Martingales bounded in L^q are shown to be amarts for M_p, but also other examples are given.

Sections 1 and 2 sketch the theory of amarts for M_p, analogous to that of amarts. Section 3 gives extensions to Banach spaces. At the end of the paper it is briefly shown how one can replace L^p spaces by Orlicz spaces.

Sections 1 and 2 are independent of other work on amarts. Section 3 depends in part on [**24**] and [**25**].

1. Real valued case without Vitali conditions. Let J be a set of indices partially ordered by $\leqq$; s, t and u are elements of J. J is a *directed set filtering to the right*, i.e., such that for each pair t_1, t_2 of elements of J, there exists an element t_3 of J such that $t_1 \leqq t_3$ and $t_2 \leqq t_3$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Functions, sets, random variables are considered equal if they are equal almost surely. Let (X_t) be a family of random variables taking values in $\bar{R}$. The *essential supremum* of (X_t) is the unique almost surely smallest random variable $e \sup_t X_t$ such that for every t, $e \sup_t X_t \geqq X_t$ a.s. The *essential infimum* of (X_t), $e \inf_t X_t$, is defined by

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$e \inf_t X_t = -e \sup_t (-X_t)$. The *essential upper limit* of (X_t), $e \lim_t \sup X_t$, is defined by

$$e \lim \sup_t X_t = e \inf_s (e \sup_{t \geqq s} X_t).$$

The *essential lower limit* of (X_t), $e \lim \inf_t X_t$, is defined by

$$e \lim \inf_t X_t = -e \lim \sup_t (-X_t).$$

The family (X_t) is said to *converge essentially* if $e \lim \sup_t X_t = e \lim \inf_t X_t$; this common value is called the *essential limit* of (X_t), $e \lim X_t$. The *stochastic upper limit* of (X_t), $s \lim \sup_t X_t$, is the essential infimum of the set of random variables Y such that $\lim P(\{Y < X_t\}) = 0$. The *stochastic lower limit* of (X_t), $s \lim \inf_t X_t$, is defined by $s \lim \inf_t X_t = -s \lim \sup_t (-X_t)$. The family (X_t) is said to *converge stochastically*, or to *converge in probability*, if $s \lim \sup_t X_t = s \lim \inf_t X_t$; this common value is called the *stochastic limit* of (X_t), $s \lim X_t$. If (A_t) is a directed family of measurable sets, the *essential upper limit* of (A_t), $e \lim \sup_t A_t$, is the set such that

$$1_{e \lim \sup_t A_t} = e \lim \sup_t 1_{A_t}.$$

A *stochastic basis* $(\mathscr{F}_t)$ is an increasing family of sub σ-algebras of $\mathscr{F}$ (i.e., for every $s \leqq t$, $\mathscr{F}_s \subset \mathscr{F}_t$). A *stochastic process* (X_t) is a family of random variables $X_t: \Omega \rightarrow R$ such that for each t, X_t is $\mathscr{F}_t$ measurable. The process is called *integrable* (*positive*) if for every t, X_t is integrable (positive). The process is L^p-bounded $(1 \leqq p \leqq \infty)$ if $\sup \|X_t\|_p < \infty$, where $\| \ \|_p$ is the L^p norm. Given a stochastic basis $(\mathscr{F}_t)$, a family of sets (A_t) is *adapted* if for every $t \in J$, $A_t \in \mathscr{F}_t$.

Denote by $\mathscr{J}$ the set of finite subsets of J. An (*incomplete*) *multivalued simple stopping time* is a map τ from Ω (from a subset of Ω called $D(\tau)$) to $\mathscr{J}$ such that $R(\tau) = \bigcup_{\omega \in D(\tau)} \tau(\omega)$ is finite, and such that for every $t \in J$,

$$\{\tau = t\} = \{\omega \in \Omega | t \in \tau(\omega)\} \in \mathscr{F}_t.$$

$R(\tau)$ will be called by extension the *range* of τ. Denote by M (IM) the set of (incomplete) multivalued simple stopping times. Denote by T the set of *simple stopping times*, i.e., of elements τ of M such that for every ω, $\tau(\omega)$ is a singleton of J. Let $\tau \in IM$; the *excess function* of τ is

$$e_\tau = \sum_{t \in R(\tau)} 1_{\{\tau = t\}} - 1_{D(\tau)}.$$

The *overlap of order* p of τ, $1 \leqq p \leqq \infty$, is $O_p(\tau) = \|e_\tau\|_p$. If (X_t) is a stochastic process, let

$$X_\tau = \sum_{t \in R(\tau)} 1_{\{\tau = t\}} X_t.$$

If (A_t) is an adapted family of sets, let $A_\tau = \bigcup (\{\tau = t\} \cap A_t)$. Let σ and τ be in M; we say that

$$\sigma \leqq \tau \text{ if } \forall s, \forall t, \{\sigma = s\} \cap \{\tau = t\} \neq \emptyset \text{ implies that } s \leqq t.$$

(In the case where σ and $\tau \in T$, it is the usual order $\leqq$.) For the order $\leqq$, M is a directed set filtering to the right. An integrable stochastic process (X_t) is an *amart for* M_p if the net $(EX_\tau)_{\tau \in M_p}$ converges when $\tau \in M$ and $O_p(\tau) \to 0$, i.e., there exists a number L such that for every $\epsilon > 0$, there exists $s \in J$ and $\alpha > 0$, such that $\tau \in M$, $\tau \geqq s$, $O_p(\tau) < \alpha$ imply $|EX_\tau - L| < \epsilon$. An amart for M_∞ is simply called an *amart*. Throughout this paper, if $1 \leqq p \leqq \infty$ we assume that $1 \leqq q \leqq \infty$ and $1/p + 1/q = 1$.

PROPOSITION 1.1. *Let (X_t) be an L^q-bounded martingale, then it is an amart for M_p. Conversely, if (X_t) is an amart for M_p and either an L^1-bounded martingale or a positive submartingale, then (X_t) is L^q-bounded.*

Proof. Let (X_t) be an L^q-bounded martingale; then (X_t) is L^1-bounded. Let $\sigma \in M$ and let t be bigger than the elements of $R(\sigma)$. Then

$$EX_\sigma = \sum_{s \in R(\sigma)} E(1_{\{\sigma = s\}} X_s) = \sum_{s \in R(\sigma)} E(1_{\{\sigma = s\}} X_t) = EX_t + E(X_t e_\sigma).$$

Since $|E(X_t e_\sigma)| \leqq O_p(\sigma) \sup \|X_t\|_q$, the net $(EX_\tau)_{\tau \in M_p}$ converges.

