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THE ROOT SYSTEM OF PRIMES OF A HAHN GROUP

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Abstract

Let Δ be a root system and let V be the Hahn group of real-valued functions on Δ. Then Δ can be order-embedded into $P(\Delta)$, the root system of prime l-ideals of V. In this note we identify $P(\Delta)$ in terms of Δ without explicit reference to V, up to the convex subgroup structure of the additive groups of real closed η_1-fields. In particular, we characterize the minimal prime l-ideals of V in terms of Δ by an ultrafilter construction which generalizes the well-known method when Δ is trivially ordered.

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1. Introduction

Throughout this introduction let G be an abelian l-group. Let $\mathscr{C}(G)$ denote the set of all *convex l-subgroups* of G (or *l-ideals*, since they are normal). If G is an l-subgroup of an l-group H, and the map $C \to C \cap G$ is a lattice isomorphism of $\mathscr{C}(H)$ onto $\mathscr{C}(G)$, then H is an *a-extension* of G. Those elements P of $\mathscr{C}(G)$ for which G/P is totally ordered are called *prime*; equivalently, the set of elements of $\mathscr{C}(G)$ larger than P is a chain. Thus, the set of primes forms a *root system*, that is, a partially ordered set in which no two incomparable elements have a lower bound. Each prime exceeds at least one minimal prime; a prime P is minimal if and only if for each $g \in P^+$ there exists $h \notin P^+$ such that $h \wedge g = 0$. If the intersection P^* of all elements of $\mathscr{C}(G)$ larger than a prime P covers P, then P is called a *value*; it is maximal with respect to not containing each element $g \in P^* \setminus P$. The root system of all values of G is denoted by $\Gamma(G)$. If Δ is any root system, then

$$V = V(\Delta, \mathbf{R}) = \{f\colon \Delta \to \mathbf{R}\colon \text{the support of } f \text{ has } ACC\}$$

is an abelian l-group, called a *Hahn group*. Each abelian l-group G may be l-embedded into $V(\Gamma(G), \mathbf{R})$ (Conrad, Harvey and Holland (1963)). For further information about l-groups, the reader may consult Conrad (1970), or Bigard, Keimel and Wolfenstein (1977).

If Δ is a trivially ordered set, then it is well known that the set of minimal primes of $V(\Delta, \mathbf{R})$ are in a one-to-one correspondence with the ultrafilters on Δ (see Conrad and McAlister (1969) and Gillman and Jerison (1960)). In the next section we generalize this to the case where Δ is any root system. In the third section, we identify the set $P(\Delta)$ of the prime l-ideals of V in two steps. First, we identify $S(\Delta) = P(\Delta)/\approx$, where $P \approx Q$ if they contain the same set of minimal primes. (For a discussion of this equivalence relation in a more general context, see Conrad (1978).) Then, each $\approx$-equivalence class is described in terms of the convex subgroup structure of the additive groups of certain real closed η_1-fields.

2. The minimal prime l-ideals of V

Throughout Δ will be a fixed root system and $V = V(\Delta, \mathbf{R})$. Let $\mathfrak{A}$ be the set of all maximal trivially ordered subsets of Δ. We partially order $\mathfrak{A}$ by declaring $A \leqslant B$ if $\delta \in A$ implies that there exists $\gamma \in B$ with $\delta \leqslant \gamma$. This is a lattice order with

$$A \vee B = \{\text{maximal elements of } A \cup B\}$$

and

$$A \wedge B = \{\text{minimal elements of } A \cup B\}.$$

We will occasionally abuse this notation by speaking of $A \vee B$ where at most one of A and B is trivially ordered but not maximal. If $A, B \in \mathfrak{A}$ and $X \subseteq A \cap (B \wedge A)$, let

$$B^{\leftarrow}(X) = \{\delta \in B\colon \text{there exists } \gamma \in X \text{ with } \gamma \leqslant \delta\}.$$

LEMMA 2.1. *Let $A, B \in \mathfrak{A}$ with $A \leqslant B$ and let $\mathscr{U}$ be an ultrafilter on A. Let $B^{\leftarrow}(\mathscr{U}) = \{B^{\leftarrow}(X)\colon X \in \mathscr{U}\}$. Then $B^{\leftarrow}(\mathscr{U})$ is an ultrafilter on B.*

