Can. J. Math., Vol. XXVI, No. 2, 1974, pp. 340–351

LIMIT POINT CRITERIA FOR DIFFERENTIAL EQUATIONS, II

DON HINTON

Introduction. We consider here singular differential operators, and for convenience the finite singularity is taken to be zero. One operator discussed is the operator L defined by

$$(0.1)\quad L(y) = (-1)^n(q_0 y^{(n)})^{(n)} + (-1)^{n-1}(q_1 y^{(n-1)})^{(n-1)} + \ldots + q_n y,$$

where $q_0 > 0$ and the coefficients q_i are real, locally Lebesgue integrable functions defined on an interval (a, b). For a given positive, continuous weight function h, conditions are given on the functions q_i for which the number of linearly independent solutions y of $L(y) = \lambda h y$ $(\operatorname{Re} \lambda = 0)$ satisfying

$$\int_a^b h|y|^2 < \infty$$

is $\leqq n$. These results parallel those of [**2**] where the singularity is at infinity. In fact, the approach used will be to modify the results of [**2**] so as to obtain criteria for finite and infinite singularities from a single framework. This work solves a certain deficiency index problem which we now describe.

Denote the Hilbert space of all complex valued measurable functions y such that

$$\int_a^b h|y|^2 < \infty$$

by $\mathscr{L}_2(h; a, b)$, and define the quasi-derivatives $y^{[i]}$ $(i = 0, \ldots, 2n)$ by: $y^{[i]} = y^{(i)}$ $(i = 0, \ldots, n-1)$, $y^{[n]} = q_0 y^{(n)}$, and $y^{[n+i]} = q_i y^{(n-i)} - (y^{[n+i-1]})'$ $(i = 1, \ldots, n)$. A function y is said to be *L-admissible* provided the quasi-derivatives $y^{[i]}$ $(i = 0, \ldots, 2n-1)$ exist and are absolutely continuous on compact intervals (then $L(y) = y^{[2n]}$). Let $\mathscr{D}$ be the set of all L-admissible $y \in \mathscr{L}_2(h; a, b)$ such that $(1/h)L(y) \in \mathscr{L}_2(h; a, b)$, and let T be the restriction of $(1/h)L$ to $\mathscr{D}$. Denote by $\mathscr{D}_0'$ the set of all $y \in \mathscr{D}$ which have compact support interior to (a, b), and let T_0' be the restriction of T to $\mathscr{D}_0'$. Then as in [**3**, § 17.3, 17.4] where $h \equiv 1$, it may be shown that T_0' is a densely defined symmetric operator in $\mathscr{L}_2(h; a, b)$; hence admits a closure T_0, and $T_0^* = T$ [**3**, § 17.4].

Received October 4, 1972 and in revised form, March 19, 1973.

The deficiency indices of T_0 are (n_1, n_2) where n_v is number m of linearly independent solutions in $\mathscr{L}_2(h; a, b)$ of $L(y) = \lambda hy$ ($\lambda = i$ for $v = 1$ and $\lambda = -i$ for $v = 2$). As in [**3**, § 14.7, 17.5] where $h \equiv 1$, it may be shown that the number m is actually the same for all non real λ, and $m \geqq n$ in either of the following two cases: $a = 0$, $b < \infty$, and $1/q_0, q_1, \ldots, q_n$ are Lebesgue integrable on (ϵ, b) for each $\epsilon > 0$ or $-\infty < a, b = \infty$, and $1/q_0, q_1, \ldots, q_n$ are Lebesgue integrable on (a, d) for each $d > a$. Thus we give conditions under which the deficiency indices of T_0 are (n, n). The case $n_1 = n_2 = n$ is called the *limit point* case.

In section 1 the necessary modifications of section 2 of [**2**] are given, and in section 2 these results are applied to a singularity at zero. To derive limit point criteria at zero from limit point criteria at infinity, it is necessary to consider a more general operator than (0.1). This operator is defined in section 1.

1. Singularities at infinity. Let r, H, p_i ($i = 0, \ldots, n$) be real functions on a ray $[c, \infty)$ which are Lebesgue integrable on compact intervals. In addition, let $r > 0$, $H > 0$, and $p_0 > 0$ satisfy

(1.0) H, r, and p_0 are respectively $n - 1$, $n - 1$, and n times continuously differentiable.

