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The Secondary Chern–Euler Class for a General Submanifold

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Abstract. We define and study the secondary Chern–Euler class for a general submanifold of a Riemannian manifold. Using this class, we define and study the index for a vector field with non-isolated singularities on a submanifold. As an application, we give conceptual proofs of a result of Chern.

The objective of this paper is to define, study, and use the secondary Chern–Euler class for a general submanifold of a Riemannian manifold. We give the definition in Definition 1.1 in Section 1. In Section 2, we study cohomologically the class and its relation with several other natural homology and cohomology classes. The cases when the codimension of the submanifold is one or greater than one are different, and we consider both cases. In Section 3, we use the secondary Chern–Euler class to define the index for a vector field with non-isolated singularities on a submanifold in Definition 3.1. To this end, we develop the notion of blow-up of the submanifold along the vector field. We then obtain three formulas in Theorem 3.2 to compute the index. Our studies, in particular, give three conceptual proofs of a classical result of Chern [3, (20)] concerning the paring of the secondary Chern–Euler class with the normal sphere bundle of the submanifold. Two of the proofs are given in Section 2 and the third in Section 3.

1 Secondary Chern–Euler Class for a General Submanifold

Let X be a connected oriented compact Riemannian manifold of dimension n. (Throughout the paper, $n = \dim X$.) The Gauss–Bonnet theorem (see [2, (9)]) asserts that

$$\int_X \Omega = \chi(X), \tag{1.1}$$

where $\chi(X)$ is the Euler characteristic of X, and Ω is the Euler curvature form defined as follows. Choose local positively oriented orthonormal frames $\{e_1, \dots, e_n\}$ of the tangent bundle TX. Let (ω_{ij}) and (Ω_{ij}) be the $\mathfrak{so}(n)$-valued connection forms and curvature forms for the Levi–Civita connection ∇ of the Riemannian metric on X

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defined by

$$\nabla e_i = \sum_{j=1}^{n} \omega_{ij} e_j, \tag{1.2}$$

$$\Omega_{ij} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik}\omega_{kj}. \tag{1.3}$$

Then [3, (10)] defines the degree n form

$$\Omega = \begin{cases} 0 & n \text{ odd}, \\ (-1)^m \dfrac{1}{2^{2m}\pi^m m!} \displaystyle\sum_i \epsilon(i)\Omega_{i_1 i_2} \cdots \Omega_{i_{n-1} i_n} & n = 2m \text{ even}, \end{cases} \tag{1.4}$$

where the summation runs over all permutations i of $\{1, 2, \ldots, n\}$, and $\epsilon(i)$ is the sign of i. The form Ω does not depend on the choice of local frames. (In this paper, we closely follow Chern's notation and convention in [2, 3]. In particular we follow his convention in choosing the row and column indices in (1.2), which may not be the most standard. Also, products of differential forms always mean "exterior products", although we omit the notation $\wedge$ for simplicity.)

Chern [3, (9)] defines a form Φ (called Π in Chern's papers) of degree $(n-1)$ on the unit sphere bundle STX, consisting of unit vectors in the tangent bundle TX, as follows. To a unit tangent vector $v \in STX$, we attach a local positively oriented orthonormal frame $\{e_1, \ldots, e_{n-1}, e_n\}$ such that $v = e_n$. For $k = 0, 1, \ldots, [\frac{n-1}{2}]$ define

$$\Phi_k = \sum_\alpha \epsilon(\alpha)\Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}}\omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{n-1}n}, \tag{1.5}$$

where the summation runs over all permutations α of $\{1, 2, \ldots, n-1\}$. Also, define

$$\Phi = \frac{1}{(n-2)!!|S^{n-1}|} \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \frac{1}{2^k k!(n-2k-1)!!} \Phi_k, \tag{1.6}$$

where

$$|S^{n-1}| = \begin{cases} \dfrac{(2\pi)^m}{(n-2)!!} & n = 2m \text{ even}, \\ \dfrac{2(2\pi)^m}{(n-2)!!} & n = 2m+1 \text{ odd} \end{cases} \tag{1.7}$$