PROOF. Clearly $B^{\leftarrow}(X) \neq \varnothing$ for each $X \in \mathscr{U}$. Suppose X, $Y \in \mathscr{U}$ and let

$$U = \bigcup \{W \subseteq A\colon B^{\leftarrow}(W) = B^{\leftarrow}(X)\}$$

and

$$Z = \bigcup \{W \subseteq A\colon B^{\leftarrow}(W) = B^{\leftarrow}(Y)\}.$$

Then, $U, Z \in \mathscr{U}, B^{\leftarrow}(U) = B^{\leftarrow}(X), B^{\leftarrow}(Z) = B^{\leftarrow}(Y)$, and $B^{\leftarrow}(U) \cap B^{\leftarrow}(Z) = B^{\leftarrow}(U \cap Z)$. Since $U \cap Z \in \mathscr{U}$, $B^{\leftarrow}(X) \cap B^{\leftarrow}(Y) = B^{\leftarrow}(U \cap Z)$, which is an element of $B^{\leftarrow}(\mathscr{U})$.

Therefore, $B^{\leftarrow}(\mathscr{U})$ has the finite intersection property. Similarly, if $X \subseteq B$, then $X \in B^{\leftarrow}(\mathscr{U})$ or $B \setminus X \in B^{\leftarrow}(\mathscr{U})$. Therefore $B^{\leftarrow}(\mathscr{U})$ is an ultrafilter.

If $A(\mathscr{U})$ and $B(\mathscr{U})$ are ultrafilters on $A, B \in \mathfrak{A}$ respectively, then $A(\mathscr{U})$ and $B(\mathscr{U})$ are said to be *compatible* if

$$(A \vee B)^{\leftarrow}(A(\mathscr{U})) = (A \vee B)^{\leftarrow}(B(\mathscr{U})).$$

If for each $A \in \mathfrak{A}$, $A(\mathscr{U})$ is an ultrafilter on A, and for each $A, B \in \mathfrak{A}$, $A(\mathscr{U})$ and $B(\mathscr{U})$ are compatible, then $\{A(\mathscr{U}): A \in \mathfrak{A}\}$ is called a *compatible system of ultrafilters on* $\mathfrak{A}$.

For each $v \in V$, let $S(v) = \{\alpha \in \Delta: v(\alpha) \neq 0\}$ and let

$$M(v) = \{\text{maximal elements of } S(v)\}.$$

Since $v \in V$, $S(v)$ satisfies the ascending chain condition and so $\delta \in S(v)$ implies that there exists $\alpha \in M(v)$ such that $\alpha \geqslant \delta$. Clearly $M(v)$ is a trivially ordered set.

THEOREM 2.2. *There is a one-to-one correspondence between minimal prime l-ideals of V and compatible systems of ultrafilters on* $\mathfrak{A}$ *given as follows*:

Let P be a minimal prime l-ideal of V. For each $A \in \mathfrak{A}$, *let* $A(\mathscr{U}) = \{A \setminus M(v): v \in P\}$. *Then* $\mathscr{C}_P = \{A(\mathscr{U}): A \in \mathfrak{A}\}$ *is a compatible system of ultrafilters on A.*

Let $\mathscr{C} = \{A(\mathscr{U}): A \in \mathfrak{A}\}$ *be a compatible system of ultrafilters on* $\mathfrak{A}$ *and let*

$$P_{\mathscr{C}} = \{v \in V: A \setminus M(v) \in A(\mathscr{U}), \textit{for all } A \in \mathfrak{A}\}.$$

Then $P_{\mathscr{C}}$ *is a minimal prime l-ideal of V.*

PROOF. Let P be a minimal prime; we first show that each element $A(\mathscr{U})$ of $\mathscr{C}_P$ is an ultrafilter on A. Suppose $\varnothing \in A(\mathscr{U})$. Then there exists $v \in P^+$ so that $M(v) \supseteq A$. Since A is a maximal trivially ordered set, $M(v) = A$. But then $w \wedge v = 0$ implies that $w = 0$, which is impossible since v is an element of the minimal prime P. Therefore, each element of $A(\mathscr{U})$ is non-empty. Now let $X, Y \in A(\mathscr{U})$ and choose $u, v \in P^+$ such that $A \setminus M(u) = X$ and $A \setminus M(v) = Y$. Let $s = \chi(A \setminus X)$ and $t = \chi(A \setminus Y)$ where $\chi(T)$ is the characteristic function on T. Then $x = s \wedge u$ and $y = t \wedge v$ are both elements of P. Moreover, $M(x) = A \cap M(u)$ and $M(y) = A \cap M(v)$ and so $M(x \vee y) = M(x) \cup M(y)$. Therefore,

$$X \cap Y = (A \setminus M(u)) \cap (A \setminus M(v)) = (A \setminus M(x)) \cap (A \setminus M(y))$$

$$= A \setminus (M(x) \cup M(y)) = A \setminus M(x \vee y).$$

Since $x \vee y \in P$, $X \cap Y \in A(\mathscr{U})$ and so $A(\mathscr{U})$ has the finite intersection property. Finally, let $X \subseteq A$, and suppose that $u = \chi(X)$ and $v = \chi(A \setminus X)$. Then $u \wedge v = 0$; so

$u \in P$ or $v \in P$. Therefore $A \setminus X \in A(\mathscr{U})$ or $X \in A(\mathscr{U})$ and so $A(\mathscr{U})$ is an ultrafilter on A.