For a sufficiently differentiable function y, we define the quasi-derivatives $y^{[i]}$ by:

$$y^{[i]} = \begin{cases} y, \; i = 0 \\ ry^{[i-1]\prime}, \; i = 1, \ldots, n-1. \\ rp_0 y^{[n-1]\prime}, \; i = n, \\ r\{p_{i-n}y^{[2n-i]} - y^{[i-1]\prime}\}, \; i = n+1, \ldots, 2n-1. \end{cases}$$

The operator S is defined by

$$S(y) = H^{-1}\{p_n y^{[0]} - y^{[2n-1]\prime}\}.$$

A function y is said to be *S-admissible* provided the quasi-derivatives $y^{[i]}$ ($i = 0, \ldots, 2n - 1$) exist and are absolutely continuous on compact subintervals of $[c, \infty)$. For $r \equiv 1$, S reduces to the familiar case

$$HS(y) = (-1)^n (p_0 y^{(n)})^{(n)} + (-1)^{n-1}(p_{n-1}y^{(n-1)})^{(n-1)} + \ldots + p_n y. \tag{1.1}$$

The equation $S(y) = \lambda y + m$ has the vector formulation

$$Y' = AY + [0, \ldots, 0, -Hm]^T \tag{1.2}$$

where $Y = (y^{[0]}, \ldots, y^{[2n-1]})^T$ and

$$A = \left[\begin{array}{cccc|cccc} 0 & 1/r & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \ddots\ 1/r & 0 & & & \\ & & & 0 & 1/rp_0 & & & \\ \hline & & & p_1 & 0 & -1/r & & \\ & & & \cdot & \ddots & \ddots & & \\ & & \cdot & & & \ddots & \ddots & \\ & \cdot & & & & & \ddots & -1/r \\ p_{n-1} & & & & & & & \cdot \\ p_n - \lambda H & & & & & & & 0 \end{array}\right].$$

The Lagrange identity for S is

$$S(y)\bar{z} - y\overline{S(z)} = H^{-1}[y, z]'$$

where

$$(1.3)\quad [y, z] = \sum_{i=0}^{n-1} \{y^{[i]}\bar{z}^{[2n-i-1]} - y^{[2n-i-1]}\bar{z}^{[i]}\}.$$

If $S(y) = \lambda y + m$, then it is easy to verify that the quadratic expression

$$(1.4)\quad -(\lambda y + m)H\bar{y} + \frac{1}{rp_0}|y^{[n]}|^2 + \sum_{i=0}^{n-1} p_{n-i}|y^{[i]}|^2 = \left\{\sum_{i=0}^{n-1} y^{[2n-i-1]}\bar{y}^{[i]}\right\}'$$

holds. Our concern in this section is with the solutions of $S(y) = \lambda y + m$ which are in $\mathscr{L}_2(H; c, \infty)$.

Much of our analysis will depend on certain *a priori* bounds on the S-admissible members of $\mathscr{L}_2(H; c, \infty)$. To establish these bounds we use a non-homogeneous version of Theorem 1.1 of [**2**]. Consider the system of differential equations.

$$(z)\qquad X' = wBX + [0, \ldots, 0, f]^T,$$

where $X = (x_1, \ldots, x_m)^T$ is a column vector, f and the entries of the $m \times m$ matrix $B = \{b_{ij}\}$ are measurable, locally integrable, complex-valued functions on $[c, \infty)$, and w is a positive, continuous function on $[c, \infty)$. In addition, suppose B satisfies

$$b_{ij} = \begin{cases} 0, & \text{if } j > i + 1 \\ \pm 1, & \text{if } j = i + 1. \end{cases}$$

THEOREM A. *Suppose X is a solution of* (z) *and that for some $k \leqq m$, b_{ij} is*

https://doi.org/10.4153/CJM-1974-035-7 Published online by Cambridge University Press

bounded on $[c, \infty)$ *for all* $i \leqq k$. *Let*

$$I_i = I_i(t) = \max\left\{1, \int_a^t w|x_i|^2 ds\right\} \quad (i = 1, \ldots, m)$$

and suppose $I_1(\infty) < \infty$.

(i) *If* $k < m$, *then for* $i = 1, \ldots, k$, *the following order relations hold as* $t \to \infty$:

$$I_i = O(I_{i+1}{}^{(i-1)/i}) \quad \text{and} \quad |x_i|^2 = O(I_{i+1}{}^{(2i-1)/2i}).$$

(ii) *If* $k = m$ *and*

$$\int_c^\infty w^{-1}|\mathrm{f}|^2 ds < \infty,$$

then for $i = 1, \ldots, m$ *and as* $t \to \infty$, $I_i = O(1)$ *and* $|x_i|^2 = O(1)$.

The proof of part (i) of Theorem A is identical to the proof of part (i) of Theorem 1.1 of [**2**]. The proof of part (ii) differs from the proof of part (ii) of Theorem 1.1 only in the consideration of the integral

$$\int_c^t \bar{x}_m{}'(b_{m-1,m})x_{m-1}$$

which now contains the addition term

$$\int_c^t f(b_{m-1,m})X_{m-1}.$$

However,

$$\left|\int_c^t f(b_{m-1,m})x_{m-1}\right| \leqslant \left(\int_c^t w^{-1}|f|^2\right)^{\frac{1}{2}}\left(\int_c^t w|x_{m-1}|^2\right)^{\frac{1}{2}}$$
$$= O(I_{m-1}{}^{\frac{1}{2}}) = O(I_m{}^{\frac{1}{2}}).$$

The proof now proceeds as that of part (ii) of Theorem 1.1. We refer the reader to [**2**] for the details.

