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ON NON-ANTICIPATIVE LINEAR TRANSFORMATIONS OF GAUSSIAN PROCESSES WITH EQUIVALENT DISTRIBUTIONS

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Let $\xi(t)$, $t \in T$, be a Gaussian process on a set T, and $H = H(\xi)$ be the closed linear manifold generated by all values $\xi(t)$, $t \in T$, with the inner product

$$\langle \eta_1, \eta_2 \rangle = E\eta_1\eta_2 ; \qquad \eta_1, \eta_2 \in H .$$

We suppose that the Hilbert space H is separable.

Let $\mathscr{A}$ be a linear operator on H; we call a random process of the form

$$\eta(t) = \mathscr{A}\xi(t) , \qquad t \in T , \tag{1}$$

a *linear transformation* of the process $\xi(t)$, $t \in T$. One says that a linear transformation $\mathscr{A}$ is *non-anticipative*, if

$$\mathscr{A}H_t(\xi) \subseteq H_t(\xi) , \qquad t \in T , \tag{2}$$

where $H_t(\xi)$ denotes the subspace in H, which is generated by all values $\xi(s)$, $s \leq t$.

Let P be a probability distribution of the Gaussian process $\xi = \xi(t)$, $t \in T$, in some measurable space $(X, \mathfrak{B}, P)$ of ⟪trajectories⟫ $x = x(t)$, $t \in T$, where σ-algebra $\mathfrak{B}$ is generated by all sets $\{x(t) \in B\}$ $(t \in T)$, B are Borel sets on the real line, so P is determined by finite-dimensional distributions of the random process $\xi = \xi(t)$, $t \in T$. Let Q be a probability distribution of the Gaussian process $\eta = \eta(t)$, $t \in T$, represented by the formula (1). According to well known *Feldman's theorem* (see, for example, [1]), *Q is equivalent to P $(Q \sim P)$ if and only if the operator*

$$\boldsymbol{B} = \mathscr{A}^*\mathscr{A} \tag{3}$$

is invertible and $I - \boldsymbol{B} \in S_2$, where S_2 denotes the class of all Hilbert-Schmidt operators in H.

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The operator $\boldsymbol{B}$ connects with the correlation function $\boldsymbol{B}(s, t)$ of the Gaussian distribution Q as

$$\boldsymbol{B}(s, t) = \langle \boldsymbol{B}\xi(s), \xi(t) \rangle , \qquad s, t \in T ; \tag{4}$$

let us call $\boldsymbol{B}$ *the correlation operator of* Q. Obviously, for any equivalent distribution Q (i.e. Q has strictly positive correlation operator $\boldsymbol{B}$, such that $I \quad \boldsymbol{B} \in S_2$.) there is a linear transformation (1), which gives us a Gaussian process $\eta(t), t \in T$, with the distribution Q: *the general operator* $\mathscr{A}$, *which satisfies the condition* (3), *has the form*

$$\mathscr{A} = V\boldsymbol{B}^{1/2} \tag{5}$$

where V *is an arbitrary unitary operator in* H.

Let us consider a linear transformation (1) with $\mathscr{A} = I - \Delta$:

$$\eta(t) = \xi(t) - \Delta\xi(t) , \qquad t \in T . \tag{6}$$

It is more convenient to reformulate Feldman's theorem in the following way: $Q \sim P$ *if and only if* $I - \boldsymbol{B}^{1/2} \in S_2$ *and* 1 *does not belong to the spectrum of* $I - \boldsymbol{B}^{1/2}$. Indeed, $I - \boldsymbol{B} \in S_2$ if and only if

$$(I - \boldsymbol{B}^{1/2}) = (I - \boldsymbol{B})(I + \boldsymbol{B}^{1/2})^{-1} \in S_2 .$$