Next, we show that $\mathscr{C}_P$ is a compatible system of ultrafilters on $\mathfrak{A}$. Let $A, B \in \mathfrak{A}$ with $A \leqslant B$. We need to show that $B(\mathscr{U}) = B^{\leftarrow}(A(\mathscr{U}))$. Suppose (by way of contradiction) that there is $B \setminus X \in B(\mathscr{U})$ with $X \in B^{\leftarrow}(A(\mathscr{U}))$. Let $u = \chi(X)$ and $v = \chi(B \setminus X)$. Then $u \wedge v = 0$ and so we may assume that $u \in P$. Let

$$U = \bigcup \{W \in A(\mathscr{U}) \colon B^{\leftarrow}(W) = X\}.$$

Then $U \in A(\mathscr{U})$ and if $W = \chi(U)$, then $0 < w \leqslant u$ and so $w \in P$. Since $U \in A(\mathscr{U})$, by the argument above there exists $x \in P^{+}$ so that $A \setminus U = M(x)$. But then $x \vee w \in P$ and $M(x \vee w) = A$, which is impossible, since P is a minimal prime. Therefore, $\mathscr{C}_P$ is a compatible system of ultrafilters on $\mathfrak{A}$.

Now, let $\mathscr{C} = \{A(\mathscr{U}) \colon A \in \mathfrak{A}\}$ be a compatible system of ultrafilters on $\mathfrak{A}$; we shall show that $P_{\mathscr{C}}$ is a minimal prime. Let

$$Q = \{v \in V \colon \text{for all } A \in \mathfrak{A} \text{ with } M(v) \subseteq A, A \setminus M(v) \in A(\mathscr{U})\}.$$

We will simplify the computations which follow by first showing that $P = Q$. Clearly $P \subseteq Q$. Suppose by way of contradiction that $v \in Q^{+} \setminus P$. Then there is a $B \in \mathfrak{A}$ with $B \setminus M(v) \notin B(\mathscr{U})$. Since $B(\mathscr{U})$ is an ultrafilter, $B \cap M(v) \in B(\mathscr{U})$. Let $A \in \mathfrak{A}$ be such that $M(v) \subseteq A$. Therefore $B \cap M(v) \subseteq A$ and so $B \cap M(v) \in (A \vee B)(\mathscr{U})$. Since $v \in Q$, $X = A \setminus M(v) \in A(\mathscr{U})$; so $(A \vee B)^{\leftarrow}(X) \in (A \vee B)(\mathscr{U})$. However,

$$(B \cap M(v)) \cap ((A \vee B)^{\leftarrow}(X)) = \varnothing,$$

which is impossible since $(A \vee B)(\mathscr{U})$ is an ultrafilter. Therefore, $Q = P$. Since $\mathscr{C}$ is a compatible system of ultrafilters,

$$P = \{v \in V \colon \text{there exists } A \in \mathfrak{A} \text{ with } A \supseteq M(v) \text{ and } A \setminus M(v) \in A(\mathscr{U})\}.$$

With this simplification of the definition of P, we will proceed with the proof.

P is a subgroup. Let $u, v \in P$ and let $x = u + v$. Let $A, B, C \in \mathscr{U}$ be such that $M(x) \subseteq A$, $M(u) \subseteq B$ and $M(v) \subseteq C$. By replacing A by $A \wedge (B \vee C)$ we may assume that $A \leqslant B \vee C$. Since $u, v \in P$, $(B \vee C) \setminus M(u) \in (B \vee C)(\mathscr{U})$ and $(B \vee C) \setminus M(v) \in (B \vee C)(\mathscr{U})$. Therefore,

$$(B \vee C) \setminus (M(u) \cup M(v)) \in (B \vee C)(\mathscr{U}).$$

Let $X = (B \vee C)^{\leftarrow}(M(x))$. Then $X \subseteq M(u) \cup M(v)$ and so $X \notin (B \vee C)(\mathscr{U})$. Therefore $M(x) \notin A(\mathscr{U})$ and thus $x \in P$. Since $M(x) = M(-x)$, $x \in P$ implies that $-x \in P$. Therefore P is a subgroup.

