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Semiclassical Limits of Eigenfunctions on Flat n-Dimensional Tori

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Abstract. We provide a proof of a conjecture by Jakobson, Nadirashvili, and Toth stating that on an n-dimensional flat torus $\mathbb{T}^n$, and the Fourier transform of squares of the eigenfunctions $|\varphi_\lambda|^2$ of the Laplacian have uniform l^n bounds that do not depend on the eigenvalue λ. The proof is a generalization of an argument by Jakobson, *et al.* for the lower dimensional cases. These results imply uniform bounds for semiclassical limits on $\mathbb{T}^{n+2}$. We also prove a geometric lemma that bounds the number of codimension-one simplices satisfying a certain restriction on an n-dimensional sphere $S^n(\lambda)$ of radius $\sqrt{\lambda}$, and we use it in the proof.

1 Introduction

We let Δ denote the Laplacian on the n-dimensional flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The eigenvalues of $-\Delta$ are denoted by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, and the corresponding eigenfunctions are denoted by φ_j. We normalize $\|\varphi_j\|_2 = 1$.

The following proposition was proved in [21] for $n = 2$, in [10] for $n = 3$, and in [11] for $n = 4$.

Proposition 1.1 *Let $2 \leq n \leq 4$. Then the Fourier series of $|\varphi_j|^2$ has uniformly bounded l^n norms, where the bound is independent of λ_j.*

We remark that it is well known that the *multiplicity* of λ_j becomes unbounded for $n \geq 2$, and therefore so does $\|\varphi_j\|_\infty$.

It was conjectured in [10] that the conclusion of Proposition 1.1 holds for arbitrary n. The main result of this paper is the proof of that conjecture.

Theorem 1.2 *For any $n \geq 5$, there exists $C = C_n < \infty$ such that for every eigenfunction $\Delta\varphi_j + \lambda_j\varphi_j = 0$ with $\|\varphi_j\|_2 = 1$, the Fourier series of $g := |\varphi_j|^2$ satisfies*

$$\|\widehat{g}\|_{l^n} \leq C(n)\|\varphi_j\|_2^2.$$

We stress that the bound C does not depend on the eigenvalue λ_j. The bound $C(n)$ is computed at the end of the proof and tends to 2 as $n \to \infty$.

Theorem 1.2 implies (by an argument in [10]) a statement about limits of eigenfunctions on $\mathbb{T}^{n+2}$. Consider weak limits of the probability measures $d\mu_j = |\varphi_j|^2 dx$, and denote the limit measure as $\lambda_j \to \infty$ by $d\nu$. One can prove that all such limit measures $d\nu$ are absolutely continuous in any dimension with respect to the Lebesgue

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measure on $\mathbb{T}^n$ (see [10]). Accordingly, by the Radon–Nikodym theorem, one can conclude that $d\nu$ has a density $h(x) \in L^1(\mathbb{T}^n)$ such that $d\nu = h(x)\,dx$. Then we consider the Fourier expansion of $h(x)$:

$$h(x) \sim \sum_{\tau \in \mathbb{Z}^n} c_\tau \, e^{i(\tau,x)}. \tag{1.1}$$

In dimension $n = 2$, it was shown in [10] that the density of every such limit is a trigonometric polynomial with at most *two* different magnitudes for the frequency. It was also shown in [10, 11] that on $\mathbb{T}^n$ for $3 \le n \le 6$, the Fourier expansion of the limit measure $d\nu$ is in l^{n-2}, that is,

$$\sum_{\tau \in \mathbb{Z}^n} |b_\tau|^{n-2} < \infty. \tag{1.2}$$

The proofs in dimensions $4 \le n \le 6$ used Proposition 1.1 and results in [10] that reduced estimates for limits on $\mathbb{T}^{n+2}$ to estimates for eigenfunctions on $\mathbb{T}^n$. The estimate (1.2) implies that on $\mathbb{T}^3$, the density of any limit $d\nu$ has an absolutely convergent Fourier series, whereas on $\mathbb{T}^4$, we conclude that $h(x) \in L^2(\mathbb{T}^4)$.