Conversely, let (X_t) be an L^1-bounded martingale (resp. a positive submartingale) which is an amart for M_p. Assume that (X_t) is not L^q-bounded. In both cases since $(\|X_t\|_q)$ is an increasing net, there exists an increasing sequence (r_n) such that if $s_n \geqq r_n \forall n$, then $\sup \|X_{s_n}\|_q = \infty$. Since (X_t) is an amart for M_p, there exists an increasing sequence (t_n) of indices, and a sequence (α_n) of numbers such that if $t_n \geqq r_n \forall n$, and if $\tau \in M$ satisfies $\tau \geqq t_n$ and $O_p(\tau) < \alpha_n$, then

$$|EX_\tau - \lim_{\tau \in M_p} EX_\tau| \leqq 2^{-n}.$$

Denote $Y_n = X_{t_n}$; the stochastic process $(Y_n, \mathscr{F}_{t_n})$ is an amart for M_p, and $\sup \|Y_n\|_q = \infty$. Since (Y_n) may be replaced by a subsequence, we may and do assume $\|Y_n^+\|_q > n^2 \forall n$. There exists a random variable Z_n such that $\|Z_n\|_p \leqq 1/n$, and $E(Y_n^+ Z_n) > n$. One may require that Z_n be $\mathscr{F}_{t_n}$ measurable, positive, and that the support of Z_n be included in the support of Y_n^+. Define S_n by $S_n = k$ on the set $\{k \leqq Z_n < k + 1\}$ for $k \leqq K_n$, and $S_n = 0$ on $\{Z_n \geqq K_n\}$. By a proper choice of K_n one has $E(S_n Y_n^+) > n - EY_n^+$. Set $\tau(\omega) = \{t_n, t_{n+1}, \ldots, t_{n+k}\}$ for $\omega \in \{S_n = k\}$; $\tau \in M$, $e_\tau = S_n$, and since for every t, $\{\tau = t\} \in \mathscr{F}_{t_n}$,

$$EX_\tau = \sum_j E(1_{\{\tau = t_{n+j}\}} X_{t_{n+j}}) \geqq \sum_j E(1_{\{\tau = t_{n+j}\}} X_{t_n}) > n - EX_{t_n}^-.$$

If (X_t) is an L^1-bounded martingale (resp. a positive submartingale), the previous inequality shows that the net $(EX_\tau)_{\tau \in M_p}$ is not bounded, which brings a contradiction.

An integrable process (X_t) is a *semiamart for* M_p if there exists $s \in J$ such that the net $(EX_\tau)_{\tau \in M_p, \tau \geqq s}$ is bounded.

The amart case of the following result is due to [**2**], [**12**], [**1**].

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THEOREM 1.2. (a) *Let* (X_t) *be a semiamart for* M_p. *If* $\liminf EX_t^- < \infty$ (*resp.* $\liminf EX_t^+ < \infty$), *then* (X_t^+) (*resp.* (X_t^-)) *is a semiamart for* M_p.

(b) *Let* (X_t) *be an* L^1*-bounded amart for* M_p; *then* (X_t^+), (X_t^-) *and* $(|X_t|)$ *are amarts for* M_p.

Proof. (a) Assume that $\liminf EX_t^- < \infty$; let $\beta \in R$, $s \in J$ and $\epsilon > 0$ be such that if $\tau \in M$, $\tau \geqq s$, $O_p(\tau) \leqq \epsilon$, then $EX_t < \beta$. Let $\sigma \in M$, $\sigma \geqq s$, $O_p(\sigma) < \epsilon$. Choose t bigger than the indices $s \in R(\sigma)$, such that

$$EX_t^- < \liminf EX_t^- + 1.$$

Define $\tau \in M$ as follows: for $s \in R(\sigma)$, $s \in \tau(\omega)$ if $\omega \in \{\sigma = s\} \cap \{X_s \geqq 0\}$; let

$$A = \bigcup (\{\sigma = s\} \cap \{X_s \geqq 0\})$$

and set $\tau = t$ on A^c. Then $O_p(\tau) \leqq O_p(\sigma)$, $\tau \geqq s$, and if we set $U_t = X_t^+$, then

$$EU_\sigma = \sum_{s \in R(\sigma)} E(X_s 1_{\{\sigma = s\} \cap \{X_s \geqq 0\}}) = EX_\tau - E(1_{A^c} X_t) \leqq \beta + \liminf EX_t^- + 1.$$

(b) Given $\epsilon > 0$, choose $s \in J$ and $\alpha > 0$ such that if $s \leqq \pi \in M$, $O_p(\pi) < 2\alpha$, then $|EX_s - EX_\pi| < \epsilon$. Next using (a) choose $\sigma_0 \in M$ with $\sigma_0 \geqq s$ and $O_p(\sigma_0) < \alpha$, such that

$$E(U_{\sigma_0}) \geqq \sup_{\tau \geqq s, O_p(\tau) < \alpha} EU_\tau - \epsilon,$$

where $U_t = X_t^+$. Set $R(\sigma_0) = \{s_1, s_2, \ldots, s_n\}$; choose $t \in J$ bigger than the elements of $R(\sigma_0)$. Let $\tau \in M$, $\tau \geqq t$, $O_p(\tau) < \alpha$; set $R(\tau) = \{t_1, \ldots, t_k\}$. Define $\tau' \in M$ as follows: Set

$$A = \bigcup_{i \leqq n} (\{\sigma_0 = s_i\} \cap \{X_{s_i} < 0\}); A \in \mathscr{F}_t.$$