It is easy to see that for any operator $\Delta \in S_2$, which has no eigenvalue equal to 1, the random process $\eta(t)$, $t \in T$, of the form (6) has the equivalent distribution Q with the correlation operator, because

$$I - \boldsymbol{B} = \Delta + \Delta^*(I - \Delta) \in S_2 .$$

But the condition $\Delta \in S_2$ is *not nesessary* for the equivalence $Q \sim P$. Namely, by the formula (5) we have

$$\Delta = I - V\boldsymbol{B}^{1/2} , \tag{7}$$

where V is some unitary operator and (for the equivalent distribution Q) $I - \boldsymbol{B}^{1/2} \in S_2$; obviously $\Delta \in S_2$ if and only if $[\Delta - (I - \boldsymbol{B}^{1/2})]\boldsymbol{B}^{-1/2} = I - V \in S_2$.

Then we shall be interested in the linear transformation (6) with operators $\Delta \in S_2$. As we have obtained, it holds true if and only if

$$I - V \in S_2 \tag{8}$$

where V is an unitary operator connected with the operator Δ by the formula (7): $\Delta = I - V\boldsymbol{B}^{1/2}$. According to Feldman's theorem any transformation (6) such that $\Delta \in S_2$ and 1 does not belong to the spectrum Δ gives a random process $\eta(t)$, $t \in T$, with an equivalent distribution Q.

We shall be interested also in a such property of the linear transformation (6) as *to be non-anticipative* that means

$$\Delta H_t(\xi) \subseteq H_t(\xi) , \qquad t \in T . \tag{9}$$

In the resent time it was paid attention for non-anticipative transformations in connection with Hitsuda's result [2] for the Wiener process $\xi(t)$, $0 \leq t \leq 1$: any Gaussian process $\eta(t)$, $0 \leq t \leq 1$, with an equivalent probability distribution can be represent in the form

$$\eta(t) = \xi(t) - \int_0^t \left[\int_0^s \Delta(u, s) d\xi(u) \right] ds \tag{10}$$

where $\Delta(t, s)$; $0 \leq t$, $s \leq 1$,

$$\Delta(t, s) = 0 , \qquad s < t , \tag{11}$$

$$\int_0^1 \int_0^1 \Delta(t, s)^2 dt ds < \infty . \tag{12}$$

Though in the paper [2] it was used some theorems on the martingales, it was clear that the representation (10) can be obtained as a result of the theory of operators in a Hilbert space: the formula (10) is given by a non-anticipative transformation (6) with $\Delta \in S_2$ in the case of Wiener process $\xi(t)$, $0 \leq t \leq 1$. The existense of such transformation in the general case follows from non-trivial Gohberg-Krein's theorems on so-called *special factorization*; namely, any positive operator $\boldsymbol{B}$ of the type

$$\boldsymbol{B} = (I - F) = (I - G)^{-1}$$
$$(F \text{ and } G = -F\boldsymbol{B}^{-1} \text{ belong } S_2)$$

can be represented in the form

$$B = (I + X)\mathscr{D}(I + X^*) \tag{13}$$

where $(I + X)$ is invertible, $X \in S_2$ and $\mathscr{D} \geq 0$; besides *the operators* X *and* $\mathscr{D}$ *satisfy the condition*

$$XH_t \subseteq H_t , \quad \mathscr{D}H_t \subseteq H_t \qquad (t \in T)$$

for a given monotone family of subspaces H_t, $t \in T$ ($H_s \subseteq H_t$ if $s \leq t$) (see the theorems 6.2 Ch. IV and 10.1 Ch. I in the book [3]). It is clear that for $H_t = H_t(\xi)$, $t \in T$, the operator

$$\mathscr{A} = (I + X)\mathscr{D}^{1/2} \tag{14}$$

satisfies the conditions (2) and (3), so the corresponding linear transformation (6) with $\varDelta = I - \mathscr{A}$ will be *non-anticipative*. This proof of the existense of non-anticipative representations (6) for Gaussian processes $\eta(t)$, $t \in T$, with equivalent distributions was suggested resently by Kallianpur and Oodaira [4] (in the case of Wiener process $\xi(t)$, $0 \le t \le 1$, it was done ealier by Kailath [5]). We should like to do the following essential note: for the operator $\mathscr{A}$, which was mentioned above (see (14)) it holds true that

$$\varDelta = I - \mathscr{A} \in S_2 , \tag{15}$$

so *for any Gaussian process* $\xi(t)$, $t \in T$, *there is a non-anticipative Gaussian process* $\eta(t) = \xi(t) - \varDelta\xi(t)$, $t \in T$ (*where* $\varDelta \in S_2$ *satisfies the condition* (9)) *with a given equivalent probability distribution.*