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P is convex. Suppose $0<x<u$ and $u \in P$. Let $A, B \in \mathscr{U}$ be chosen so that $M(x) \subseteq A$ and $M(u) \subseteq B$. Without loss of generality, $A \leqslant B$. Since $B^{\leftarrow}(M(x)) \subseteq M(u)$, $B^{\leftarrow}(M(x)) \notin B(\mathscr{U})$ and thus $M(x) \notin A(\mathscr{U})$. Therefore $x \in P$.

P is a minimal prime. Since $M(u) = M(|u|)$, u in P implies that $|u| \in P$. Since P is a convex subgroup, this means that P is an l-subgroup. Let $u, v \in V$ be chosen so that $u \wedge v = 0$. Pick $A \in \mathfrak{A}$ so that $M(u) \cup M(v) \subseteq A$. Since $M(u) \cap M(v) = \varnothing$, $A \setminus M(u)$ or $A \setminus M(v)$ is an element of $A(\mathscr{U})$. Thus u or v is in P and so P is prime. A similar argument will show that $v \in P^{+}$ implies the existence of $u \notin P$ such that $u \wedge v = 0$; thus P is a minimal prime.

3. The structure of $P(\Delta)$

Let $P(\Delta)$ be the set of prime l-ideals of $V(\Delta, \mathbf{R})$, and $m(\Delta)$ the set of minimal primes of V. From Section 2, we know that $m(\Delta)$ is order-isomorphic to $\bar{m}(\Delta)$, the set of compatible systems of ultrafilters on $\mathfrak{A}$, and so is completely determined in terms of Δ. For each $P \in P(\Delta)$ let $m(P) = \{Q \in m(\Delta)\colon Q \subseteq P\}$ and for $P, Q \in P(\Delta)$, let $P \approx Q$ if and only if $m(P) = m(Q)$. This is clearly an equivalence relation on $P(\Delta)$. Let $S(\Delta) = P(\Delta)/\approx$; this root system is called the *skeleton* of $P(\Delta)$.

A *branch point* of a root system Γ is an element η of Γ so that $\eta = \alpha \vee \beta$ for some pair of incomparable elements α, β of Γ. Therefore, $S(\Delta)$ is obtained from $P(\Delta)$ by identifying all elements of $P(\Delta)$ strictly between two adjacent branch points with the smaller branch point. Consequently, each σ in $S(\Delta)$ is a totally ordered set. We let $P_\sigma = \bigcap \{Q\colon Q \in \sigma\}$. This is the smaller branch point and hence is the minimal element of σ. (Notice that $\bigcup \{Q\colon Q \in \sigma\}$ need not be an element of σ.) Thus $\sigma \to P_\sigma$ is a natural embedding of $S(\Delta)$ into $P(\Delta)$ which takes P to P for each minimal prime P.

Our next step in the identification of $P(\Delta)$ in terms Δ is the identification of the skeleton in those terms. To this end, we need a way to determine when a collection of minimal primes is contained in a proper prime of V. The following theorem gives the technique which we will use:

THEOREM 3.1. *Let $\{P_\varphi\colon \varphi \in \Phi\}$ be a collection of minimal prime l-ideals of V. For each φ, let $\mathscr{C}_\varphi$ be the compatible system of ultrafilters corresponding to P_φ and denote the ultrafilters on $A \in \mathfrak{A}$ belonging to $\mathscr{C}_\varphi$ by $A(\mathscr{C}_\varphi)$. Then there exists a proper prime Q containing $\bigcup \{P_\varphi\colon \varphi \in \Phi\}$ if and only if there exists $A \in \mathfrak{A}$ so that $A(\mathscr{C}_\varphi) = A(\mathscr{C}_\eta)$ for all $\varphi, \eta \in \Phi$.*

PROOF. First, suppose that $Q \supseteq \bigcup \{P_\varphi\colon \varphi \in \Phi\}$. Choose $x \in V^{+} \setminus Q$ and $A \in \mathfrak{A}$ so that $M(x) \subseteq A$. We claim that $A(\mathscr{C}_\varphi) = A(\mathscr{C}_\eta)$ for all $\varphi, \eta \in \mathfrak{A}$. Suppose by way of contradiction that there exist $\varphi, \eta \in \Phi$ so that $A(\mathscr{C}_\varphi) \neq A(\mathscr{C}_\eta)$. Then there exists