For every $i \leqq n$, $s_i \in \tau'(\omega)$ if $\omega \in \{\sigma_0 = s_i\} \cap \{X_{s_i} < 0\}$. For every $j \leqq k$, $t_j \in \tau'(\omega)$ if $\omega \in \{\tau = t_j\} \cap A^c$. Then $e_{\tau'} \leqq e_{\sigma_0} + e_\tau$, and $\tau' \geqq s$. Furthermore,

$$\begin{aligned} U_{\sigma_0} - U_\tau &= \sum_{i \leqq n} 1_{\{\sigma_0 = s_i\} \cap \{X_{s_i} \geqq 0\}} X_{s_i} - \sum_{j \leqq k} 1_{\{\tau = t_j\} \cap \{X_{t_j} \geqq 0\}} X_{t_j} \\ &= \sum_{i \leqq n} 1_{\{\sigma_0 = s_i\}} X_{s_i} - \sum_{i \leqq n} 1_{\{\sigma_0 = s_i\} \cap \{X_{s_i} < 0\}} X_{s_i} - \sum_{j \leqq k} 1_{\{\tau = t_j\} \cap \{X_{t_j} \geqq 0\}} X_{t_j} \\ &= X_{\sigma_0} - X_{\tau'} - 1_A \sum_{j \leqq k} 1_{\{\tau = t_j\} \cap \{X_{t_j} \geqq 0\}} X_{t_j} \\ &\quad + 1_{A^c} \sum_{j \leqq k} 1_{\{\tau = t_j\} \cap \{X_{t_j} < 0\}} X_{t_j} \leqq X_{\sigma_0} - X_{\tau'}. \end{aligned}$$

Hence $EU_{\sigma_0} - EU_\tau \leqq 2\epsilon$. From the definition of σ_0, $EU_{\sigma_0} \geqq EU_\tau - \epsilon$, and therefore

$$|E_{\sigma_0} - EU_\tau| \leqq 2\epsilon.$$

A similar proof shows that (X_t^-) is an amart for M_p, and since $|X_t| = X_t^+ + X_t^-$, $(|X_t|)$ is an amart for M_p.

The amart case of the following result is due to [**12**].

THEOREM 1.3. (Riesz decomposition of amarts for M_p). *Let (X_t) be an amart for M_p. Then X_t can be uniquely written as $X_t = Y_t + Z_t$, where (Y_t) is a martingale and an amart for M_p, and $(|Z_t|)$ is an amart for M_p which converges to 0 in L^1.*

Proof. Fix $s \in J$; let $A \in \mathcal{F}_s$ and $s' \geqq s$. Given $\sigma, \tau \in M$, $\sigma \geqq s'$, $\tau \geqq s'$, define σ' and τ' as follows: Let $t \in J$ be bigger than all the elements of $R(\sigma)$ and $R(\tau)$; set $\sigma' = \sigma$ and $\tau' = \tau$ on A, $\sigma' = \tau' = t$ on A^c. Since $\sigma' \geqq s'$, $\tau' \geqq s'$, $e_{\sigma'} \leqq e_\sigma$, $e_{\tau'} \leqq e_\tau$, and since

$$|E(1_A X_\sigma - 1_A X_\tau)| = |E(X_{\sigma'} - X_{\tau'})|,$$

the net $(E(1_A X_\tau))_{\tau \in M_p}$ is Cauchy uniformly in $A \in \mathcal{F}_s$. Hence the net $(E[1_A X_\tau])_{\tau \in M_p}$ converges to $\mu_s(A)$ uniformly in $A \in \mathcal{F}_s$, and μ_s is finitely additive on $\mathcal{F}_s$. Let $A_n \searrow \emptyset$, $A_n \in \mathcal{F}_s$; given $\epsilon > 0$, there exists s' such that for every n,

$$|\mu_s(A_n)| \leqq \epsilon + |E(1_{A_n} X_{s'})|.$$

Hence there exists n such that $|\mu_s(A_n)| < 2\epsilon$, so that μ_s is σ-additive on $\mathcal{F}_s$, and absolutely continuous with respect to P. Let Y_s be the Radon-Nikodym derivative of μ_s with respect to P; clearly (Y_s) is a martingale. Let $\tau \in M$, $\tau \geqq s$, and denote $R(\tau) = \{t_1, \ldots, t_n\}$. Given $\epsilon > 0$, choose $u_1 \leqq \ldots \leqq u_n$, such that for every $i \leqq n$, $u_i \geqq t_i$, and

$$|E[1_{\{\tau = t_i\}}(Y_{t_i} - X_{u_i})]| \leqq \epsilon/n.$$

Define $\pi \in M$ as follows: for every $i \leqq n$, $\{\pi = u_i\} = \{\tau = t_i\}$; then $e_\pi = e_\tau$ and $\pi \geqq s$. Furthermore,

$$\begin{aligned} EY_\tau &= \textstyle\sum_{i \leqq n} E(1_{\{\tau = t_i\}} X_{u_i}) + \sum_{i \leqq n} E[1_{\{\tau = t_i\}}(Y_{t_i} - X_{u_i})] \\ &= EX_\pi + \textstyle\sum_{i \leqq n} E[1_{\{\tau = t_i\}}(Y_{t_i} - X_{u_i})]. \end{aligned}$$

Hence $|EY_\tau - EX_\pi| \leqq \epsilon$, which proves that $\lim_{\tau \in M_p} EY_\tau = \lim_{\tau \in M_p} EX_\tau$. For every t, set $Z_t = X_t - Y_t$. Since $E[1_A(X_t - Y_t)]$ converges to 0 uniformly in $A \in \mathcal{F}_t$, Z_t converges to 0 in L^1. Since (Z_t) is an amart for M_p, $(|Z_t|)$ also is by Theorem 1.2.