Indeed, in the representation (13) we have $(I + X)^{-1} = I + \mathscr{T}$, $\mathscr{T} = -X(I + X)^{-1} \in S_2$, and the operator $\mathscr{D}$ has a form

$$\mathscr{D} = (I + \mathscr{T})(I - F)(I + \mathscr{T}^*) = I + V$$

where

$$V = \mathscr{T}(I - F)(I + \mathscr{T}^*) - F(I + \mathscr{T}^*) + \mathscr{T}^* \in S_2 .$$

From relations

$$\mathscr{D}^{1/2} = (I + V)^{1/2} = I + W ,$$
$$I + V = (I + W)^2 = I + W(2I + W) = I + W(I + \mathscr{D}^{1/2}) ,$$

we obtain that

$$W = V(I + \mathscr{D}^{1/2})^{-1} \in S_2 ,$$

so

$$\begin{aligned}\varDelta &= I - \mathscr{A} = I - (I + X)\mathscr{D}^{1/2} \\ &= I - (I + X)(I + W) = -X(I + W) - W \in S_2 .\end{aligned}$$

It is worth to pay attention for the following fact: *the linear transformation* (6) *with the operator* $\mathscr{A} = I - \varDelta$ *of the form* (14) *is such that*

$$H_t(\xi) = H_t(\eta) , \qquad t \in T . \tag{16}$$

Indeed, for the invertible positive operator $\mathscr{D}^{1/2}: \mathscr{D}^{1/2}H_t(\xi) \subseteq H_t(\xi)$, we have

$$\mathscr{D}^{1/2}H_t(\xi) = H_t(\xi)$$

because in a contrary case there is an element $h \in H_t(\xi)$, such that $h \perp \mathscr{D}^{1/2}H_t(\xi)$ and $\mathscr{D}^{1/2}h = 0$. Remind that a Volterra operator X has only one point of a spectra equal to 0, so for the operator $(I + X)$ in the formula (14), $(I + X)H_t(\xi) \subseteq H_t(\xi)$, we have

$$(I + X)H_t(\xi) = H_t(\xi) .$$

Now it is obvious that the operator $\mathscr{A} = (I + X)\mathscr{D}^{1/2}$ satisfies the condition (16).

Let us consider a few examples of non-anticipative representations (6) with $\Delta \in S_2$.

EXAMPLE 1. Let $\xi(t)$, $0 \leq t \leq 1$, be a Gaussian process with stationary increments:

$$\xi(t) = \int_{-\infty}^{\infty} \frac{e^{i\lambda t} - 1}{i\lambda} \Phi(d\lambda) ,$$

which has a spectral density $f(\lambda)$ of the type:

$$0 < \varliminf_{\lambda \to \infty} f(\lambda) \leq \varlimsup_{\lambda \to \infty} f(\lambda) < \infty$$

(if $f(\lambda) = 1/2\pi$, we deal with Wiener process $\xi(t)$, $0 \leq t \leq 1$).