$X \subset A$ so that $X \in A(\mathscr{C}_\varphi)$, while $A \setminus X \in A(\mathscr{C}_\eta)$. Define $u, v \in V$ as follows:

$$u(\gamma) = \begin{cases} 0, & \gamma \in (\Delta \setminus A) \cup X \\ 2x(\gamma), & \gamma \in M(x) \setminus X \\ 1, & \gamma \in A \setminus (X \cup M(x)), \end{cases}$$

$$v(\gamma) = \begin{cases} 0, & \gamma \in \Delta \setminus X, \\ 2x(\gamma), & \gamma \in M(x) \cap X, \\ 1, & \gamma \in X \setminus M(x). \end{cases}$$

Since $M(u) = X \subseteq A$, $A \setminus M(u) = A \setminus X \in A(\mathscr{C}_\varphi)$ and so $u \in P_\varphi$. Similarly, $v \in P_\eta$. But then $u, v \in Q$ and so $u \vee v \in Q$. But $u \vee v \geqslant x$, which is not an element of Q, which is a contradiction. Thus, $A(\mathscr{C}_\varphi) = A(\mathscr{C}_\eta)$ for all $\varphi, \eta \in \Phi$.

Conversely, suppose that there exists $A \in \mathfrak{A}$ with $A(\mathscr{C}_\varphi) = A(\mathscr{C}_\eta)$ for all $\varphi, \eta \in \Phi$. Let

$$S = \{v \in V\colon \gamma \in M(v) \text{ implies that there exists } \delta \in A \text{ with } \delta > \gamma\}.$$

Then S is a convex l-subgroup of V. Let $\varphi \in \Phi$ and let Q be the convex l-subgroup of V generated by S and P_φ. Since $Q \supset P_\varphi$, Q is prime. Let $\eta \in \Phi$ with $\eta \neq \varphi$, and choose $x \in P_\eta^+$. Pick $B \in \mathfrak{A}$ so that $M(x) \subseteq B$. Since $\mathscr{C}_\varphi$ and $\mathscr{C}_\eta$ are compatible systems of ultrafilters and $A(\mathscr{C}_\varphi) = A(\mathscr{C}_\eta)$, then $(A \vee B)(\mathscr{C}_\varphi) = (A \vee B)(\mathscr{C}_\eta)$. Define $v \in V$ as follows:

$$v(\gamma) = \begin{cases} x(\gamma) & \text{if there exists } \delta \in (A \vee B) \cap M(x) \text{ with } \delta \geqslant \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$M(v) \subseteq (A \vee B) \cap M(x) \quad \text{and} \quad (A \vee B) \setminus M(v) = (A \vee B) \setminus M(x).$$

Since $(A \vee B) \setminus M(x) \in (A \vee B)(\mathscr{C}_\varphi)$, $v \in P_\varphi$. Clearly, $x - v \in S$. Thus $x \in Q$ and so $P_\eta \subseteq Q$. Since $\chi(A) \notin Q$, Q is a proper prime of V which contains $\bigcup \{P_\varphi\colon \varphi \in \Phi\}$.

For $A \in \mathfrak{A}$, $\mathscr{C}_1, \mathscr{C}_2 \in \bar{m}(\Delta)$, we define $\mathscr{C}_1 \sim_A \mathscr{C}_2$ if $A(\mathscr{C}_1) = A(\mathscr{C}_2)$. This is an equivalence relation on $\bar{m}(\Delta)$. Notice that if $B \geqslant A$, $A, B \in \mathfrak{A}$, then $\mathscr{C}_1 \sim_A \mathscr{C}_2$ implies that $\mathscr{C}_1 \sim_B \mathscr{C}_2$. Given $\mathscr{C} \in \bar{m}(\Delta)$, let $[\mathscr{C}]_A$ denote the equivalence class of $\mathscr{C}$ under $\sim_A$. Let

$$\bar{S}(\Delta) = \{\sigma \subseteq \bar{m}(\Delta)\colon \text{for all } \mathscr{C} \in \bar{m}(\Delta) \setminus \sigma, \text{ there exists } A \in \mathfrak{A} \text{ and } \mathscr{D} \in \bar{m}(\Delta) \text{ so that } [\mathscr{D}]_A \supseteq \sigma \text{ and } \mathscr{C} \notin [\mathscr{D}]_A\}.$$

Partially order $\bar{S}(\Delta)$ by set inclusion. For $[P] \in S(\Delta)$, let

$$f([P]) = \{\mathscr{C}_Q \in \bar{m}(\Delta)\colon Q \in m(P)\}$$

and if $\sigma \in \bar{S}(\Delta)$, let $g(\sigma) = [P_\sigma]$, where $P_\sigma = \bigcap \{P \in P(\Delta)\colon P \supseteq P_{\mathscr{C}}, \text{ for all } \mathscr{C} \in \sigma\}$.