THEOREM 1.4. *Let (X_t) be an L^1-bounded amart for M_p. Then the net $(X_\tau)_{\tau \in M_p}$ converges stochastically.*

Proof. Assume at first that (X_t) is an L^∞-bounded amart for M_p. Define by induction (α_n), $\alpha_1 > \alpha_2 > \ldots, \alpha_n \to 0$, and an increasing sequence of indices (s_n) such that if $\sigma \in M$, $\sigma \geqq s_n$, $O_p(\sigma) \leqq \alpha_n$, then $|EX_\sigma - L| \leqq 1/n$, where L denotes the limit of $(EX_\tau)_{\tau \in M_p}$. Set $\beta_n = \alpha_n - \alpha_{n+1}$; let (σ_n) be an increasing sequence of elements of M, such that $\sigma_n \geqq s_n$, $O_p(\sigma_n) \leqq \beta_n$, and such that there exists an increasing sequence of indices (t_n), $\sigma_n \leqq t_n \leqq \sigma_{n+1}$ for all n. Set $V = \liminf X_{\sigma_n}$, $W = \limsup X_{\sigma_n}$. Given $\epsilon > 0$, choose K_0 such that $1/K_0 < \epsilon$. Given any δ, $0 < \delta < \alpha_{K_0}$, there exists an index t_k and two $\mathcal{F}_{t_k}$ measurable random variables V' and W' such that

$$P(\{|V - V'| > \delta\}) < \delta, P(\{|W - W'| > \delta\}) < \delta.$$

We also assume that $\alpha_k \leqq \delta$ and $k > K_0$. Choose $k' \geqq k$ such that

$$P(\bigcup_{k \leqq n \leqq k'} \{|X_{\sigma_n} - V'| < 2\delta\}) \geqq 1 - 2\delta.$$

Set

$$A = \bigcup_{k \leqq n \leqq k'} \bigcup_{t \in R(\sigma_n)} [\{\sigma_n = t\} \cap \{|X_t - V'| < 2\delta\}].$$

For each n, $k \leqq n \leqq k'$, the cardinality of $\sigma_n(\omega)$ is strictly larger than 1 for $\omega \in B_n$; $1_{B_n} \leqq e_{\sigma_n}$, so that $P(B_n) \leqq \|e_{\sigma_n}\|_1 \leqq O_p(\sigma_n) \leqq \beta_n$. Hence

$$P(A) \geqq 1 - 2\delta - \sum_{k \leqq n} \beta_n \geqq 1 - 2\delta - \alpha_n \geqq 1 - 3\delta.$$

Set for each n, $k \leqq n \leqq k'$,

$$A_n = [\bigcup_{t \in R(\sigma_n)} (\{\sigma_n = t\} \cap \{|X_t - V'| < 2\delta\})] \\ \cap [\bigcap_{k \leqq j \leqq n-1} \bigcap_{s \in R(\sigma_j)} \{|X_s - V'| \geqq 2\delta\}].$$

Define $\tau \in M$ as follows: For every $\omega \in A_n$, $k \leqq n \leqq k'$, let $t \in \tau(\omega)$ if $\omega \in \{\sigma_n = t\} \cap \{|X_t - V'| < 2\delta\}$, and set $\tau = t_{k'+1}$ on A^c. Hence

$$s_k \leqq \tau, e_\tau \leqq \sum_{k \leqq n \leqq k'} e_{\sigma_n}$$

so that $O_p(\tau) \leqq \alpha_k$, and $\tau(\omega)$ has a cardinality strictly larger than 1 on a set of probability less than δ. Since $P(\{|X_\tau - V'| < 2\delta\}) \geqq 1 - 4\delta$,

$$|EX_\tau - EV'| \leqq 2\delta + 8\delta \sup \|X_t\|_\infty.$$

In a similar way we define $\tau' \in M$, $O_p(\tau') \leqq \alpha_k$, $s_k \leqq \tau'$, such that

$$|EX_{\tau'} - EW'| \leqq 2\delta + 8\delta \sup \|X_t\|_\infty.$$

Since $|EX_\tau - EX_{\tau'}| \leqq 2/K_0$, we have

$$|EW - EV| \leqq 2\epsilon + \delta(6 + 18 \sup \|X_t\|_\infty).$$

Since ϵ and δ are arbitrarily small, $V = W$ a.s. Hence the sequence X_{σ_n} converges stochastically, which proves that the net X_τ converges stochastically when $\tau \in M$, $O_p(\tau) \to 0$. Let (X_t) be an L^1-bounded amart for M_p, and assume that the net $(X_\tau)_{\tau \in M_p}$ does not converge stochastically. If $s \lim_{\tau \in M_p} X_\tau = \infty$ (resp. $-\infty$) on a set of positive measure, then $s \lim X_t = \infty$ (resp. $-\infty$) on this set. Hence by Fatou's lemma there exists $a < b$ such that

$$P(\{s \lim_{\tau \in M_p} \inf X_\tau < a < b < s \lim \sup_{\tau \in M_p} X_\tau\}) > 0.$$

Set $X_t' = (a - 1) \vee [X_t \wedge (b + 1)]$; by Theorem 1.2 (X_t') is an L^∞-bounded amart for M_p. The argument above shows that $(X_\tau')_{\tau \in M_p}$ converges stochastically. Since for every $\tau \in M$, $X_\tau' = (a - 1) \vee [X_\tau \wedge (b + 1)]$ on a set of probability larger than $1 - O_p(\tau)$, the net

$$((a - 1) \vee [X_\tau \wedge (b + 1)])_{\tau \in M_p}$$

converges stochastically, which brings a contradiction.

2. Real valued case: convergence with Vitali conditions. A stochastic basis $(\mathscr{F}_t)$ satisfies the *Vitali condition* V_p if for every adapted family of sets (A_t) and for every $\epsilon > 0$, there exists $\tau \in IM$ such that $O_p(\tau) < \epsilon$ (overlap limitation), $P(e \limsup A_t \backslash A_\tau) < \epsilon$ (deficiency of covering limitation), and for every $t \in R(\tau)$, $\{\tau = t\} \subset A_t$. (It is easy to see that one gets an equivalent formulation by replacing the condition $P(e \limsup A_t \backslash A_\tau) < \epsilon$ with $P(e \limsup A_t) - P(A_\tau) < \epsilon$. This definition is equivalent to the one given in [**23**]. It generalizes the definition of $V = V_\infty$ given in [**24**], [**25**].) In this section we characterize V_p in terms of essential convergence of amarts for M_p, and give an example of an amart for M_p which converges essentially.

The following theorem is a generalization of Krickeberg's results [**20**], [**22**], and of Astbury's result [**1**].