is the surface area of the unit $(n-1)$ sphere. The Φ_k and hence Φ do not depend on the choice of $e_1, \ldots, e_{n-1}$. Note in particular that the zeroth term

$$\widetilde{\Phi_0} = \frac{1}{(n-2)!!|S^{n-1}|} \frac{1}{(n-1)!!} \sum_\alpha \epsilon(\alpha)\omega_{\alpha_1 n} \cdots \omega_{\alpha_{n-1}n} = \widetilde{d\,\mathrm{vol}_{n-1}} \tag{1.8}$$

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is the unit volume form when restricted to a fiber sphere ST_xX for $x \in X$.

Then [3, (11)] proves that

$$d\Phi = -\Omega. \tag{1.9}$$

The form Φ, Stokes' theorem, (1.8), and the Poincaré–Hopf theorem were then used in [3, (18)] and [2, (25)] to give an intrinsic proof for the Gauss–Bonnet theorem (1.1).

Let $M \subset X$ be a connected oriented submanifold of dimension m. Then Φ in (1.6) is a closed form when restricted to $STX|_M$ in view of (1.9) and (1.4), since even if n is even, $\Omega|_M = 0$ for dimensional reasons.

Definition 1.1 We call the restriction of Φ to $STX|_M$ the *secondary Chern–Euler form of M in X*, and its cohomology class $[\Phi] \in H^{n-1}(STX|_M, \mathbb{R})$ *the secondary Chern–Euler class.*

2 Cohomological Studies

In this section, we first prove the following theorem.

Theorem 2.1 *We consider three cases concerning the secondary Chern–Euler class* $[\Phi] \in H^{n-1}(STX|_M, \mathbb{R})$.

(i) *When* $\operatorname{codim} M \geq 2$, $[\Phi] \in H^{n-1}(STX|_M, \mathbb{Z})$ *is integral and independent of the choice of the connection and hence of the Riemannian metric on X.*

(ii) *When* $\operatorname{codim} M = 1$ *and* $\dim X$ *is odd,* $[\Phi] \in H^{n-1}(STX|_M, \mathbb{Z}) \otimes \frac{1}{2}$ *is half-integral and independent of the choice of the connection.*

(iii) *When* $\operatorname{codim} M = 1$ *and* $\dim X$ *is even,* $[\Phi] \in H^{n-1}(STX|_M, \mathbb{R})$ *is only real and depends on the connection.*

Proof We prove the theorem by computing the integrals of Φ over generators of $H_{n-1}(STX|_M, \mathbb{Z})$. Consider the following cofibration sequence

$$STX|_M \to DTX|_M \to Th(TX)|_M, \tag{2.1}$$

where $DTX|_M$ is the unit disk bundle over M, which is homotopic to M, and $Th(TX)|_M$ is the Thom space of TX restricted to M. This gives rise to an exact sequence

$$0 \to H_n(Th(TX)|_M, \mathbb{Z}) \to H_{n-1}(STX|_M, \mathbb{Z}) \to H_{n-1}(M, \mathbb{Z}) \to 0,$$

when $n \geq 2$, which is obviously the only interesting case. Therefore one has

$$H_{n-1}(STX|_M, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \operatorname{codim} M \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \operatorname{codim} M = 1. \end{cases} \tag{2.2}$$

One generator of $H_{n-1}(STX|_M, \mathbb{Z})$ is a fiber sphere ST_xX, for $x \in M$. (For simplicity of notation, we do not distinguish cycles from the homology classes, although

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we do distinguish closed forms from the cohomology classes.) The pairing of the secondary Chern–Euler class $[\Phi] \in H^{n-1}(STX|_M)$ with ST_xX is

$$\int_{ST_xX} \Phi = 1, \tag{2.3}$$

by (1.8) since all curvature forms vanish when restricted to x.