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THEOREM 3.2. *f is an order isomorphism of $S(\Delta)$ onto $\bar{S}(\Delta)$ with inverse g.*

PROOF. We first show that $f([P]) \in \bar{S}(\Delta)$. Let $\mathscr{C} \in \bar{m}(\Delta) \setminus f([P])$. Since $P_{\mathscr{C}} \nsubseteq P$, there exists $x \in P_{\mathscr{C}}^{+} \setminus P$. Choose $A \in \mathfrak{A}$ so that $M(x) \subseteq A$. By the proof of Theorem 3.1, $[\mathscr{C}_Q]_A \supseteq f([P])$, for each $Q \in m(P)$. Let $Q \in m(P)$. Since $x \notin Q$, there exists $B \in \mathfrak{A}$ so that $B \setminus M(x) \notin B(\mathscr{C}_Q)$. Therefore $M(x) \cap B \in B(\mathscr{C}_Q)$. Since $M(x) \subseteq A$ and

$$M(x) \notin A(\mathscr{C}), \quad A(\mathscr{C}) \neq A(\mathscr{C}_Q).$$

Thus, $\mathscr{C} \notin [\mathscr{C}_Q]_A$ and so $f([P]) \in \bar{S}(\Delta)$.

Clearly $gf([P]) = [P]$ and both f and g preserve order; it remains to show that $fg(\sigma) = \sigma$. We need only check that if $\sigma \in \bar{S}(\Delta)$, then $m(P_\sigma) = \{P_{\mathscr{C}} : \mathscr{C} \in \sigma\}$. One containment is clear. Suppose (by way of contradiction) that $P_\sigma \supseteq Q$ for some $Q \in m(\Delta)$ with $\mathscr{C}_Q \notin \sigma$. Then there is an $A \in \mathfrak{A}$ and $\mathscr{D} \in \bar{m}(\Delta)$ so that $[\mathscr{D}]_A \supseteq \sigma$ but $\mathscr{C}_Q \notin [\mathscr{D}]_A$. By the proof of Theorem 3.1, there is a prime $N \supset \bigcup \{P_{\mathscr{C}} : \mathscr{C} \in \sigma\}$ with $Q \nsubseteq N$. Therefore $N \subset P_\sigma$ which contradicts the definition of

$$P_\sigma = \bigcap \{P : P \supseteq P_{\mathscr{C}}, \mathscr{C} \in \sigma\}.$$

We have now seen that the skeleton $S(\Delta)$ is describable entirely in terms of Δ. It now remains to describe the primes in each $[P] \in S(\Delta)$.

Let $[P] \in S(\Delta)$ and suppose $\sigma = f([P]) \subseteq \bar{m}(\Delta)$. Define

$$B(\sigma) = \{A \in \mathfrak{A} : \mathscr{C}_1 \sim_A \mathscr{C}_2, \text{ for all } \mathscr{C}_1, \mathscr{C}_2 \in \sigma\}.$$

Notice that if $A \in B(\sigma)$ and $B \geqslant A$, then $B \in B(\sigma)$. For notational convenience, for each $B \in B(\sigma)$, let $B(\mathscr{U}) = B(\mathscr{U}_{P_{\mathscr{C}}})$ where $\mathscr{C}$ is any element of σ. (This is possible by the definition of $B(\sigma)$.) For $A, B \in B(\sigma)$, let $A^B = \{\alpha \in A : \alpha < \beta \in B\}$, and let $A_B = \{\alpha \in A : \alpha > \beta \in B\}$. (Another description of A^B is $A^B = ((A \wedge B) \cap A) \setminus (A \cap B)$). Now, $A = A^B \cup A_B \cup (A \cap B)$ and precisely one of these sets is in $A(\mathscr{U})$. Define $A \sim B$ if $A \cap B \in A(\mathscr{U})$ $(A, B \in B(\sigma))$. This is an equivalence relation on $B(\sigma)$. If $A^B \in A(\mathscr{U})$, then we write $[A] \prec [B]$, where the brackets denote the equivalence class under $\sim$. A routine computation shows that this relation is well defined and forms a total order on $B(\sigma)/\sim$.

LEMMA 3.3. *If $A, B \in B(\sigma)$ with $[A] \succcurlyeq [B]$, then $A \vee B \sim A$. Therefore, if $[A] \succcurlyeq [B]$, we may assume that $A \geqslant B$.*

PROOF. Suppose that $[A] \succcurlyeq [B]$. Now, $(A \vee B) \cap A \supseteq A_B \cup (A \cap B)$. If $[A] \succ [B]$, then $A_B \in A(\mathscr{U})$; if $[A] = [B]$, then $A \cap B \in A(\mathscr{U})$. In either case, $(A \vee B) \cap A \in A(\mathscr{U})$ and so $A \vee B \sim A$.