THEOREM 2.1. *Let p be fixed, $1 \leqq p \leqq \infty$. Let $(\mathscr{F}_t)$ be a stochastic basis; the following conditions are equivalent:*

(1) *$(\mathscr{F}_t)$ satisfies the Vitali condition V_p.*

(2) *For any process (X_t), the stochastic convergence of the net $(X_\tau)_{\tau \in M_p}$ implies the essential convergence of X_t.*

(3) *Every L^1-bounded amart for M_p converges essentially.*

(4) *Every amart for M_p of the form (1_{A_t}) with $\lim P(A_t) = 0$, converges essentially to 0.*

Proof. (1) $\Rightarrow$ (2). Denote $X_\infty = s \lim_{\tau \in M_p} X_\tau$, let $a > 0$, and set

$$A = e \limsup \{|X_t - X_\infty| > a\}.$$

Given ϵ, $0 < \epsilon < a/3$, there exists $s \in J$, $X \in \mathscr{F}_s$ such that $P(\{|X_\infty - X| > \epsilon\}) \leqq \epsilon$. Choose $s' \geqq s$ and α, $0 < \alpha < \epsilon$ such that if $s' \leqq \tau \in M$ and $O_p(\tau) \leqq \alpha$, then $P(\{|X_\tau - X_\infty| \geqq \epsilon\}) \leqq \epsilon$. For every $t \in J$, set $A_t = \{|X_t - X| > a - \epsilon\}$ if $t \geqq s'$, and $A_t = \emptyset$ otherwise; then

$$P(e \limsup A_t) \geqq P(A) - \epsilon.$$

By the Vitali condition V_p, we can define $\sigma \in IM$, $\sigma \geqq s'$, $O_p(\sigma) < \alpha$, such that

$$P(e \limsup A_t \backslash A_\sigma) < \epsilon,$$

and $\{\sigma = u\} \subset A_u$ for every $u \in R(\sigma)$. Furthermore, since $A_\sigma \subset \{|X_\sigma - X| > a - \epsilon\} \cup$ support e_σ, and $P(\text{support } e_\sigma) \leqq \|e_\sigma\|_p < \alpha$, we have

$$P(A) - 2\epsilon \leqq P(e \limsup A_t) - \epsilon \leqq P(A_\sigma) \leqq P(\{|X_\sigma - X| > a - \epsilon\}) + \alpha \leqq P(\{|X_\sigma - X_\infty| > a - 2\epsilon\}) + 2\epsilon \leqq 3\epsilon.$$

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Since this inequality holds for every $\epsilon > 0$, $P(A) = 0$ and $e \lim X_t = X_\infty$.

(2) $\Rightarrow$ (3). Let (X_t) be an L^1-bounded amart for M_p; by Theorem 1.4 the net $(X_\tau)_{\tau \in M_p}$ converges stochastically. Hence if (2) holds (X_t) converges essentially.

(3) $\Rightarrow$ (4). This implication is obvious.

(4) $\Rightarrow$ (1). A similar argument appears in [**1**].

Let (A_t) be an adapted family of sets and let $A = e \limsup A_t$. Set

$$\Lambda = \{\tau \in IM | \forall t \in R(\tau), \{\tau = t\} \subset A_t\}.$$

Define by induction two sequences (τ_k) in Λ and (r_k) in R as follows:

$$r_0 = \sup \{P[D(\tau)] | \tau \in \Lambda, O_p(\tau) < 1\}.$$

τ_1 is any element of Λ such that $O_p(\tau_1) < 1$ and $P[D(\tau_1)] \geqq r_0/2$; set

$$r_1 = \sup \{P[D(\tau) \backslash D(\tau_1)] | \tau \in \Lambda, O_p(\tau) < 1/2, D(\tau) \supset D(\tau_1)\}.$$

If τ_{k-1} and r_{k-1} have been defined, τ_k is any element of Λ such that $O_p(\tau_k) < 1/k$, $D(\tau_k) \supset D(\tau_{k-1})$, and $P[D(\tau_k) \backslash D(\tau_{k-1})] \geqq r_{k-1}/2$. Set

$$r_k = \sup \{P[D(\tau) \backslash D(\tau_k)] | \tau \in \Lambda, O_p(\tau) < 1/k + 1, D(\tau) \supset D(\tau_k)\}.$$

Let $\tau \in \Lambda$, $O_p(\tau) < 1/(k+1)$, $D(\tau) \supset D(\tau_k)$; then

$$r_{k-1} \geqq P[D(\tau) \backslash D(\tau_k)] + P[D(\tau_k) \backslash D(\tau_{k-1})] \geqq P[D(\tau) \backslash D(\tau_k)] + r_{k-1}/2.$$

Hence $r_k \leqq r_{k-1}/2$. Set

$$C_t = A_t \backslash \cup_{u \leqq t} \cup_{k \in N} \{\tau_k = u\}, X_t = 1_{C_t}.$$

Let $k \in N$, and choose $t' \in J$ such that t' is larger than all the elements of $\cup_{j \leqq k} R(\tau_j)$. Let $\tau \in M$, $\tau \geqq t'$, $O_p(\tau) < 1/k - O_p(\tau_k)$. Define $\sigma \in M$ as follows: $\sigma = \tau_k$ on $D(\tau_k)$, $t \in \sigma(\omega)$ if $\omega \in \{\tau = t\} \cap C_t$ for $t \in R(\tau)$. Then $\sigma \in \Lambda$. $D(\sigma) \supset D(\tau_{k-1})$, $e_\sigma \leqq e_{\tau_k} + e_\tau$; hence

$$P[D(\sigma) \backslash D(\tau_{k-1})] \leqq r_{k-1} \leqq 2^{-k+1}.$$

Furthermore, since

$$X_\tau = \sum_{t \in R(\tau)} 1_{C_t \cap \{\tau = t\}} \leqq 1_{D(\sigma) \backslash D(\tau_k)} + e_\tau,$$

$E(X_\tau) \leqq 2^{-k+1} + k^{-1}$. Hence (X_t) is an amart for M_p which converges essentially to 0 under the assumption (4). Hence if $B = \cup D(\tau_k)$,

$$A \backslash B \subset e \limsup (A_t \backslash B) \subset e \limsup C_t.$$

Hence $P(A \backslash B) = 0$; since $D(\tau_k)$ increases to B, given $\epsilon > 0$ there exists k such that $O_p(\tau_k) < \epsilon$, and

$$P(e \limsup A_t \backslash D(\tau_k)) = P(e \limsup A_t \backslash A_{\tau_k}) < \epsilon.$$