Combining Theorem 1.2 with the results in [10], we immediately obtain the following result.

Theorem 1.3 *Given the Fourier expansion* (1.1) *of the limit measure $d\nu$ on $\mathbb{T}^{n+2}$, we have*

$$\Big(\sum_{\tau \in \mathbb{Z}^{n+2}} |b_\tau|^n \Big)^{1/n} \le C(n) < \infty.$$

A generalization of B. Connes' result [6], proved in [10], shows that the constant $C(n)$ appearing in Theorem 1.3 on $\mathbb{T}^{n+2}$ coincides with the constant in Theorem 1.2 on $\mathbb{T}^n$. The bound $C(n)$ will be computed at the end of the proof and we will find that it tends to 2 as $n \to \infty$.

An important question about eigenfunctions of the Laplacian is the following: given $\varphi(x)$ satisfying $\Delta\varphi_j + \lambda_j\varphi_j = 0$ and $\|\varphi\|_2 = 1$ on a general n-dimensional smooth Riemannian manifold $\mathcal{M}$, what is the asymptotic growth rate of the L^p norms of the eigenfunction? That is, how fast does $\|\varphi_j\|_{L^p}$ grow as the eigenvalue $\lambda_j \to \infty$?

On a two-dimensional compact boundaryless Riemannian manifold, C. Sogge showed in [17] that for $2 \le p \le \infty$, $\|\varphi_j\|_p \le C\lambda_j^{\delta(p)}\|\varphi_j\|_2$, where

$$\delta(p) = \begin{cases} \dfrac{1}{4}\Big(\dfrac{1}{2} - \dfrac{1}{p}\Big), & 2 \le p \le 6, \\[2ex] \dfrac{1}{2}\Big(\dfrac{1}{2} - \dfrac{2}{p}\Big), & 6 \le p \le \infty. \end{cases}$$

This bound turned out to be sharp on the round sphere S^2.

In a remarkable result, Zygmund [21] provided a uniform bound for the L^4-norm of the eigenfunctions of the Laplacian on $\mathbb{T}^2$. That is,

$$\frac{\|\varphi\|_4}{\|\varphi\|_2} \leq 5^{1/4}. \tag{1.3}$$

The bound (1.3) provided in [21] is independent of the eigenvalue.

Before we mention the next result, we define

$$M_{n,p}(\lambda) := \sup_{\substack{(\Delta+\lambda)\varphi=0 \\ \varphi \text{ on } \mathbb{T}^n}} \frac{\|\varphi\|_p}{\|\varphi\|_2}. \tag{1.4}$$

The question of the growth rate mentioned earlier can be translated into: what is the asymptotic behavior of $M_{n,p}(\lambda)$? It is sometimes possible to obtain uniform bounds (independent of λ) for $M_{n,p}(\lambda)$ for a restricted set of eigenvalues.

In particular, Mockenhaupt proved in [13] the following. Given a finite subset $D = \{q_1, q_2, \dots, q_k\}$ of prime integers with $q_j \equiv 1 \pmod 4$, we consider the set Λ_D consisting of all eigenvalues $\lambda \in \mathbb{N}$ such that all prime divisors q of λ with the property $q \equiv 1 \pmod 4$, belong to D. Then for all $\lambda \in \Lambda_D$ and for all $p < \infty$, we have $M_{2,p} \leq C(p,k) < \infty$, where $C(p,k)$ is a constant.

A legitimate question to ask is whether or not there exists a uniform bound for $M_{n,p}$ for general n and p. The question is still open, although there exist results about the rate of growth of $M_{n,p}(\lambda)$ as $\lambda \to \infty$. Bourgain showed in [3] that on $\mathbb{T}^n$ with $n \geq 4$, we have $M_{n,p} \ll \lambda^{(n-2)/4-n/2+\varepsilon}$ for $p \geq 2(n+1)/(n-3)$.