For each $A \in B(\sigma)$, let

$$P_A = \{v \in V : \text{for all } B \succcurlyeq A, \; B \setminus M(v) \in B(\mathscr{U})\}.$$

LEMMA 3.4. *P_A is a convex l-subgroup of V containing P_σ.*

PROOF. Let $S = \{v \in V: \gamma \in M(v) \Rightarrow$ there exists $\delta \in A$ with $\delta > \gamma\}$ and let $\mathscr{C} \in \sigma$. A routine argument shows that P_A is the convex l-subgroup generated by S and $P_\mathscr{C}$, and the proof of Theorem 3.1 shows that $P_A \supseteq \bigcup \{P_\mathscr{C}: \mathscr{C} \in \sigma\}$. Since P_σ is the intersection of all such $P_\mathscr{C}$, $P_A \supseteq P_\sigma$.

PROPOSITION 3.5. *Let $A, B \in B(\sigma)$. Then*
(i) *$P_B = P_A$ if and only if $[B] = [A]$.*
(ii) *$P_B \subset P_A$ if and only if $[B] \prec [A]$.*

PROOF. We will first show that if $[A] = [B]$ and $B \geqslant A$, then $P_A = P_B$. By the definition of P_A, $P_A \subseteq P_B$. Let $D \geqslant A$ and let $x \in V$ be chosen so that

$$D \cap M(x) \in D(\mathscr{U})$$

(that is, $x \notin P_A$). Since $[A] = [B]$, $A \cap B \in A(\mathscr{U})$ and since $D \geqslant A$, $D^{\leftarrow}(A \cap B) \in D(\mathscr{U})$. But then $D^{\leftarrow}(A \cap B) \cap M(x) \in D(\mathscr{U})$. Since

$$D^{\leftarrow}(A \cap B) \subseteq B \vee D, \quad D^{\leftarrow}(A \cap B) \cap M(x) \in (B \vee D)(\mathscr{U}).$$

Therefore $x \notin P_B$, and so $P_A = P_B$.

If $[B] \succcurlyeq [A]$, we may assume that $B \geqslant A$, by Lemma 3.3 and the above. Then $P_B \supseteq P_A$, by definition. This shows that if $P_B \subset P_A$, then $[B] \prec [A]$. Now, suppose that $[B] \prec [A]$. We may assume that $B < A$, and so $A = A_B \cup (A \cap B)$. Since $[B] \prec [A]$, $B^A \in B(\mathscr{U})$. Therefore, $\chi(B^A) \notin P_B$. Since $C \setminus M(\chi(B^A)) = C$ for all $C \geqslant A$, $\chi(B^A) \in P_A \setminus P_B$. Because P_A and P_B are comparable, $P_A \supset P_B$.

Finally, if $[A] \neq [B]$, then without loss of generality $[A] \prec [B]$. Consequently, by part (ii), $P_A \neq P_B$ and so $P_A = P_B$ implies that $[A] = [B]$.

PROPOSITION 3.6. $P_\sigma = \bigcap \{P_A: A \in B(\sigma)\}$.

PROOF. Let $0 < v \in \bigcap \{P_A: A \in B(\sigma)\}$. If $M(v) \subseteq A \in B(\sigma)$, then $v \in P$ for all $\mathscr{C} \in \sigma$. Thus $v \in P_\sigma$. If $M(v) \subseteq A \notin B(\sigma)$, then there exist $\mathscr{C}_1, \mathscr{C}_2 \in \sigma$ such that $\mathscr{C}_1 \nsim_A \mathscr{C}_2$. Thus, there exists $X \subset A$ so that $X \in A(\mathscr{C}_1)$ and $A \setminus X \in A(\mathscr{C}_2)$. Define $w_1, w_2 \in V$ as follows:

$$w_1(\gamma) = \begin{cases} v(\gamma) & \text{if } \gamma \in X, \\ 0 & \text{if } \gamma \in \Delta \setminus X, \end{cases}$$

$$w_2(\gamma) = \begin{cases} v(\gamma) & \text{if } \gamma \in A \setminus X, \\ 0 & \text{if } \gamma \in \Delta \setminus (A \setminus X). \end{cases}$$

Then $A \setminus M(w_1) \supseteq A \setminus X \in A(\mathscr{C}_2)$ and so $w_1 \in P_{\mathscr{C}_2}$; similarly $w_2 \in P_{\mathscr{C}_1}$. Therefore

$w_1+w_2 \in P_\sigma$. Since $w_1+w_2 \geq v > 0$, $v \in P_\sigma$. Thus $P_\sigma \supseteq \bigcap \{P_A: A \in B(\sigma)\}$. The other containment follows from Lemma 3.4.