Example. Let J be a family of finite (countable) measurable partitions of $(\Omega, \mathscr{F}, P)$, and order J by refinement (i.e., $s \leqq t$ if every atom of s is a union of atoms of t). Assume that $\sup \{P(A)|A \in t\}$ converges to 0, and for every t let $\mathscr{F}_t$ be the σ-algebra generated by t. Let Q be a measure of density X with respect to P, $X \in L^q$, $1 < q \leqq \infty$. Let f and g be real functions having derivatives at 0, $g'(0) \neq 0$, such that $f(0) = g(0) = 0$, and set

$$X_t = \sum_{A \in t} \frac{f[Q(A)]}{g[P(A)]} 1_A.$$

(X_t) is an amart for M_p. In the classical case where $\Omega = [0, 1]^n$ with the Borel σ-algebra and Lebesgue measure, and where J is the family of finite (countable) partitions of $[0, 1]^n$ into parallepipeds, $(\mathscr{F}_t)$ satisfies the Vitali conditions V_p for $1 \leqq p < \infty$ if $n > 1$, and $(\mathscr{F}_t)$ satisfies V_∞ if $n = 1$ (see [**22**] p. 298). Then if $1 < q \leqq \infty$, (X_t) converges essentially to $(f'(0)/g'(0))X$.

Indeed, set

$$Y_t = \frac{f'(0)}{g'(0)} \left[\sum_{A \in t} \frac{Q(A)}{P(A)} 1_A \right].$$

(Y_t) is an L^q-bounded martingale:

$$E|Y_t|^q \leqq \frac{|f'(0)|^q}{|g'(0)|^q} \sum_{A \in t} \frac{(E[|1_A X|])^q}{P(A)^q} P(A) \leqq |f'(0)|^q |g'(0)|^{-q} ||X||_q^q.$$

Hence (Y_t) is an amart for M_p. Set $Z_t = X_t - Y_t$; let $f(x) = xf'(0) + xF(x)$, $g(x) = xg'(0) + xG(x)$, with $\lim_{x \to 0} F(x) = \lim_{x \to 0} G(x) = 0$. Given ϵ, $0 < \epsilon < |g'(0)|$, choose α such that $|x| < \alpha$ implies $|F(x)| < \epsilon$ and $|G(x)| < \epsilon$. Choose s such that for every $A \in s$, $P(A) < \alpha$ and $|Q|(A) < \alpha$. For $t \geqq s$,

$$|Z_t| \leqq \sum_{A \in t} \frac{\epsilon[|f'(0)| + |g'(0)|]}{|g'(0)|[|g'(0)| - \epsilon]} \frac{|Q(A)|}{P(A)} 1_A.$$

Hence if $\tau \in M$, $\tau \geqq s$, then

$$E|Z_\tau| \leqq \frac{\epsilon[|f'(0)| + |g'(0)|]}{|g'(0)|[|g'(0)| - \epsilon]} [|Q|([0, 1]^n) + ||X||_q O_p(\tau)].$$

3. Banach-valued case. We now assume that the random variables X_t take values in a Banach space $\mathscr{E}$, are strongly measurable and Pettis integrable. Other definitions remain the same. Amarts for M_p are defined by the convergence of $(EX_\tau)_{\tau \in M_p}$ in the norm topology.

The case $J = N$ of the following result for amarts is due to [**13**]; see also [**1**].

THEOREM 3.1. *Suppose that the Banach space $\mathscr{E}$ has the Radon-Nikodym property. Let (X_t) be an $\mathscr{E}$-valued amart for M_p such that* $\liminf E|X_t| < \infty$. *Then X_t can be uniquely written as $X_t = Y_t + Z_t$, where (Y_t) is a martingale and an amart for M_p, and (Z_t) is an amart for M_p which converges to zero in Pettis norm.*

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Proof. Recall that the norm defined on the set of random variables measurable with respect to $\mathscr{F}_s$ as $\|X\|_{pe} = \sup_{A\in\mathscr{F}_s}|E(1_A X)|$, is equivalent with the Pettis norm. The argument given in the proof of Theorem 1.3 above extends to the Banach-valued case, showing that $(E[1_A X_\tau])_{\tau\in M_p}$ converges uniformly in $A \in \mathscr{F}_s$; hence $(X_\tau)_{\tau\in M_p}$ is Cauchy in Pettis norm. In general this does not imply convergence, but μ_s defined by $\mu_s(A) = \lim E(1_A X_t)$, $A \in \mathscr{F}_s$, is of bounded variation because of the assumption $\liminf E|X_t| < \infty$, and is countably additive because $E(1_A X_t)$ converges uniformly in $A \in \mathscr{F}_s$ (cf. the proof of Theorem 1.2). Since $\mathscr{E}$ has the Radon-Nikodym property, there exists a random variable $Y_s \in L^1(\mathscr{E})$ such that for every $A \in \mathscr{F}_s$, $\mu_s(A) = E(1_A Y_s)$. (Y_s) is easily seen to be a martingale. Set $Z_t = X_t - Y_t$; the argument given above in the real-valued case shows that (Y_t) and hence (Z_t) are amarts for M_p, and Z_t converges to zero in Pettis norm.

We say that X_t *converges weakly essentially* if there exists a random variable X_∞ such that $e\lim f(X_t) = f(X_\infty)$for every $f \in \mathscr{E}'$. It should be pointed out that in the case $J = N$ this need not imply weak almost sure convergence, which holds under more stringent assumptions (cf. [**7**], [**5**]).