Equations (2.2) and (2.3) imply case (i).

When $\operatorname{codim} M = 1$, let $\vec{n}$ denote the unit normal vector field of M such that for a positively oriented frame $\{e_1, \dots, e_{n-1}\}$ of TM, $\{e_1, \dots, e_{n-1}, \vec{n}\}$ is a positively oriented frame of TX. Then the image $\vec{n}(M)$, which we will call M^+, defines the other generator of $H_{n-1}(STX|_M, \mathbb{Z})$ in (2.2).

Equations (2.2), (2.3) and Lemma 2.2 below imply case (ii).

A simple example like a general circle S^1 on a sphere S^2 shows case (iii). In this example, the relative Gauss–Bonnet theorem [3, (19)] asserts

$$\int_{(S^1)^+} \Phi = \chi(D) - \int_D \Omega,$$

where D is the region of S^2 bounded by S^1 such that $\vec{n}$ points outward to D. Therefore $\chi(D) = 1$, but $\int_D \Omega$ takes real values and depends on the metric. ■

Lemma 2.2 *When* $\operatorname{codim} M = 1$ *and* $\dim X$ *is odd,*

$$\int_{M^+} \Phi = \frac{1}{2}\chi(M). \tag{2.4}$$

Proof This can be seen in two ways. A direct way using ideas from Chern [3] is by showing

$$\Phi|_{M^+} = \frac{1}{2}\tilde{\Omega}, \tag{2.5}$$

where $\tilde{\Omega}$ is the pullback of the Euler curvature form of M for the induced metric to $M^+ \subset STX|_M$. Then the Gauss–Bonnet theorem (1.1) gives (2.4). (2.5) is proved as follows.

Choose local frames $\{e_1, \dots, e_{n-1}, e_n\}$ for $TX|_M$ such that $\{e_1, \dots, e_{n-1}\}$ are local frames for TM and $e_n = \vec{n}$. Assume that $\dim X = n = 2m + 1$. Then by (1.6), (1.5), and (1.7) one has

$$\begin{aligned} \Phi|_{M^+} &= \frac{1}{2^{m+1}\pi^m} \sum_{k=0}^{m} \frac{(-1)^k}{2^k k!(2m-2k)!!} \sum_{\alpha} \epsilon(\alpha) \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}} \omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{2m}n} \\ &= \frac{1}{2^{2m+1}\pi^m} \sum_{k=0}^{m} \frac{(-1)^k}{k!(m-k)!} \sum_{\alpha} \epsilon(\alpha) \Omega_{\alpha_1\alpha_2} \cdots \Omega_{\alpha_{2k-1}\alpha_{2k}} \omega_{\alpha_{2k+1}n} \cdots \omega_{\alpha_{2m}n}, \end{aligned}$$

where one uses $(2m-2k)!! = 2^{m-k}(m-k)!$.

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Using (1.3), one has

$$\tilde{\Omega}_{\alpha\beta} = \Omega_{\alpha\beta} - \omega_{\alpha n}\omega_{\beta n}$$

for $1 \le \alpha, \beta \le n - 1 = 2m$, where the $\Omega_{\alpha\beta}$ and $\tilde{\Omega}_{\alpha\beta}$ are the curvature forms of X and M. Therefore by (1.4) and multinomial theorem, one has

$$\begin{aligned}\tilde{\Omega} &= \frac{(-1)^m}{2^{2m}\pi^m m!}\sum_{\alpha}\epsilon(\alpha)(\Omega_{\alpha_1\alpha_2} - \omega_{\alpha_1 n}\omega_{\alpha_2 n})\cdots(\Omega_{\alpha_{2m-1}\alpha_{2m}} - \omega_{\alpha_{2m-1}n}\omega_{\alpha_{2m}n})\\ &= \frac{(-1)^m}{2^{2m}\pi^m m!}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}(-1)^{m-k}\sum_{\alpha}\epsilon(\alpha)\Omega_{\alpha_1\alpha_2}\cdots\Omega_{\alpha_{2k-1}\alpha_{2k}}\omega_{\alpha_{2k+1}n}\cdots\omega_{\alpha_{2m}n}\\ &= \frac{1}{2^{2m}\pi^m}\sum_{k=0}^{m}\frac{(-1)^k}{k!(m-k)!}\sum_{\alpha}\epsilon(\alpha)\Omega_{\alpha_1\alpha_2}\cdots\Omega_{\alpha_{2k-1}\alpha_{2k}}\omega_{\alpha_{2k+1}n}\cdots\omega_{\alpha_{2m}n}.\end{aligned}$$