The corresponding space H consists of all random variables

$$\eta = \int_{-\infty}^{\infty} \varphi(\lambda)\Phi(d\lambda) = \int_0^1 c(t)\dot{\xi}(t)dt \qquad (18)$$
$$\left(\varphi(\lambda) = \int_0^1 e^{i\lambda t} c(t)dt\right)$$

where functions $c(t)$, $0 \leq t \leq 1$, belonging to $L^2[0,1]$ and $\dot{\xi}(t)$ is the generalized delivative of process $\xi(t)$; besides[1)]

$$\|\eta\|^2 = \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d\lambda \underset{\cap}{\cup} \int_0^1 c(t)^2 dt \qquad (19)$$

[1)] The relation $\alpha \underset{\cap}{\cup} \beta$ between variables α, β means that

$$0 < c_1 \leq \frac{\alpha}{\beta} \leq c_2 < \infty .$$

(see, for example, [1] or [6]), and the formula (18) gives us the isomorphism between H and $L^2[0,1]$ such that

$$H_t(\xi) \leftrightarrow L^2[0,t] , \qquad 0 \le t \le 1 , \tag{20}$$

where $L^2[0,t]$ denotes the subspace of all functions $c(s)$, $0 \le s \le 1$: $c(s) = 0$ for $s > t$. As it follows from the conditions (19) and (20), the formula

$$\Delta\eta = \int_0^1 [\tilde{\Delta} c(t)]\dot{\xi}(t)dt$$

gives us the isomorphism $\Delta \leftrightarrow \tilde{\Delta}$ between Hilbert-Schmidt operators in H and $L^2[0,1]$, and an operator Δ satisfies the condition (9) if and only if

$$\tilde{\Delta} L^2[0,t] \subseteq L^2[0,t] , \qquad 0 \le t \le 1 ,$$

that is equivalent to the condition (11) for a corresponding kernel $\Delta(t,s)$:

$$\tilde{\Delta} c(t) = \int_0^1 \Delta(t,s)c(s)ds , \qquad 0 \le t \le 1 ,$$

(remind $\tilde{\Delta} \in S_2$ if and only if $\Delta(t,s)$; $0 \le t$, $s \le 1$, satisfies the condition (12)). Thus any *non-anticipative* operator $\Delta \in S_2$ can be discribed by the formula

$$\Delta\eta = \int_0^1 \left[\int_t^1 \Delta(t,s)c(s)ds\right]\dot{\xi}(t)dt \tag{21}$$

with a Volterra, Hilbert-Schmidt kernel $\Delta(t,s)$; $0 \le t \le 1$. For variables $\xi(t)$, $0 \le t \le 1$, which correspond to the functions

$$c(s) = \begin{cases} 1 , & 0 \le s \le t , \\ 0 , & s > t , \end{cases}$$

we obtained from the formula (21) a general *non-anticipative* transformation (6) with $\Delta \in S_2$ as

$$\eta(t) = \xi(t) + \int_0^t \left[\int_0^s \Delta(u,s)\dot{\xi}(u)du\right]ds , \qquad 0 \le t \le 1 , \tag{22}$$

that gives us Hitsuda's representation (10) in the case of Wiener process $\xi(t)$, $0 \le t \le 1$.

EXAMPLE 2. Let $\xi(t)$, $0 \le t \le 1$, be a Gaussian stationary process:

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\Phi(d\lambda)$$

with a spectral density $f(\lambda)$ of the type

$$0 < \varliminf_{\lambda\to\infty} \lambda^{2n} f(\lambda) \leq \varlimsup_{\lambda\to\infty} \lambda^{2n} f(\lambda) < \infty \ . \tag{23}$$

It will be convenient to introduce the process

$$\begin{aligned} \zeta(t) &= \sum_{k=0}^{n-1} {}_nC_{k+1}[\xi^{(k)}(t) - \xi^{(k)}(0)] + \int_0^t \xi(s)ds \\ &= \int_{-\infty}^{\infty} \frac{e^{i\lambda t}-1}{i\lambda}(1+i\lambda)^n\Phi(d\lambda) \ , \qquad 0 \leq t \leq 1 \ . \end{aligned} \tag{24}$$

Obviously, the spectral density of this process $\zeta(t)$ with stationary increments satisfies the condition (17) and we can use results of our example 1 for the process $\zeta(t)$, $0 \leq t \leq 1$.