THEOREM 3.2. *Let $(\mathscr{F}_t)$ satisfy V_p and let $\mathscr{E}$ have the Radon-Nikodym property. Then an L^1-bounded amart for M_p converges weakly essentially.*

Proof. Applying Theorem 3.1, write $X_t = Y_t + Z_t$. For each $f \in \mathscr{E}'$, $f(Z_t)$ is a real-valued L^1-bounded amart for M_p, which converges essentially by Theorem 2.1, necessarily to zero. Hence Z_t converges weakly essentially to zero. It remains to discuss the convergence of the martingale (Y_t). For each $f \in \mathscr{E}'$, $f(Y_t)$ is an L^1-bounded amart for M_p, and hence converges essentially to a random variable depending on f, say R_f. At the same time, for every increasing sequence (τ_n) in T, (Y_{τ_n}) is an L^1-bounded martingale which converges by Chatterji's theorem ([**8**]; see e.g. [**27**], p. 112) almost surely, hence stochastically in the norm topology. Since the stochastic convergence is defined by a complete metric, this implies that $(Y_\tau)_{\tau\in T}$ converges stochastically, say to Y_∞. Therefore for each $f \in \mathscr{E}'$, $f(Y_t)$ converges stochastically to $f(Y_\infty) = R_f$. Thus X_t converges weakly essentially to Y_∞.

For L^q-bounded martingales, a stronger result is obtained. We at first prove the following maximal inequality:

LEMMA 3.3. *If $X \in L^q(\mathscr{E})$ and if $(\mathscr{F}_t)$ satisfies V_p, then given any $a > 0$, $P[e\limsup\{|E^{\mathscr{F}_t}X| > a\}] \leqq (1/a)E|X|$.*

Proof. Set $A_t = \{E^{\mathscr{F}_t}|X| > a\}$, $A = e\limsup A_t$, and let $\epsilon > 0$; there exists $\tau \in IM$, such that for every t, $\{\tau = t\} \subset A_t$, $P(A\backslash D(\tau)) \leqq \epsilon$, and $O_p(\tau) \leqq \epsilon$. Then

$$a[P(A) - \epsilon] \leqq \sum_{t\in R(\tau)} E[1_{\{\tau=t\}} E^{\mathscr{F}_t}|X|] \leqq E|X| + O_p(\tau)\|X\|_q \leqq E|X| + \epsilon\|X\|_q,$$

which gives the maximal inequality when ϵ approaches 0.

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THEOREM 3.4. *Let* $(\mathscr{F}_t)$ *satisfy the Vitali condition* V_p, *let* $\mathscr{E}$ *be a Banach space with the Radon-Nikodym property, and let* (X_t) *be an* L^q*-bounded* $\mathscr{E}$*-valued martingale. Then* X_t *converges essentially in the norm topology.*

Proof. We prove that the net $(X_\tau)_{\tau \in M_p}$ converges stochastically in the norm topology of $\mathscr{E}$, and then apply the implication (1) $\Rightarrow$ (2) in Theorem 2.1, which extends to Banach-valued X_t without change of proof.

First in the case $p = \infty$ one shows, as in the proof of Theorem 3.2, that $(X_\tau)_{\tau \in T}$ converges stochastically; it follows that X_t converges essentially. Assume now that $1 \leqq p < \infty$; an L^q-bounded martingale is uniformly integrable, therefore it admits a representation $X_t = E^{\mathscr{F}_t}X$, with $X \in L^q[\cup \mathscr{F}_t)$ (cf. [**18**]; [**27**], p. 113). Let Λ be the vector space of functions $X \in L^q$, measurable with respect to some $\mathscr{F}_s$, $s \in J$. Λ is dense in $L^q(\cup \mathscr{F}_t)$, and for $X \in \Lambda$, $E^{\mathscr{F}_t}X$ obviously converges essentially to X. Let X be in $L^q(\cup \mathscr{F}_t)$, $Y \in \Lambda$; then for every $t \in J$,

$$|E^{\mathscr{F}_t}X - X| \leqq E^{\mathscr{F}_t}|X - Y| + |E^{\mathscr{F}_t}Y - Y| + |Y - X|.$$

Hence

$$e \lim\sup |E^{\mathscr{F}_t}X - X| \leqq e \lim\sup E^{\mathscr{F}_t}|X - Y| + |X - Y|.$$

Let $a > 0$; given $\epsilon > 0$, choose $Y \in \Lambda$ such that $\|X - Y\|_q < \epsilon$. Lemma 3.3 yields that under V_p,

$$P[e \lim\sup \{|E^{\mathscr{F}_t}X - X| > a\}] \leqq P[e \lim\sup \{E^{\mathscr{F}_t}|X - Y| > a/2]$$
$$+ P[|X - Y| > a/2] \leqq 2/a[E|X - Y| + \|X - Y\|_q] \leqq 4\epsilon/a.$$

Since a and ϵ are arbitrary, it follows that $e \lim E^{\mathscr{F}_t}X = X$.

Our final result concerns the behavior of $\mathscr{E}$-valued pramarts under the condition V_∞. *Pramarts*, introduced in [**24**], are defined by the property

$$s \lim_{\sigma \leqq \tau, \sigma, \tau \in T}|X_\sigma - E^{\mathscr{F}_\sigma}X_\tau| = 0.$$

Recall that $M_\infty = T$, and stopping times now considered are single-valued. If (X_t) is a real-valued amart, it is a pramart; however, this implication fails in every infinite-dimensional Banach space [**24**], [**25**]. Banach-valued pramarts, unlike amarts, converge strongly. Pramarts (or *mils*: cf. [**24**] and [**26**]) such that $\sup|X_t| \in L^1$ can be shown to be A. Bellow's uniform amarts (cf. [**4**], [**16**]).

THEOREM 3.5. *Let* $(\mathscr{F}_t)$ *satisfy* V_∞, *and let* $\mathscr{E}$ *have the Radon-Nikodym property. A pramart* (X_t) *converges essentially in the norm topology if either* (a) *or* (b) *holds:*

(a) $(|X_t|)$ *is uniformly integrable.*

(b) (X_t) *is of class* (B), *i.e.*, $\sup_{\tau \in T} E|X_\tau| < \infty$.