We assume that ρ is a positive function with n continuous derivatives. The function g is defined by $g = (rH)^{1/2n}$ and we consider the conditions:

$$\text{(1.5)} \quad \frac{|p_i|\rho^{4i}r}{p_0 g^{2i}} = O(1) \quad \text{as } t \to \infty, \quad i = 1, \ldots, n-1.$$

$$\text{(1.6)} \quad \text{For some } K > 0, \frac{-p_n\rho^{4n}r}{p_0 g^{2n}} \leqslant K.$$

$$\text{(1.7)} \quad \frac{\rho^2 r}{tg} = O(1) \text{ and } \frac{\rho^2 r}{g}\left[\frac{|\rho'|}{\rho} + \frac{|g'|}{g} + \frac{|p_0'|}{p_0}\right] = O(1) \quad \text{as } t \to \infty.$$

$$\text{(1.8)} \quad \int_c^\infty \frac{g\rho^{4n-2}}{rp_0} dt = \infty.$$

https://doi.org/10.4153/CJM-1974-035-7 Published online by Cambridge University Press

(1.9) As $t \to \infty$,

$$[g\rho^{4n-2}/rp_0]^{(j)} = O(g^{j+1}\rho^{4n-2-2j}/r^{j+1}p_0), \qquad j = 1, \ldots, n-1,$$

and

$$[\rho^{4n}/p_0]^{(j)} = O(g^j\rho^{4n-2j}/r^jp_0), \qquad j = 1, \ldots, n.$$

(1.10) For $j = 1, \ldots, n-1$, $r^{(j)} = O(g^j/r^{j-1}\rho^{2j})$ as $t \to \infty$.

Note that in (1.9) and (1.10), the order relations are equalities for $j = 0$. The vector spaces $\mathscr{D}_S$, $V_1(\lambda)$, and $V_2(\lambda)$ are defined by

$$\mathscr{D}_S = \{y|y \text{ is } S\text{-admissible and } y \in \mathscr{L}_2(H; c, \infty)\},$$
$$V_1(\lambda) = \{y|S(y) = \lambda y \text{ and } y \in \mathscr{L}_2(H; c, \infty)\},$$
$$V_2(\lambda) = \{z|S(z) = \bar{\lambda} z \text{ and } z \in \mathscr{L}_2(H; c, \infty)\}.$$

In order to apply Theorem A, we transform the equation (1.2) by $X = MY$ where M is the diagonal matrix

$$M = \text{diagonal}\left\{g^\alpha\rho, g^{\alpha-1}\rho^3, \ldots, g^{\alpha-n+1}\rho^{2n-1}, \frac{g^{\alpha-n}\rho^{2n+1}}{p_0}, \ldots, \frac{g^{\alpha-2n+1}\rho^{4n-1}}{p_0}\right\}$$

with $\alpha = (2n-1)/2$. The vector X satisfies

(1.11) $X' = (g/r\rho^2)BX + [0, \ldots, 0, -g^{\alpha-2n+1}\rho^{4n-1}Hm/p_0]^T$

where $B = (r\rho^2/g)[MAM^{-1} + M'M^{-1}]$. Calculations show $B = \{b_{ij}\}$ satisfies $b_{i,i+1} = \pm 1$, b_{ii} is bounded (by (1.7)),

$$b_{n+i,n+1-i} = rp_i\rho^{4i}/p_0g^{2i} \quad (i = 1, \ldots, n-1),$$

$b_{2n,1} = r(p_n - \lambda H)\rho^{4n}/p_0g^{2n}$, and otherwise $b_{ij} = 0$. The integral relations between $X = (x_1, \ldots, x_{2n})^T$ and Y are

$$(1.12) \quad \int_c^t (g/r\rho^2)|x_i|^2\,ds = \begin{cases} \displaystyle\int_c^t \frac{\rho^{4i-4}g^{2(n-i+1)}}{r}|y^{[i-1]}|^2ds, & i = 1, \ldots, n, \\ \displaystyle\int_c^t \frac{\rho^{4i-4}g^{2(n-i+1)}}{rp_0^2}|y^{[i-1]}|^2ds, & i = n+1, \ldots, 2n. \end{cases}$$