We notice that Theorem 1.2 does not imply a bound on eigenfunctions since there is no converse to the Hausdorff–Young inequality. For $1 < p \leq 2 \leq q < \infty$ with $p^{-1} + q^{-1} = 1$, we have $\|b_\tau\|_{l^q} \ll \|\varphi\|^2_{L^{2p}}$.

Although the bound $C(n)$ from Theorem 1.2 does not depend on the eigenvalue λ, it does not give us information about the bound $M_{n,p}$ in (1.4).

In recent papers, J. Bourgain and Z. Rudnick [4, 5] considered upper and lower bounds for the L^p norms of the the restriction of eigenfunctions of the Laplacian to smooth hypersurfaces of $\mathbb{T}^n$ with nonvanishing $\|\varphi_\lambda\|_{L^2(\Sigma)} \asymp \|\varphi_\lambda\|_2$ for all eigenfunctions φ_λ of the Laplacian on $\mathbb{T}^n$ with $\lambda \geq \Lambda$ for some Λ that depends only on the hypersurface Σ.

There exist bounds for the L^∞ norm of the eigenfunctions as well. Hörmander showed (see [8, 9]) that on any compact Riemannian manifold M, we have

$$\|\varphi_\lambda\|_\infty \leq C\,\lambda^{(n-1)/4},$$

where n is the dimension of the manifold M. This bound is attained for some manifolds, such as S^n, but not for others, such as $\mathbb{T}^n$. Manifolds for which this bound is sharp are called manifolds with *maximal eigenfunction growth*.

Y. Safarov studied the asymptotic behavior of the spectral function, the remainder in Weyl's law, and of eigenfunctions, in many papers including [14, 15].

In a series of papers C. Sogge, J. Toth, and S. Zelditch [18–20] studied the following question: what characterizes the manifolds with maximal eigenfunction growth?

They established that the manifolds with maximal eigenfunction growth must have a point x where the set of geodesic loops at that point has a positive measure in S_x^*M. The converse turned out to be false, as they constructed a counterexample in [19].

An older question of the same type is: how fast does the spectral function and the remainder term in Weyl's formula grow as $\lambda \to \infty$? The spectral function is given by

$$N_{x,y}(\lambda) = \sum_{0<\sqrt{\lambda_j}<\sqrt{\lambda}} \varphi_j(x)\overline{\varphi_j(y)}.$$

If we consider the diagonal when $x = y$, we obtain $N_{x,x}(\lambda)$. If we integrate the latter over the volume of the manifold M (assumed to be compact), we obtain the eigenvalue counting function $N(\lambda)$ defined by $N(\lambda) = \#\{\lambda_i < \lambda\}$. The remainder term in Weyl's formula is given by $R(\lambda) = N(\lambda) - c_n \operatorname{vol}(M)\,\lambda^{n/2}$, where c_n is a constant that depends on the dimension n.

The asymptotic behavior of the spectral function and the remainder term were studied by many people; see [1, 7, 9, 12, 16] and the references therein for a detailed exposition of the subject.

The results of this paper appear in [2].

2 Proof of the Main Result

Let us define the notation that will be used throughout the argument. For $\varphi_j(x)$, an L^2-normalized eigenfunction of the Laplacian on an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with eigenvalue λ_j, we let its Fourier expansion be:

$$\varphi_j(x) \sim \sum_{\substack{\eta\in\mathbb{Z}^n\\ |\eta|^2=\lambda_j}} a_\eta e^{i(x,\eta)}.$$

The Fourier series of $g(x) = |\varphi_j(x)|^2$ (recall the definition from the introduction) is as follows:

$$|\varphi_j(x)|^2 \sim \sum_{\substack{\tau=\xi-\eta\\ |\xi|^2=|\eta|^2=\lambda_j}} b_\tau e^{i(x,\tau)},$$

$$b_\tau = \sum_{\substack{\xi-\eta=\tau\\ |\xi|^2=|\eta|^2=\lambda_j}} a_\xi \bar{a}_\eta,$$

$$\sum_{\substack{\eta\in\mathbb{Z}^n\\ |\eta|^2=\lambda_j}} |a_\eta|^2 \equiv 1.$$