As is known (see, for example, [1] or [6]) the Hilbert space $H = H(\xi)$ consists of all variables

$$\eta = \int_{-\infty}^{\infty} \varphi(\lambda)\Phi(d\lambda)$$

$$\left(\varphi(\lambda) = \sum_{k=0}^{n-1} c_k (i\lambda)^k + (1+i\lambda)^n \int_0^1 e^{i\lambda t} c(t)dt\right)$$

where $c_1, \cdots, c_{n-1}$ are arbitrary constants and $c(t)$, $\in L^2[0, 1]$ or

$$\eta = \sum_{k=0}^{n-1} c_k \xi^{(k)}(0) + \int_0^1 c(t)\dot{\zeta}(t)dt \tag{25}$$

where $\dot{\zeta}(t)$ denotes the generalized derivative of the process $\zeta(t)$ determined by the transformation (24).

If we consider in the general formula (25) only functions $c(s)$, $\in L^2[0, t]$, we obtain the corresponding subspace $H_t(\xi)$, $0 \leq t \leq 1$, and it shows that $H_t(\xi)$ is a *direct sum* of the subspace

$$H_{0+}(\xi) = \bigcap_{t>0} H_t(\xi) \ ,$$

which consists of all variables $\eta = \sum_{k=0}^{n-1} c_k \xi^{(k)}(0)$, and the subspace $H_t(\zeta)$ of all variables $\eta = \int_0^t c(s)\dot{\zeta}(s)ds$:

$$H_t(\xi) = H_{0+}(\xi) + H_t(\zeta) \ , \qquad 0 \leq t \leq 1 ;$$

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in particular

$$H(\xi) = H_{0+}(\xi) + H(\zeta) .$$

Let P be a projector on the subspace $H(\zeta)$ parallel to the subspace $H_{0+}(\xi)$. If $\varDelta \in S_2$ then $P\varDelta P \in S_2$; obviously, if $\varDelta$ satisfies the condition (9) then $P\varDelta P$ satisfies to the similar condition with respect to $H_t(\zeta)$, $0 \leq t \leq 1$. As it has been shown (see (21)), the non-anticipative operator $P\varDelta P$ in $H(\zeta)$ can be discribed by a Volterra, Hilbert-Schmidt kernel $\varDelta(t, s)$; $0 \leq t,\ s \leq 1$:

$$P\varDelta P\eta = \int_0^1 \left[\int_t^1 \varDelta(t, s) c(s) ds \right] \dot{\zeta}(t) dt \tag{26}$$

where $\eta \in H$ is given by the formula (25) and

$$P\eta = \int_0^1 c(t) \dot{\zeta}(t) dt .$$

For any non-anticipative operator $\varDelta$ in $H(\xi)$ we have

$$\varDelta H_{0+} = \varDelta(\bigcap_{t>0} H_t) \subseteq \bigcap_t (\varDelta H_t) \subseteq \bigcap_t H_t = H_{0+}$$

that is equivalent to the condition

$$(I - P)\varDelta(I - P) = \varDelta(I - P) .$$

Then

$$\varDelta = (I - P)\varDelta P + \varDelta(I - P) + P\varDelta P = (I - P)\varDelta + P\varDelta P$$

where the finite-dimensional operator $(I - P)\varDelta$, mapping $H(\xi)$ on the subspace $H_{0+}(\xi)$, has the form

$$(I - P)\varDelta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) \tag{27}$$

($\eta_0, \eta_1, \cdots, \eta_{n-1}$ are some fixed elements in H_{0+}). Combining formulas (26) and (27), we obtain a general *non-anticipative* operator $\varDelta \in S$ as

$$\varDelta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^1 \left[\int_t^1 \varDelta(t, s) c(s) ds \right] \dot{\zeta}(t) dt ; \tag{28}$$

in particular, for $\eta \in H_t(\xi)$

$$\varDelta\eta = \sum_{k=0}^{n-1} \langle \eta, \eta_k \rangle \xi^{(k)}(0) + \int_0^t \left[\int_0^s \varDelta(u, s) \dot{\zeta}(u) du \right] c(s) ds . \tag{29}$$

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