For Lemma 1.1 below we need the functions G_k and H_k which for fixed t are defined for $c \leqq s \leqq t$. Their definitions are:

$$G_0(s) = (1 - s/t)^{n-1}\left[\frac{g\rho^{4n-2}}{rp_0}\right](s),$$

$$G_k(s) = \frac{d}{ds}[rG_{k-1}], \quad k = 1, \ldots, n-1,$$

$$H_0(s) = \frac{d}{ds}\left\{(1 - s/t)^n\frac{\rho^{4n}(s)}{p_0(s)}\right\},$$

$$H_k(s) = \frac{d}{ds}[rH_{k-1}], \quad k = 1, \ldots, n-1.$$

A property of G_k which follows from (1.7), (1.9), and (1.10) that we shall need is that for some c_{kj}

$$(1.13)\quad |G_k^{(j)}| \leqslant c_{kj}\left(\frac{g^{j+1+k}\rho^{4n-2-2k-2j}}{r^{j+1}p_0}\right),\quad j = 0,\dots,n-k-1,$$

where the constant in (1.13) is independent of t.

For $k = 0$ in (1.13), $1 \leqq j \leqq n-1$ (for $k = j = 0$ we may take $c_{00} = 1$), and from (1.7), (1.9), and $s \leqq t$,

$$\begin{aligned} G_0^{(j)}(s) &= \sum_{u=0}^{j}\binom{j}{u}\frac{d^{j-u}}{ds^{j-u}}(1-s/t)^{n-1}\frac{d^u}{ds^u}\left[\frac{g\rho^{4n-2}}{rp_0}\right] \\ &= \sum_{u=0}^{j} O\left(\frac{1}{t^{j-u}}\left[\frac{g^{u+1}\rho^{4n-2-2u}}{r^{u+1}p_0}\right](s)\right) \\ &= \left[\frac{g^{j+1}\rho^{4n-2-2j}}{r^{j+1}p_0}\right](s)\sum_{u=0}^{j} O\left(\frac{\rho^2(s)r(s)}{sg(s)}\right)^{j-u} \\ &= O\left(\left[\frac{g^{j+1}\rho^{4n-2-2j}}{r^{j+1}p_0}\right](s)\right) \end{aligned}$$

and the constant in the order relation is independent of t.

Assuming now (1.13) holds for some k, $0 \leqq k < n-1$, we have by application of (1.10) that

$$\begin{aligned} G_{k+1}^{(j)} &= (rG_k)^{(j+1)},\quad j = 0,\dots,n-k-2 \\ &= \sum_{u=0}^{j+1}\binom{j+1}{u} r^{(j+1-u)}G_k^{(u)} \\ &= \sum_{u=0}^{j+1} O\left(\frac{g^{j+1-u}}{r^{j-u}\rho^{2(j+1-u)}}\cdot\frac{g^{u+1+k}\rho^{4n-2-2k-2u}}{r^{u+1}p_0}\right) \\ &= O\left(\frac{g^{j+k+2}\rho^{4n-4-2k-2j}}{r^{j+1}p_0}\right), \end{aligned}$$

and again the constant in the order relation is independent of t. This induction establishes (1.13), and in a similar manner we may show there are constants d_{kj} such that

$$(1.14)\quad |H_k^{(j)}| \leqslant d_{kj}\frac{g^{j+k+1}\rho^{4n-2j-2k-2}}{r^{j+1}p_0},\quad j = 0,\dots,n-k-1,$$

and the constant d_{jk} is independent of t. For a later integration by parts, we note that $G_k(t) = H_k(t) = 0$ for $k = 0,\dots,n-2$.

LEMMA 1.1. *Suppose conditions* (1.0), (1.5), (1.7), (1.9), *and* (1.10) *hold and assume y and z are nontrivial members of $\mathscr{D}_S$. Let*

$$J_1 = J_1(t) = \int_c^t \frac{\rho^{4n}}{rp_0^2}|y^{[n]}|^2 \quad and \quad J_2 = J_2(t) = \int_c^t \frac{\rho^{4n}}{rp_0^2}|z^{[n]}|^2.$$

https://doi.org/10.4153/CJM-1974-035-7 Published online by Cambridge University Press

Then for $i = n, \ldots, 2n-1,$

$$\text{(i)} \quad \left| \int_c^t y^{[i]} \bar{z}^{[j]} G_k ds \right| = O([J_1 J_2]^{\frac{1}{2}}) \quad \text{as } t \to \infty$$

for all j, k *such that* $i + j + k = 2n - 1$, *and*

$$\text{(ii)} \quad \left| \int_c^t y^{[i]} \bar{y}^{[j]} H_k ds \right| = O(J_1^{(2n-1)/2n}) \quad \text{as } t \to \infty$$

for all j, k *such that* $i + j + k = 2n - 1$.