We will write $\mathbf{S}^{n-1}(\lambda_j)$ for the $(n-1)$-sphere of radius $\sqrt{\lambda_j}$ and S_{n-1,λ_j} for the set of lattice points on $\mathbf{S}^{n-1}(\lambda_j)$. In the spirit of this new notation, the last three equations may be written as follows:

$$|\varphi_j(x)|^2 \sim \sum_{\substack{\tau=\xi-\eta \\ \xi,\eta\,\in S_{n-1,\lambda_j}}} b_\tau\, e^{i(x,\tau)},$$

$$b_\tau = \sum_{\substack{\xi,\eta\,\in S_{n-1,\lambda_j} \\ \xi-\eta=\tau}} a_\xi \bar{a}_\eta,$$

$$\sum_{\eta\,\in S_{n-1,\lambda_j}} |a_\eta|^2 \equiv 1.$$

Without loss of generality, we can assume the coefficients a_ξ to be real, and then we have $|a_\xi| = |\bar{a}_\xi| = |a_{-\xi}|$. For the case where $\tau = \mathbf{0}$, we have

$$b_{\mathbf{0}} = \sum_{0=\tau=\xi-\eta} a_\xi \bar{a}_\eta = \sum_{\xi\,\in S_{n-1,\lambda_j}} |a_\xi|^2 = 1. \tag{2.1}$$

The proof of Theorem 1.2 requires a lemma that will be proved at the end of this section.

Lemma 2.1 *Given n points $\{\xi_i\}_{i=1}^n$ on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, no two of which are diametrically opposite, that form a codimension-one simplex, assume that there exist $\tau \in \mathbb{Z}^n$ and another n points $\{\eta_i\}_{i=1}^n$ on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$ such that*

$$\xi_i - \eta_i = \pm\tau, \quad 1 \le i \le n. \tag{2.2}$$

Then there can be at most 2^{n-1} such different *vectors τ satisfying* (2.2).

Remark 2.2 Given $m > n$ points on $\mathbf{S}^{n-1}(\lambda_j) \cap \mathbb{Z}^n$, we will still have the same bound, 2^{n-1}, on the number of possible τ's. In other words, adding more points augments the number of restrictions, which, in principle, might reduce the number of possibilities for the *different* τ's.

Remark 2.3 We also notice that the bound we obtained is independent of the eigenvalue λ_j. This fact is crucial in the proof of Theorem 1.2.

The proof of Theorem 1.2 is by strong induction, the base case being done in [10] for the case $n = 3$ and in [11] for the case $n = 4$. First we will provide a proof for the case $n = 5$. This will give a feeling of how the proof of the general case goes.

Proof of Theorem 1.2 for the case $n = 5$ The aim of the following calculations is to bound the sum $\sum_\tau |b_\tau|^5$. Given (2.1), we will consider the sum with nonzero τ:

$$\sum_{\tau\neq\mathbf{0}} |b_\tau|^5 \le \sum_{\tau\neq\mathbf{0}} \Big(\sum_{\xi_j-\eta_j=\tau} \prod_{j=1}^{5} |a_{\xi_j}||a_{\eta_j}| \Big) \tag{2.3}$$

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The trick that we shall use is to bound the right-hand side of (2.3) by

$$\sum_{\tau\neq 0}\sum_{\xi_i-\eta_i=\tau}\frac{1}{2}\Big(\prod_{i=1}^{5}|a_{\xi_i}|^2+\prod_{i=1}^{5}|a_{\eta_i}|\Big). \tag{2.4}$$

Then we interchange the order of summation in (2.4), and finally we use Lemma 2.1 to obtain a finite upper bound.