Direct comparison gives (2.5).

Another way follows from Remark 2.7, which says that in this case (2.4) is equivalent to (2.12), and our Proofs 1 and 3 of Theorem 2.6, which work in all codimensions. ■

Remark 2.3 Case (i) is actually a manifestation of a general phenomenon for Chern–Simons forms as stated in [4, Theorem 3.9, Corallary 3.17].

Remark 2.4 In general, when $\operatorname{codim} M = 1$ and $\dim X$ is even, $\int_{M^+}\Phi$ stands for the geodesic curvature of M. It vanishes for a totally geodesic submanifold M, since then all the $\omega_{\alpha n} = 0$ by (1.2) and the total geodesicity of M (recall that we choose $e_n = \vec{n}$), but each summand of all the Φ_k in (1.5) contains at least one $\omega_{\alpha n}$, since it is of an odd degree $n - 1$.

Remark 2.5 The cohomology $H^{n-1}(STX|_M, \mathbb{Z})$ and the class $[\Phi] \in H^{n-1}(STX|_M)$ have already been studied in [8] for $M = \partial X$, under the condition that the metric on X is *locally product* near M. This in particular means that M is a totally geodesic submanifold of X. Therefore [8, pp 1156 Special Cases] are special cases of our cases (ii) and (iii), in view of Remark 2.4.

We now study the relation of $[\Phi]$ with some other natural homology and cohomology classes. From (2.1), one has the following dual homomorphisms:

$$\delta\colon H^{n-1}(STX|_M, \mathbb{R}) \to H^n(Th(TX)|_M, \mathbb{R}), \tag{2.6}$$

$$\delta'\colon H_n(Th(TX)|_M, \mathbb{R}) \to H_{n-1}(STX|_M, \mathbb{R}). \tag{2.7}$$

Then

$$\delta[\Phi] = \gamma_{TX}, \tag{2.8}$$

where $\gamma_{TX} \in H^n(Th(TX)|_M, \mathbb{Z}) \cong \mathbb{Z}$ is the Thom class of $TX|_M$. To see this, note that in (2.7) by definition

$$\delta'(DT_xX/ST_xX) = ST_xX, \tag{2.9}$$

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where $DT_xX/ST_xX \in H_n(Th(TX)|_M, \mathbb{Z}) \cong \mathbb{Z}$ is a generator dual to the Thom class. Then (2.8) follows from

$$\int_{DT_xX/ST_xX} \delta[\Phi] = \int_{\delta'(DT_xX/ST_xX)} [\Phi] = \int_{ST_xX} [\Phi] = 1$$

by (2.3), where we denote the pairing of cohomology and homology by integration.

Let V be a generic vector field on X, and consider its restriction $V|_M$ on M. Generically, $V|_M$ has no singularities. Let $\alpha_V : M \to STX|_M$ be defined by rescaling $V|_M$, *i.e.*, $\alpha_V(x) = \frac{V(x)}{|V(x)|}$, $\forall x \in M$. Then $\alpha_V(M)$ is a dimension m cycle in $STX|_M$, and hence defines a homology class in $H_m(STX|_M, \mathbb{Z})$. For $x \in M$, the intersection $\alpha_V(M) \cdot ST_xX = \alpha_V(x)$ is one point. Therefore when $\operatorname{codim} M \geq 2$,

$$\alpha_V(M) = \text{P.D.}([\Phi]) \tag{2.10}$$

is the Poincaré dual of $[\Phi]$, in view of (2.3) and (2.2).