Proof. Applying part (i) of Theorem A to (1.11), we have from (1.12) that for $1 \leqq i \leqq n$ (note that $g^{2n}/r = H$),

$$(1.15) \quad \int_c^t \frac{\rho^{4i-4} g^{2(n+1-i)}}{r} |y^{[i-1]}|^2 ds = \int_c^t \frac{g}{r\rho^2} |x_i|^2 ds$$
$$= O\left(\left[\int_c^t \frac{g}{r\rho^2} |x_{n+1}|^2\right]^{n-1/n}\right)$$
$$= O(J_1^{(n-1)/n}) = O(J_1),$$

and similarly for z and $1 \leqq i \leqq n$,

$$(1.16) \quad \int_c^t \frac{\rho^{4i-4} g^{2(n+1-i)}}{r} |z^{[i-1]}|^2 ds = O(J_2^{(n-1)/n}) = O(J_2).$$

Consider now (i). With $j + k = n - 1$, it follows from (1.13) that

$$(1.17) \quad \left| \int_c^t y^{[n]} \bar{z}^{[j]} G_k ds \right| \leqslant \int_c^t |y^{[n]} \bar{z}^{[j]}| O\left(\frac{g^{k+1} \rho^{4n-2-2k}}{r p_0}\right) ds$$
$$= \int_c^t O\left(\frac{\rho^{2n}}{p_0 r^{\frac{1}{2}}} |y^{[n]}| \frac{\rho^{2j} g^{n-j}}{r^{\frac{1}{2}}} |z^{[j]}|\right).$$

Since $j \leqq n - 1$, and application of the Cauchy inequality and (1.16) to the right hand side of (1.17) establishes (i) for $i = n$.

Assume now (i) holds for some i, $n \leqq i < 2n - 1$ and that $(i + 1) + j + k = 2n - 1$. Then

$$(1.18) \quad \left| \int_c^t y^{[i+1]} \bar{z}^{[j]} G_k ds \right| = \left| \int_c^t r\{p_{i+1-n} y^{[2n-i-1]} - y^{[i]\prime}\} \bar{z}^{[j]} G_k ds \right|$$
$$= \left| \int_c^t r p_{i+1-n} y^{[2n-i-1]} \bar{z}^{[j]} G_k ds + O(1) \right.$$
$$\left. + \int_c^t y^{[i]} \{r \bar{z}^{[j]} G_k\}' ds \right|.$$

Since $\{r\bar{z}^{[j]} G_k\}' = \bar{z}^{[j+1]} G_k + z^{[j]} G_{k+1}$, the induction hypothesis applies to the

https://doi.org/10.4153/CJM-1974-035-7 Published online by Cambridge University Press

last integral on the right hand side of (1.18). From (1.5), (1.13), and $i + 1 + j + k = 2n - 1$ we obtain

$$|rp_{i+1-n}y^{[2n-i-1]}\bar{z}^{[j]}G_k| = O\left(\frac{p_0 g^{2(i+1-n)}}{\rho^{4(i+1-n)}}|y^{[2n-i-1]}\bar{z}^{[j]}|\frac{g^{k+1}\rho^{4n-2-2k}}{rp_0}\right)$$
$$= O\left(\frac{\rho^{2(2n-i-1)}g^{(i+1-n)}}{r^{\frac{1}{2}}}|y^{[2n-i-1]}|\cdot\frac{\rho^{2j}g^{n-j}}{r^{\frac{1}{2}}}|\bar{z}^{[j]}|\right).$$

Hence an application of the Cauchy inequality, (1.15), and (1.16) yields that the first integral on the right hand side of (1.18) is $O([J_1J_2]^{1/2})$. This inductive step completes the proof of part (i). Part (ii) follows from a similar inductive argument.

LEMMA 1.2. *Suppose* (1.0) *holds*, $y \in \mathscr{D}_S$, *and* $S(y) = \lambda y + m$ *with* $\operatorname{Re}\lambda = 0$ *and* $(\rho^{4n}m/p_0) \in \mathscr{L}_2(H; c, \infty)$.

(i) *If* (1.5), (1.6), (1.7), (1.9), *and* (1.10) *hold, then*

$$(1.19)\quad \int_c^\infty \frac{\rho^{4i-4}g^{2(n-i+1)}}{r}|y^{[i-1]}|^2 ds < \infty, \quad i = 1, \dots, n,$$

$$(1.20)\quad \int_c^\infty \frac{\rho^{4n}}{rp_0^{\,2}}|y^{[n]}|^2 ds < \infty,$$

and for $i = 1, \dots, n$,

$$(1.21)\quad |g^{(2n+1-2i)/2}\rho^{2i-1}y^{[i-1]}| = O(1) \quad as\ t \to \infty.$$