In doing so, we will encounter several configurations of the points ξ_i's on $\mathbf{S}^4(\lambda_j)\cap\mathbb{Z}^5$. Each configuration needs to be studied separately. An obvious case is when two or more points ξ_i coincide. Then equation (2.3) reduces to

$$\sum_{\tau\neq 0}\sum_{\xi_0-\eta_0=\tau}|a_{\xi_0}|^2|a_{\eta_0}|^2\Big(\sum_{\xi_i-\eta_i=\xi_0-\eta_0}\Big(\prod_{i=3}^{5}|a_{\xi_i}||a_{\eta_i}|\Big)\Big) \tag{2.5}$$

and one can bound the terms $|a_{\xi_i}||a_{\eta_i}|$ inside the product of (2.5) by $\frac{1}{2}(|a_{\xi_i}|^2+|a_{\eta_i}|^2)$. Then we can bound this case by

$$\frac{1}{2^3}\sum_{\tau\neq 0}\sum_{\xi_0-\eta_0=\tau}|a_{\xi_0}|^2|a_{\eta_0}|^2\Big(\sum_{\xi,\eta\in S_{4,\lambda_j}}|a_{\xi}|^2|a_{\eta}|^2\Big),$$

where the former is bounded by $\frac{1}{2^3}$.

Now we may suppose that no two points coincide. We end up with five points in $\mathbb{R}^5$. These points will either lie in a 4-dimensional affine subspace (where they will form a 4-simplex), a 3-dimensional affine subspace, or a 2-dimensional affine subspace.

In the case where the points form a 4-simplex, we can use Lemma 2.1 and interchange the order of summation in (2.4) as follows:

$$\frac{1}{2}\sum_{\xi_i\in S_{4,\lambda_j}}\sum_{\tau\neq 0}\sum_{\xi_i-\eta_i=\pm\tau}\Big(\prod_{i=1}^{5}|a_{\xi_i}|^2+\prod_{i=1}^{5}|a_{\eta_i}|^2\Big).$$

The former will be bounded by

$$\frac{1}{2}\sum_{\xi_i\in S_{4,\lambda_j}}2^4\cdot 2\prod_{i=1}^{5}|a_{\xi_i}|^2,$$

which by the L^2 normalization will not exceed 2^4.

In the case where the points ξ_i lie in a 3-dimensional affine subspace, namely α, they will form a codimension-2 simplex. There will be three different configurations that need to be considered.

The first case is when $\{\xi_i\}_{i=1,\dots,5}\in\alpha$ and at least one of the $-\eta_i\notin\alpha$. Without loss of generality, we may suppose that $-\eta_5\notin\alpha$. Then the simplex formed by $(\xi_1,\xi_2,\xi_3,\xi_4,-\eta_5)$ is a parallel translate of the simplex formed by $(\eta_1,\eta_2,\eta_3,\eta_4,-\xi_5)$

and these simplices do *not* lie in a 3-dimensional subspace. They form a non-degenerate 4-simplex. Hence, we are reduced to the case just studied above and we obtain the same bound, that is, 2^4.

In the next case, we suppose that the points $\{\xi_i\} \in \alpha$, $\{-\eta_i\} \in \alpha$, but $\{\eta_i\} \notin \alpha$ for all $i = 1, \dots, 5$. The trick we will be using is to consider the subspace that contains both α and η_1 say, namely, γ. The subspace γ is a 4-dimensional subspace that contains $\mathbf{0}$, since both η_1 and $-\eta_1$ lie in γ. Thus, $\gamma \cap \mathbf{S}^4(\lambda_j)$ is the great 3-sphere, where the great k-sphere is defined to be the intersection of $\mathbf{S}^n(\lambda_j)$ with a k-dimensional hyperplane passing through the origin. Hence, by Lemma 2.1 and Remark 2.2, we have the same bound on the number of τ's as to have four points on S_{3,λ_j}, and this will lead to a bound of 2^3.

The last scenario that needs to be considered in the case where $\{\xi_i\}_{i=1,\dots,5} \in \alpha$ is when $\{-\eta_i\}_{i=1,\dots,5} \in \alpha$ and at least one of the $\eta_i \in \alpha$, say η_1. Since both η_1 and $-\eta_1$ are in α, $\mathbf{0} \in \alpha$ and all of $\pm\eta_i, \pm\xi_i \in \alpha$. Hence, $\alpha \cap \mathbf{S}^4(\lambda_j)$ is the great 2-sphere. Once again, Lemma 2.1 and Remark 2.2 will lead us to a bound that is equal to 2^2.