Proof. (a) From the pramart property of (X_t), the net $(|E^{\mathscr{F}_s}X_t - X_s|)_{s \leqq t}$ of real-valued random variables converges to 0 in probability. Since this net is uniformly integrable (because $|X_t|$ is), it converges to 0 in L^1. If $s_0 \leqq s \leqq t$,

$$|E^{\mathscr{F}_{s_0}}X_t - E^{\mathscr{F}_{s_0}}X_s| = |E^{\mathscr{F}_{s_0}}(E^{\mathscr{F}_s}X_t - X_s)|;$$

hence for a fixed s_0 the net

$$(E^{\mathscr{F}_{s_0}}X_t)_{t \geqq s_0}$$

is Cauchy in $L^1(\mathscr{E})$, and converges to a Bochner integrable random variable Y_{s_0}, $(Y_s)_{s \in J}$ is an L^1-bounded martingale, and if we set $Z_s = X_s - Y_s$, (Z_s) is a pramart such that $\lim E|Z_s| = 0$ (a similar argument appears in [**1**]). Now observe that if (Z_t) is any pramart, then $(|Z_t|)$ is necessarily a real-valued *subpramart*, i.e., satisfies

$$s \lim_{\tau \geqq \sigma \to \infty} \sup \,[|Z_\sigma| - E^{\mathscr{F}_\sigma}|Z_\tau|] \leqq 0.$$

Indeed, if $\sigma \leqq \tau$, σ, $\tau \in T$, then for every $\epsilon > 0$

$$P[\{|Z_\sigma| - E^{\mathscr{F}_\sigma}|Z_\tau| > \epsilon\}] > \epsilon] \leqq P[\{|Z_\sigma| - |E^{\mathscr{F}_\sigma}Z_\tau| > \epsilon\}]$$
$$\leqq P[\{|Z_\sigma - E^{\mathscr{F}_\sigma}Z_\tau| > \epsilon\}].$$

Since under V_∞ an L^1-bounded subpramart converges essentially [**24**], [**25**], $\lim E|Z_s| = 0$ implies that $e \lim Z_s = 0$. Also Y_s converges essentially by Theorem 3.3 with $p = \infty$. Hence X_t converges essentially.

(b) Consider at first a pramart of class (B) $(X_n)_{n \in N}$. Let $\lambda > 0$ be given; set $A = \cup\{|X_n| > \lambda\}$, and define $\sigma(\omega) = \inf\{n| \, |X_n| > \lambda\}$ if $\omega \in A$, and $\sigma(\omega) = \infty$ if $\omega \in A^c$. Then σ is a possibly infinite stopping time. Set $X_n' = X_{n \wedge \sigma}$. Theorem 2.4 [**24**], valid also in the Banach-valued case, shows that (X_n') is a pramart. By Fatou's lemma,

$$E(1_A|X_\sigma|) \leqq \liminf E(1_A|X_{n\wedge\sigma}|) \leqq \sup_{\tau \in T} E|X_\tau|.$$

Thus $E(\sup|X_n'|) < \lambda + \sup_{\sigma \in T} E|X_\sigma|$. Furthermore,

$$P(A) \leqq \lambda^{-1} \sup_{\tau \in T} E|X_\tau|$$

(see [**7**]), and on A^c, $X_n = X_n'$ for every n. Hence to prove that X_n converges a.s. in the norm topology, it suffices to show that X_n' does. Since $\sup |X_n'|$ is integrable, this follows from part (a).

Let (X_t) be a pramart of class (B). Choose a sequence of indices (s_n) such that $s_n \leqq \sigma \leqq \tau$ implies

$$P(\{|X_\sigma - E^{\mathscr{F}_\sigma}X_\tau| > 1/n\}) < 1/n,$$

and let (σ_n) be an increasing sequence of elements of T, such that $s_n \leqq \sigma_n$ for all n. Set $X_n' = X_{\sigma_n}$, and $\mathscr{G}_n = \mathscr{F}_{\sigma_n}$; $(X_n', \mathscr{G}_n)$ is a pramart of class (B). Hence X_n' converges a.s. and stochastically in the norm topology. Therefore $(X_\tau)_{\tau \in T}$ converges stochastically in the norm topology. If $(\mathscr{F}_t)$ satisfies V_∞, we deduce the strong essential convergence of X_t.

Finally, we observe how our results extend to Orlicz spaces. Let us first recall some properties of Orlicz spaces (see [**27**], Appendix).

Let $\varphi: R^+ \to R^+$ be an increasing left-continuous function which is zero at the origin, such that $\lim_{s\to\infty} \varphi(s) = \infty$. Let ψ be the function inverse to φ, i.e., defined by $\psi(u) = \sup \{s | \varphi(s) < u\}$ for every $u > 0$. Let Φ (resp. Ψ) be the indefinite integral of φ (resp. ψ), i.e.,

$$\Phi(t) = \int_0^t \varphi(s)\, ds.$$

Φ is said to be conjugate to Ψ. Let L^Φ be the set of random variables for which there exists a number $a > 0$ such that $E[\Phi(a^{-1}|X|)] \leqq 1$, and set

$$\|X\|_\Phi = \inf \{a | a > 0, E[\Phi(a^{-1}|X|)] \leqq 1\}.$$

The normed vector space L^Φ is a Banach space. There exists a constant $c > 0$ such that $c\|X\|_1 \leqq \|X\|_\Phi$ for every random variable X of L^Φ. Furthermore, if Φ and Ψ are conjugate Young functions, for every pair $X \in L^\Phi$, $Y \in L^\Psi$, the product XY is integrable and satisfies the inequality $\|XY\|_1 \leqq 2\|X\|_\Phi \|Y\|_\Psi$. Φ satisfies Δ_2 if $\sup \Phi(2t)/\Phi(t) < \infty$.

An integrable stochastic process (X_t) is an amart for M_Φ if the net $(EX_\tau)_{\tau \in M_\Phi}$ converges when $\tau \in M$ and $O_\Phi(\tau) = \|e_\tau\|_\Phi \to 0$. A stochastic basis $(\mathscr{F}_t)$ satisfies the Vitali condition V_Ψ if for every adapted family of sets (A_t) and for every $\epsilon > 0$, there exists $\tau \in IM$ such that $O_\Psi(\tau) < \epsilon$, $P(e \limsup A_t \backslash A_\tau) < \epsilon$, and for every $t \in R(\tau)$, $\{\tau = t\} \subset A_t$.

It is easy to see that the statements and proofs of the theorems remain the same if the real L^p and L^q spaces are replaced by Orlicz spaces L^Φ and L^Φ, and Φ satisfies the condition Δ_2.

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