(ii) *If* (1.5) *and* (1.7) *hold, and* (1.6) *is replaced by* $|(p_n - H)\rho^{4n}r/p_0^{\,2n}| \leqq K$ $(K > 0)$, *then in addition to* (1.19) *and* (1.21) *we have for* $i = n + 1, \dots, 2n$,

$$(1.22)\quad \int_c^\infty \frac{\rho^{4i-4}g^{2(n-i+1)}}{rp_0^{\,2}}|y^{[i-1]}|^2 ds < \infty$$

and

$$(1.23)\quad |g^{(2n+1-2i)/2}\rho^{2i-1}y^{[i-1]}/p_0| = O(1) \quad as\ t \to \infty.$$

Proof. It is sufficient to have $y \not\equiv 0$, and from (1.15), (1.19) will follow from (1.20). Let J_1 be as in Lemma 1.1. From (1.4) and an integration by parts,

$$(1.24)\quad \int_c^t \left[-(\lambda y + m)H\bar{y} + \frac{|y^{[n]}|^2}{rp_0} + \sum_{i=0}^{n-1} p_{n-i}|y^{[i]}|^2\right](1 - s/t)^n(\rho^{4n}/p_0)ds$$
$$= O(1) - \int_c^t \sum_{i=0}^{n-1} y^{[2n-i-1]}\bar{y}^{[i]}H_0(s)ds.$$

By part (ii) of Lemma 1.1, the right hand side of this equation is $O(J_1^{(2n-1)/(2n)})$.

Also by (1.5) and (1.15) for $1 \leqq i \leqq n - 1$,

$$\int_c^t p_{n-i}|y^{[i]}|^2(1 - s/t)^n(\rho^{4n}/p_0)ds = O\left(\int_c^t \frac{\rho^{4i}g^{2(n-i)}}{r}|y^{[i]}|^2 ds\right)$$
$$= O(J_1^{(n-1)/n}) = O(J_1^{(2n-1)/2n}).$$

By (1.6), and $(\rho^{4n}m/p_0) \in \mathcal{L}_2(H; c, \infty)$, there is a $K_1 > 0$ such that

$$\operatorname{Re} \int_c^t [(-\lambda H + p_n)|y|^2 - mH\bar{y}](1 - s/t)^n(\rho^{4n}/p_0)ds \geqslant -K_1.$$

Using these inequalities in (1.24) gives

$$\int_c^t (\rho^{4n}/rp_0^2)|y^{[n]}|^2(1 - s/t)^n ds = O(J_1^{(2n-1)/2n}).$$

Applying Lemma 2.3 of [**2**] with $F = \rho^{4n}|y^{[n]}|^2/rp_0^2$ now yields $J_1(\infty) < \infty$.
Applying part (i) of Theorem A to the system (1.11) and using

$$\int_c^\infty (g/r\rho^2)|x_{n+1}|^2 ds = J_1(\infty) < \infty$$

gives $|x_i| = O(1)$ as $t \to \infty$ for $i = 1, \ldots, n$. From the transformation $X = MY$, this gives (1.21).

For the proof of part (ii), we need only note that with B as in (1.11), part (ii) of Theorem A applies to give for $i = 1, \ldots, 2n$,

$$\int_c^\infty (g/r\rho^2)|x_i|^2 ds < \infty \quad \text{and} \quad |x_i| = O(1) \quad \textit{as}\ t \to \infty.$$

Lemma 1.2 has a number of conclusions independent of our use of it. A straightforward application is to consider (1.1) with $p_0 \equiv 1$ and $H \equiv 1$. Choosing $\rho = 1$, we may conclude that if the coefficients in (1.1) are bounded, then $[S(y) - \lambda y]$ and y both in $\mathcal{L}_2\ (1; c, \infty)$ implies that

$$\int_c^\infty |y^{[i-1]}|^2 ds < \infty \quad \text{and} \quad |y^{[i-1]}| = O(1) \quad \text{as}\ t \to \infty$$

for $i = 1, \ldots, 2n$. The reader may compare this with the lemmas in [**1**, pp. 1425 and 1428].

For the equation $(H = 1)$

(1.25) $\quad (-1)^n y^{(2n)} + py = 0,$

Lemma 1.2 applies with $\rho = t^\Delta$ provided $\Delta \leqq 1/2$. Hence we may conclude that if $-p(t) \leqq K/t^{4n\Delta} (K > 0)$, then an $\mathcal{L}_2(1; c, \infty)$ solution y of (1.25) also satisfies for $i = 0, \ldots, n - 1$,

(1.26) $$\int_c^\infty t^{4i\Delta}|y^{(i)}|^2 ds < \infty \quad \text{and} \quad t^{(2i+1)\Delta}|y^{(i)}| = O(1) \quad \textit{as}\ t \to \infty;$$

https://doi.org/10.4153/CJM-1974-035-7 Published online by Cambridge University Press

while if $|p(t)| \leqq K/t^{4n\Delta}$, then (1.26) holds for $i = 0, \ldots, 2n - 1$. In this case (1.26) also holds for $i = 2n$ since $t^{(4n+1)\Delta}|y^{(2n)}| = t^{4n\Delta}|p|t^{\Delta}|y|$ and $t^{8n\Delta}|y^{(2n)}|^2 = t^{8n\Delta}|p|^2|y|^2$.