It may happen that the points lie in a 2-dimensional affine subspace say, β. We will study the possible cases in the same manner we did previously. In the first case, we suppose that $\{\xi_i\}_{i=1,\dots,5} \in \beta$ with $\{-\eta_i\} \in \beta$ for all i. We consider the 3-dimensional subspace γ_1 that contains both β and η_1 say. Then $\mathbf{0} \in \gamma_1$, which implies that $\pm\eta_i, \pm\xi_i$ all lie in $\gamma_1 \cap \mathbf{S}^4(\lambda_j)$, which is the great 2-sphere. We are back in one of the cases studied previously and once again, Lemma 2.1 and Remark 2.2 will guarantee us a bound of 2^2.

In the very last case, we lose a bit of control on where the η_i might be. We let $\xi_i \in \beta$, but at least one of the $-\eta_i \notin \beta$, $-\eta_5$ say. Then the points $\{\xi_1, \xi_2, \xi_3, \xi_4, \eta_5\}$ lie in a 3-dimensional affine subspace and we are back to the case where the $\xi_i \in \alpha$. Hence, we have a total bound equal to $2^2 + 2^3 + 2^4 = 28$.

Summing all the bounds, we obtain $C(n) \approx 2.384729\dots$. ∎

Proof of the General Case We shall now turn into the proof of the general case, that is, the sum (2.6) given below is convergent for any n. The proof is by strong induction. That is, we suppose that the sum (2.6) is bounded in any dimension $k < n$.

$$\sum_{\tau \in \mathbb{Z}^n \cap \mathbf{S}^{n-1}(\lambda_j)} |b_\tau|^n = 1 + \sum_{0 \neq \tau \in \mathbb{Z}^n \cap \mathbf{S}^{n-1}(\lambda_j)} |b_\tau|^n. \tag{2.6}$$

As in the proof of the $n = 5$ case, we have

$$\sum_{\tau \neq \mathbf{0}} |b_\tau|^n \leq \sum_{\tau \neq \mathbf{0}} \sum_{\xi_i - \eta_i = \tau} \prod_{i=1}^{n} |a_{\xi_i}||a_{\eta_i}|. \tag{2.7}$$

The same trick is used as before, that is, we will bound the right-hand side of (2.7) by (2.8), then interchange the order of summation in the latter, and finally use Lemma 2.1 to obtain a finite upper bound,

$$\sum_{\tau \neq \mathbf{0}} \sum_{\xi_i - \eta_i = \tau} \frac{1}{2} \Big(\prod_{i=1}^{n} |a_{\xi_i}|^2 + \prod_{i=1}^{n} |a_{\eta_i}|^2 \Big). \tag{2.8}$$

Once again, several cases need to be studied. We will do so in the same manner as for the $n = 5$ case. Instead of five points, we now have n points $\{\xi_i\}_{i=1}^n$ on the surface of the sphere $\mathbf{S^{n-1}}(\lambda_j)$

The trivial case where two or more points coincide gives a bounded contribution to the sum (2.6) that is equal to $\frac{1}{2^{n-2}}$, by the same computations as in the $n = 5$ case. In the subsequent cases, we may assume that no two points ξ_i coincide.

The second trivial case is when the points $\{\xi_i\}$ form a non-degenerate codimension-1 simplex. A change of order of summation in (2.8) and Lemma 2.1 yields a bound equal to 2^{n-1}.