One has the decomposition

$$TX|_M = TM \oplus NM, \tag{2.11}$$

where NM is the normal bundle of M in X. The normal sphere bundle SNM, consisting of unit normal vectors, defines another homology class in $H_{n-1}(STX|_M, \mathbb{Z})$.

As application of our cohomological studies, we get two conceptual proofs of the following result about the pairing between $[\Phi]$ and SNM, which was first proved in [3] in a computational way.

Theorem 2.6 ([3, (20)]) *We have*

$$\int_{SNM} \Phi = \chi(M). \tag{2.12}$$

Proof 1 of Theorem 2.6. We use the notation as before. Note that $Th(NM)$ defines a homology class in $H_n(Th(TX)|_M, \mathbb{Z})$. Similar to (2.9), we have $\delta'(Th(NM)) = SNM$ in (2.7). Therefore,

$$\int_{SNM} \Phi = \int_{\delta'(Th(NM))} \Phi = \int_{Th(NM)} \delta[\Phi] = \int_{Th(NM)} \gamma_{TX} = \chi(M),$$

by (2.8). Here the last equality follows from some basic knowledge about Thom classes. In more detail, we have the following commutative diagrams in view of (2.11)

$$\begin{array}{ccc}
H^0(M) & & \\
\cong \big\downarrow \cup\gamma_{TM} & & \\
H^m(Th(TM)) & \xrightarrow{i^*} & H^m(M) \\
\cong \big\downarrow \cup\gamma_{NM} & & \cong \big\downarrow \cup\gamma_{NM} \\
H^n(Th(TX)|_M) & \xrightarrow{i^*} & H^n(Th(NM));
\end{array}
\qquad
\begin{array}{ccc}
1 & & \\
\big\downarrow \cup\gamma_{TM} & & \\
\gamma_{TM} & \xmapsto{i^*} & e_{TM} \\
\big\downarrow \cup\gamma_{NM} & & \big\downarrow \cup\gamma_{NM} \\
\gamma_{TX} & \xmapsto{i^*} & i^*\gamma_{TX},
\end{array}$$

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where the i^* are induced by $i\colon M \to Th(TM)$ and $i\colon Th(NM) \to Th(TX)|_M$ defined by the zero section of TM, and $e_{TM} \in H^m(M, \mathbb{Z})$ is the Euler class of M. Therefore,

$$\int_{Th(NM)} \gamma_{TX} = \int_{Th(NM)} i^*\gamma_{TX} = \int_M e_{TM} = \chi(M). \qquad \blacksquare$$

Remark 2.7 When $\operatorname{codim} M = 1$, $SNM = M^+ - M^-$ as a homology class in $H_{n-1}(STX|_M, \mathbb{Z})$, where $M^- = (-\vec{n})(M)$. One has $\int_{M^-} \Phi = (-1)^n \int_{M^+} \Phi$ by an analysis of (1.5) and how one chooses local frames. Therefore when $n = \dim X$ is even, both sides of (2.12) are zero, and when n is odd, (2.12) is just a doubling of (2.4). This was also asserted in [3, p. 682]. Therefore when $\operatorname{codim} M = 1$ and $\dim X$ is even, the two statements (2.4) and (2.12) are equivalent. Hence Lemma 2.2 implies this case of Theorem 2.6 and vice versa. So our Proofs 1 and 3 of Theorem 2.6, which work in all codimensions, also imply Lemma 2.2. However, the following Proof 2 only works when $\operatorname{codim} M \geq 2$.