THEOREM 1.1 *Suppose conditions* (1.0) *and* (1.5)–(1.10) *hold and* Re $\lambda = 0$. *Then* dim $V_1(\lambda) =$ dim $V_2(\lambda) \leqq n$ *with equality for* $\lambda \neq 0$.

Proof. The correspondence $y \to \bar{y}$ is one-one from $V_1(\lambda)$ onto $V_2(\lambda)$; thus dim $V_1(\lambda) =$ dim $V_2(\lambda)$. Suppose to the contrary that dim $V_1(\lambda) > n$. Then the proof of Lemma 2.1 of [**2**] applies to yield a $y \in V_1(\lambda)$ and $z \in V_2(\lambda)$ such that $[y, z] = 1$. From (1.3) then follows

$$(1.27) \quad \int_c^t (1 - s/t)^{n-1}(g\rho^{4n-2}/rp_0)ds = \int_c^t \sum_{i=0}^{n-1} [y^{[i]}\bar{z}^{[2n-i-1]} - y^{[2n-i-1]}\bar{z}^{[i]}](1 - s/t)^{n-1}(g\rho^{4n-2}/rp_0)ds.$$

By part (i) of Lemma 1.1 (with $k = 0$), the right hand side of (1.27) is $O([J_1J_2]^{1/2})$, where J_1 and J_2 are as in Lemma 1.1. By Lemma 1.2, $J_1(\infty) < \infty$ and $J_2(\infty) < \infty$; thus the right hand side of (1.27) is bounded independent of t. This is a contradiction to (1.8) and the inequality is proved. The equality follows from our earlier remark that dim $V_1(\lambda) \geqq n$ if λ is not real.

COROLLARY 1.1. *Suppose S is as in* (1.1), $H = t^{\delta}$, *and* $p_0 = t^{\eta}$ $(\eta \leqq 2n + \delta)$. *If* Re $\lambda = 0$, $|p_i| = O(t^{\gamma_i})$ $(1 \leqq i \leqq n - 1)$, $-p_n(t) \leqq Kt^{\gamma_n}$ $(K > 0)$, *where*

$$\gamma_i = [4i + \eta(4n - 4i - 2) + 4i\delta]/(4n - 2) \ (i = 1, \ldots, n),$$

then dim $V_1(\lambda) \leqq n$ *with equality for* $\lambda \neq 0$.

Proof. It may be verified that conditions (1.5)–(1.10) hold with $\rho = t^{\Delta}$, $\Delta = (\eta - 1 - \delta/2n)/(4n - 2)$.

For $(-1)^n y^{(2n)} + py = \lambda Hy$ Corollary 1.1 yields the limit point condition at infinity $(H = t^{\delta})$ if $-p(t) \leqq Kt^{2n(1+\delta)/(2n-1)}$. The 2nd order equation $(t^{\eta}y')' + py = \lambda Hy$ is in the limit point condition at infinity $(H = t^{\delta})$ if $\eta \leqq 2 + \delta$ and $p(t) \leqq Kt^{2+2\delta-\eta}$. This reduces to the well-known criterion $p(t) \leqq Kt^2$ for $y'' + py$ with $H \equiv 1$. We note that Corollary 1.1 requires that p_0 can not be too large with respect to the weight function H.

Corollary 1.1 indicates that with a large weight function H, the coefficients p_i $(i = 1, \ldots, n)$ also may be large and preserve the inequality dim $V_1(\lambda) \leqq n$. This conclusion parallels the work of Walker [**4**], where the asymptotic behavior of solutions of $S(y) = \lambda y$ is given for a large weight function H.

2. Singularities at zero. We return now to equation (0.1) where $a = 0$. Let the coefficients q_i be as before and assume also $1/q_0, q_1, \ldots, q_n$ are Lebesgue integrable on (ϵ, b) for each $\epsilon > 0$. Let h be a positive function $(0, b)$. The quasi-derivatives $y^{[i]}$ are defined as in the introduction. The equation

$L(y) = \lambda hy$ has the vector formulation $\tilde{Y}' = \tilde{A}\tilde{Y}$ where $\tilde{Y} = (y^{[0]}, \ldots, y^{[2n-1]})^T$ and $\tilde{A}$ is analogous to A in section 2. We transform $\tilde{Y}$ by $Z(t) = -\tilde{Y}(1/t)$; then Z satisfies (1.2) where $r = t^2$, $p_0(t) = q_0(1/t)$, $H(t) = (1/t^2)h(1/t)$, $m = 0$, and $p_i(t) = (1/t^2)q_i(1/t)$ for $i = 1, \ldots, n$.