The nontrivial cases are when the points $\{\xi_i\}$ lie in smaller subspaces. Providing an upper bound to each of these cases finishes the proof. The first of such nontrivial cases is when the points $\{\xi_i\}$ lie in an $(n-2)$-dimensional affine subspace, namely α_{n-2}. Let us suppose $\{\xi_i\}_{i=1}^n \in \alpha_{n-2}$ with all the $\{-\eta_i\} \in \alpha_{n-2}$ as well. If any one of the η_i's or $-\xi_i$'s is an element of α_{n-2}, then the origin $\mathbf{0} \in \alpha_{n-2}$, which implies that $\alpha_{n-2} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-2)$-sphere. Hence, we have n points on S_{n-2,λ_j} and by the induction hypothesis, this gives us a bounded contribution to the sum (2.6). Suppose now that none of the η_i's or $-\xi_i$'s is an element of α_{n-2}. Then we consider the subspace β_{n-2} containing both α_{n-2} and η_1 say. We get an $(n-1)$-dimensional subspace including $\mathbf{0}$, and $\beta_{n-2} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-2)$-sphere. Remark 2.2 implies that the resulting case is one of the cases in our induction hypothesis and this gives a bounded contribution to the sum (2.6).

In order to prove it for the rest of the cases, *i.e.,* when the points $\{\xi_i\}$ lie in a $(n-k) < (n-2)$-dimensional affine subspace, namely α_{n-k}, we will use a *second* (reversed) induction on the dimension of the affine subspace α_{n-k} where the points $\{\xi_i\}$ might lie. That is, assuming we have a bounded contribution from all the α_{n-k+1} for some k with $3 < k < (n-1)$, we will prove that we have a bounded contribution from the case where the $\{\xi_i\} \in \alpha_{n-k}$. Once again, we have the two subcases, depending on whether or not $-\eta_j$ belong to α_{n-k}.

For the first subcase, we may assume without loss of generality that $-\eta_1 \notin \alpha_{n-k}$. Then the simplex $(-\eta_1, \xi_2, \dots, \xi_n)$ is a parallel translate of $(-\xi_1, \eta_2, \dots, \eta_n)$ and the last two simplices lie in an $(n-k+1)$-dimensional subspace. Hence, we are reduced to the *second* induction hypothesis which yields a bounded contribution to the sum (2.6).

Let us now turn our attention to the second subcase. If all the $\{\xi_i\}_{i=1}^n$ and $\{-\eta_i\}_{i=1}^n$ lie in α_{n-k} with *none* of the η_i's in α_{n-k}, we consider the subspace β_{n-k} containing both α_{n-k} and η_1 say. This is an $(n-k+1)$-dimensional subspace that includes $\mathbf{0}$. We can see that $\beta_{n-k} \cap \mathbf{S}^{n-k+1}(\lambda_j)$ is the great $(n-k)$-sphere. Hence, we have n points on S_{n-k,λ_j} and by the strong *first* induction hypothesis, we obtain a finite contribution from this subcase to the sum (2.6).

We note that if all the $\{\xi_i\}_{i=1}^n$ and $\{-\eta_i\}_{i=1}^n$ lie in α_{n-k} with at least one of the η_i's in α_{n-k}, then $\mathbf{0} \in \alpha_{n-k}$ and $\alpha_{n-k} \cap \mathbf{S}^{n-1}(\lambda_j)$ is the great $(n-k-1)$-sphere, and this case gives a bounded contribution to the sum (2.6) by once again the strong *first* induction hypothesis.

We have exhausted all the possible cases, each giving a bounded contribution to the sum (2.6). Therefore, the sum is bounded and this finishes the proof of the conjecture in [10]. ∎

We may now provide a proof for the geometric Lemma 2.1.

Proof of Lemma 2.1 Suppose we are given $\{\xi_i\}_{i=1}^n$, n points on S_{n-1,λ_j}, no two of which are diametrically opposite, and such that the simplex with vertices $\{\xi_i\}_{i=1}^n$ is non-degenerate. That is, the points $\{\xi_i\}_{i=1}^n$ cannot be in any (affine) subspace of dimension *strictly* less than $n-1$. Then given n equal parallel "chords" $\{\mathbf{v_i}\}_{i=1}^n$ of S_{n-1,λ_j} (not equal to $\overline{\xi_i\xi_j}, \forall i, j$) such that ξ_i is an endpoint of $\mathbf{v_i}$, we denote the other endpoint of $\mathbf{v_i}$ by η_i and the diametrically opposite points of ξ_i (respectively η_i) by ξ_i' (respectively η_i'). The question we would like to pose is: *where on S_{n-1,λ_j} can $\{\eta_i\}_{i=1}^n$ lie?* We will see that there are *finitely* many places where the $\{\eta_i\}_{i=1}^n$ can be. In fact, there are $[n/2]$ different scenarios, and we will study each of them.