Proof 2 of Theorem 2.6 when $\operatorname{codim} M \geq 2$**.** We continue to work with the generic vector field V introduced above. Consider the projection ∂V of $V|_M$ to TM. Generically, ∂V has only isolated singularities with indices ± 1, and the sum of its indices is $\chi(M)$ by the Poincaré–Hopf theorem. Suppose p is a singular point of ∂V. Then V is perpendicular to M at p, and hence $\alpha_V(p) \in SNM$. Since $\operatorname{Ind}_p \partial V = \pm 1$, it can be seen that one has transversal intersection $\alpha_V(M) \pitchfork_{\alpha_V(p)} SNM$. Furthermore, the intersection index $\iota_{\alpha_V(p)}(\alpha_V(M), SNM) = (-1)^m \operatorname{Ind}_p \partial V$, where $m = \dim M$, for suitable orientations. Therefore the intersection number $\#(\alpha_V(M), SNM) = (-1)^m \chi(M)$, which implies by (2.10)

$$\int_{SNM} \Phi = \#(\alpha_V(M), SNM) = (-1)^m \chi(M) = \chi(M),$$

where the last equality holds by the obvious reason that $\chi(M) = 0$ when m is odd. $\blacksquare$

3 Index of Vector Field with Non-Isolated Singularities

In this section, we first use the secondary Chern–Euler class to define the index for a vector field with non-isolated singularities. This also involves the notion of blow-up of a submanifold along a vector field V which vanishes on it.

Let V be a vector field on X with non-isolated singularities on a submanifold M. Let U be a closed neighborhood of M in X, and suppose its boundary ∂U is smooth. Assume that V has no singularities on $U - M$. Consider $\alpha_V\colon U - M \to STX$ by rescaling V.

Definition 3.1 The closure of the image $\overline{\alpha_V(U - M)}$ defines a homology class in $H_n(STX|_U, STX|_{\partial U} \cup STX|_M, \mathbb{Z})$. Under the connecting homomorphism for relative homology

$$\partial\colon H_n(STX|_U, STX|_{\partial U} \cup STX|_M, \mathbb{Z}) \longrightarrow H_{n-1}(STX|_{\partial U} \cup STX|_M, \mathbb{Z}),$$

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one has

$$\partial(\overline{\alpha_V(U-M)}) = \alpha_V(\partial U) - \mathrm{Bl}_V(M), \tag{3.1}$$

where $\mathrm{Bl}_V(M) \in H_{n-1}(STX|_M, \mathbb{Z})$ is defined by the above. We call it the *blow-up of M along V* as a homology class. The *index of V at M* is defined by

$$\mathrm{Ind}_M V = \int_{\mathrm{Bl}_V(M)} \Phi, \tag{3.2}$$

where $[\Phi] \in H^{n-1}(STX|_M)$ is the secondary Chern–Euler class .

The following theorem gives three ways of evaluating the index $\mathrm{Ind}_M V$. In particular, it shows that $\mathrm{Ind}_M V$ is *always* an integer independent of the metric.

Theorem 3.2 (i) *In terms of the Euler curvature form and the secondary Chern–Euler class, one has*

$$\mathrm{Ind}_M V = \int_{\alpha_V(\partial U)} \Phi + \int_U \Omega. \tag{3.3}$$

(ii) *Extend $V|_{\partial U}$ to a vector field $\tilde{V}$ on U with isolated singularities. Then*

$$\mathrm{Ind}_M V = \mathrm{Ind}\, \tilde{V}, \tag{3.4}$$

where $\mathrm{Ind}\, \tilde{V}$ *denotes the sum of local indices of $\tilde{V}$ at its isolated singularities.*