If z_1 denotes the first component of Z, then

$$\int_{1/b}^{\infty} H(t)|z_1(t)|^2 dt = \int_0^b h(s)|y(s)|^2 ds;$$

hence dim $V_1(\lambda)$ is the number of linearly independent solutions y of $L(y) = \lambda hy$ in $\mathscr{L}_2(h; 0, b)$.

THEOREM 2.1. *Suppose q_0 and h have n and $n-1$ continuous derivatives, respectively and there is a positive n times continuously differentiable function σ on $(0, b)$ such that the following conditions hold.*

(2.1) $$\frac{|q_i|\sigma^{4i}}{q_0 h^{i/n}} = O(1) \quad \text{as } s \to 0, \quad i = 1, \ldots, n-1.$$

(2.2) *For some* $K > 0$, $$\frac{-q_n\sigma^{4n}}{q_0 h} \leqq K.$$

(2.3) $$\frac{\sigma^2}{sh^{1/2n}} = O(1) \quad \text{and} \quad \frac{\sigma^2}{h^{1/2n}}\left[\frac{|\sigma'|}{\sigma} + \frac{|h'|}{h} + \frac{|q_0'|}{q_0}\right] = O(1) \quad \text{as } s \to 0.$$

(2.4) $$\int_0^b \frac{h^{1/2n}\sigma^{4n-2}}{q_0} ds = \infty$$

(2.5) *As* $s \to 0$,

$$\frac{d^j}{ds^j}[s^2 h^{1/2n}\sigma^{4n-2}/q_0] = O(s^2 h^{(j+1)/2n}\sigma^{4n-2-2j}/q_0), \quad j = 1, \ldots, n-1,$$

and

$$\frac{d^j}{ds^j}[\sigma^{4n}/q_0] = O(h^{j/2n}\sigma^{4n-2j}/q_0), \quad j = 1, \ldots, n.$$

Then the number of linearly independent $\mathscr{L}_2(h; 0, b)$ solutions y of $L(y) = \lambda hy$ (Re $\lambda = 0$) is $\leqq n$ with equality for $\lambda \neq 0$.

Proof. Let $\rho(t) = \sigma(1/t)$. Then calculations show that (1.5)–(1.9) follow from (2.1)–(2.5) respectively. Since $r = t^2$, condition (1.10) reduces to showing

$$t = O\left(\frac{(rH)^{1/2n}}{\rho^2}\right) \quad \text{and} \quad 1 = O\left(\frac{(rH)^{1/n}}{r\rho^4}\right) \quad \text{as } t \to \infty.$$

However, both of these order relations follow from $\sigma^2/sh^{1/2n} = O(1)$ as $s \to 0$ which follows from (2.3). Thus Theorem 1.1 applies and the proof is complete.

COROLLARY 2.1. *If* $h = s^{\delta}$, $q_0 = s^{\eta}$ $(\eta \geqq 2n + \delta)$, $\operatorname{Re} \lambda = 0$, $|q_i| = O(s^{\gamma_i})$ *as* $s \to 0$ $(1 \leqq i \leqq n - 1)$, $-q_n(t) \leqq Ks^{\gamma_n} (K > 0)$, *where*

$$\gamma_i = [4i + \eta(4n - 4i - 2) + 4i\delta]/(4n - 2) \qquad (i = 1, \ldots, n),$$

then the equation $L(y) = \lambda hy$ *has at most n linearly independent solutions in* $\mathscr{L}_2(h; 0, b)$.

Proof. If σ is chosen by $\sigma = s^{\Delta}$, $\Delta = (\eta - 1 - \delta/2n)/(4n - 2)$, then conditions (2.1)–(2.5) hold.

Application of Corollary 2.1 to $(s^{\eta}y')' + qy$ yields the limit point condition at 0 $(H = s^{\delta})$ if $\eta \geqq 2 + \delta$ and $q \leqq Ks^{2+2\delta-\eta}$. For $H \equiv 1$, this requires $\eta \geqq 2$ and thus no criterion for $y'' + qy$ is obtained. Similar restrictions are imposed on higher order equations.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear operators*, Part II (Interscience, New York, 1963).
2. D. Hinton, *Limit point criteria for differential equations*, Can. J. Math. *24* (1972), 293–305.
3. M. A. Naimark, *Linear differential operators*, Part II (Ungar, New York, 1968).
4. P. W. Walker, *Asymptotics for a class of weighted eigenvalue problems*, Pacific J. Math. *40* (1972), 501–510.

The University of Tennessee,
Knoxville, Tennessee