If $\overline{\xi_i\eta_i}$ are equal for all i, then $\eta_1 = \eta_i + \overline{\xi_i\xi_1}$ for all $i = 1, \dots, n$. Hence, the points $\eta_1 + \overline{\xi_1\xi_i}$ lie on S_{n-1,λ_j} for all i. Since $\mathbf{S}^{n-1}(\lambda_j)$ is strictly convex, there is at most *one* point (other than ξ_1), namely η_1, for which the points $\eta_1 + \overline{\xi_1\xi_i}$ for all $i = 1, \dots, n$ lie on S_{n-1,λ_j}.

In the next scenario, we suppose $\overline{\xi_i\eta_i}$ are equal for all i, except at one point k, where $\overline{\xi_i\eta_i} = \overline{\eta_k\xi_k}$. Then the points $\eta_1 + \overline{\xi_1\xi_i}$ for all $i \neq k$ and $\eta_1 + \overline{\xi_1\xi_k'}$ lie on S_{n-1,λ_j}. Again by the convexity of $\mathbf{S}^{n-1}(\lambda_j)$ and the fact that $\{\xi_i\}$ forms a codimension-1 simplex, there is at most one point (other than ξ_1), namely η_1, for which the points $\eta_1 + \overline{\xi_1\xi_i}$ for $i \neq k$ and $\eta_1 + \overline{\xi_1\xi_k'}$ lie on S_{n-1,λ_j}. However, the last equation gives us at most one possibility for η_1 for every $k = 1, \dots, n$. Hence, we have a total of $n = \binom{n}{1}$ possibilities for η_1.

In the next case, we assume $\overline{\xi_i\eta_i}$ are equal for all $i \neq k, l$, where $\overline{\xi_i\eta_i} = \overline{\eta_k\xi_k} = \overline{\eta_l\xi_l}$. Here again $\eta_1 = \eta_i + \overline{\xi_i\xi_1}$ for all $i \neq k, l$ and $\eta_1 = \eta_k' + \overline{\xi_k'\xi_1} = \eta_l' + \overline{\xi_l'\xi_1}$, making the points $\eta_1 + \overline{\xi_1\xi_i}$ for $i \neq k, l$, $\eta_1 + \overline{\xi_1\xi_k'}$, and $\eta_1 + \overline{\xi_1\xi_l'}$ lie on S_{n-1,λ_j}. The convexity of $\mathbf{S}^{n-1}(\lambda_j)$ implies the uniqueness of such $\eta_1 \neq \xi_1$ for every pair k, l. Hence, we have $\binom{n}{2}$ possibilities for η_1 in this scenario.

Similarly, we will get $\binom{n}{3}$ for the next and so on, until $\binom{n}{n}$. However, the $\binom{n}{n}$ case is the same as the very first case $\binom{n}{0}$ in which we will simply change the sign of all the vectors $\overline{\xi_i\eta_i}$. The $(n-1)$-th scenario is similar to the second scenario, and so on; hence, counting every case twice. The total number of possibilities will be the sum of the possibilities in every scenario and is

$$\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} = 2^{n-1}. \qquad \blacksquare$$

The bound follows from the proof of Theorem 1.2, using the bounds given by Lemma 2.1. We do not claim that $C(n)$ is a sharp bound. The result will be:

$$C(n) = \left(2^{2-n} + \left(\frac{5n}{4} - 4\right)2^n + 5\right)^{1/n}.$$

It is clear that $C(n) \to 2$ as $n \to \infty$.

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