(iii) *Let ∂V denote the tangential projection of V to the tangent space of the boundary ∂U, and $\partial_- V$ the restriction of ∂V to the parts where V points inward to U. Then generically one has*

$$\mathrm{Ind}_M V = \chi(U) - \mathrm{Ind}\, \partial_- V. \tag{3.5}$$

Proof Following [2, (25)], and using (1.9), Stokes' theorem, and (3.1), one has

$$\int_U -\Omega = \int_{\overline{\alpha_V(U-M)}} -\Omega = \int_{\overline{\alpha_V(U-M)}} d\Phi = \int_{\alpha_V(\partial U)} \Phi - \int_{\mathrm{Bl}_V(M)} \Phi, \tag{3.6}$$

which then gives (3.3) in view of (3.2). Note that from (3.3), one immediately gets

$$\mathrm{Ind}_M V = \lim_{U \to M} \int_{\alpha_V(\partial U)} \Phi.$$

Applying the standard procedure, as in [2, (25)] and (3.6), and using (1.8), one gets

$$\mathrm{Ind}\, \tilde{V} = \int_{\alpha_V(\partial U)} \Phi + \int_U \Omega. \tag{3.7}$$

The index $\operatorname{Ind}\tilde{V}$ clearly does not depend on the extension $\tilde{V}$ from this formula, once one fixes the metric. Therefore comparison of (3.3) with (3.7) gives (3.4). This also implies that $\operatorname{Ind}_M V$ in Definition 3.1 is an integer independent of the metric, since it is equal to $\operatorname{Ind}\tilde{V}$.

From [7], one knows that the right-hand side of (3.3)

$$\int_{\alpha_V(\partial U)} \Phi + \int_U \Omega = \chi(U) - \operatorname{Ind}(\partial_- V),$$

which proves (3.5). Note that in the topology literature, (3.5) is called the law of vector fields, and was first proved in [6]. ■

As an application, we give a third proof of Theorem 2.6.

Proof 3 of Theorem 2.6 We will apply (3.4) to the radial vector field around M.

For $r > 0$ small, let $U = B_r(M) = \{x \in X \mid d(x, M) \leq r\}$ be a tubular neighborhood of M in X. Then its boundary is $\partial U = S_r(M) = \{x \in X \mid d(x, M) = r\}$. For $x \in B_r(M)$, let $p(x)$ be the point on M such that $d(x, p(x)) = d(x, M)$. Choose r sufficiently small so that $p(x)$ is unique and there is a unique shortest geodesic connecting $p(x)$ and x. Denote $s(x) = d(x, M)$ and treat s as a coordinate on $B_r(M)$. Let $\vec{n} := \frac{\partial}{\partial s}$. Note that $\vec{n}(x) = \frac{\partial}{\partial s}(x)$ is the unit tangent vector at x of the unique shortest geodesic starting from $p(x)$ and passing x. Now consider the vector field $V = \frac{s}{r}\vec{n} = \frac{s}{r}\frac{\partial}{\partial s}$. Then V has singularities on M corresponding to $s = 0$, and on $S_r(M)$ corresponding to $s = r$, $V = \vec{n}$ is the outward normal vector field. After rescaling, $\alpha_V = \vec{n} \colon B_r(M) - M \to STX$.

Consider the closure of the image $\overline{\vec{n}(B_r(M) - M)}$ in STX. Its boundary is

$$\partial(\overline{\vec{n}(B_r(M) - M)}) = \vec{n}(S_r(M)) - SNM.$$

Therefore $\operatorname{Bl}_V(M) = SNM$, and

$$\int_{SNM} \Phi = \operatorname{Ind}_M V = \operatorname{Ind}\tilde{V},$$

by Definition 3.1 and (3.4). Here $\tilde{V}$ is a generic extension with isolated singularities to $B_r(M)$ of the vector field $\vec{n}$ on $S_r(M)$. By definition, $\operatorname{Ind}\tilde{V} = \chi(B_r(M))$. Therefore one is done by the homotopy invariance of Euler characteristic. ■

Remark 3.3 Chern's computation [3] proves that $\int_{SNM} \Phi = \int_M \Omega_M$. This and Theorem 2.6, with our conceptual proofs, would prove the Gauss–Bonnet theorem (1.1) for M, $\int_M \Omega_M = \chi(M)$, if one did not know it. Such a route was taken historically by [1, 5] to prove the Gauss-Bonnet theorem for a submanifold of higher codimension in a Euclidean space from the known result of a hypersurface.

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