PROPOSITION 3.7. *Suppose $A, B \in B(\sigma)$ and $A \sim B$. Let*

$$Q_A = \{f \in \Pi_A \mathbf{R}: A \setminus S(f) \in A(\mathscr{U})\}$$

and

$$Q_B = \{f \in \Pi_B \mathbf{R}: B \setminus S(f) \in B(\mathscr{U})\}.$$

(These are minimal primes of $\Pi_A \mathbf{R}$ and $\Pi_B \mathbf{R}$ respectively.) Then $\Pi_A \mathbf{R}/Q_A$ and $\Pi_B \mathbf{R}/Q_B$ are isomorphic o-groups.

PROOF. We define $\mu: \Pi_A \mathbf{R}/Q_A \to \Pi_B \mathbf{R}/Q_B$ as follows: Given $Q_A + v \in \Pi_A \mathbf{R}/Q_A$, define $w \in \Pi_B \mathbf{R}$ by

$$w(\beta) = \begin{cases} v(\alpha) & \text{if } \beta \leq \alpha \in A,\ \beta \in B^A \cup (A \cap B), \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\mu(Q_A + v) = Q_B + w$.

First we show that μ is well defined: If $v \in Q_A$, then $A \setminus S(v) \in A(\mathscr{U})$. Now,

$$\begin{aligned} B \setminus S(w) &= B \setminus \{\beta \in B: \beta \leq \alpha \in S(v)\} \\ &= \{\beta \in B: \beta > \alpha \in A\} \cup \{\beta \in B: \beta \leq \alpha \in A \setminus S(v)\} \\ &\supseteq \{\beta \in B: \beta > \alpha \in A \setminus S(v)\} \cup \{\beta \in B: \beta \leq \alpha \in A \setminus S(v)\}. \end{aligned}$$

Since $A(\mathscr{U})$ and $B(\mathscr{U})$ are compatible ultrafilters and $A \setminus S(v) \in A(\mathscr{U})$,

$$\{\beta \in B: \beta > \alpha \in A \setminus S(v)\} \cup \{\beta \in B: \beta \leq \alpha \in A \setminus S(v)\} \in B(\mathscr{U}).$$

Hence $w \in Q_B$ and so μ is well defined.

We define $\nu: \Pi_B \mathbf{R}/Q_B \to \Pi_A \mathbf{R}/Q_A$ similarly, and claim that $\nu\mu(Q_A + v) = Q_A + v$. By definition, $\nu\mu(Q_A + v) = Q_A + v|_{A \cap B}$ where $v|_{A \cap B}(\gamma) = v(\gamma)$ if $\gamma \in A \cap B$ and is 0 if $\gamma \notin A \cap B$. Therefore, we need to show that $A \setminus S(v - v|_{A \cap B}) \in A(\mathscr{U})$. Since $S(v - v|_{A \cap B}) \subseteq A \setminus (A \cap B)$,

$$A \setminus S(v - v|_{A \cap B}) \supseteq A \setminus (A \setminus A \cap B) = A \cap B \in A(\mathscr{U}).$$

Thus $\nu\mu(Q_A + v) = Q_A + v$. Similarly, $\mu\nu$ is the identity on $\Pi_B \mathbf{R}/Q_B$. Since μ clearly preserves order, μ is an o-isomorphism.

This proposition enables us to define an o-group which we will use to analyze the order structure of $[P] \in S(\Delta)$. Let $\sigma = f([P]) \in \bar{S}(\Delta)$. Then set $G_{[A]} = \Pi_A \mathbf{R}/Q_A$, for each $[A] \in B(\sigma)/\sim$. This is well defined by Proposition 3.7. Let

$$H_\sigma = V(B(\sigma)/\sim, G_{[A]}) = \{k \in \Pi \{G_{[A]}: A \in B(\sigma)/\sim\}: S(k) \text{ satisfies the } ACC\},$$

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where $S(k)$ is given the total order $\prec$ inherited from $B(\sigma)/\sim$, and H has the obvious o-group structure.

THEOREM 3.8. *There exists an o-monomorphism*

$$\iota\colon V/P_\sigma \to H_\sigma$$

so that H_σ *is an a-extension of* $\iota(V/P_\sigma)$.

PROOF. We define $\iota(P_\sigma + v)([A]) = Q_A + v|_A$.

We first show ι is well defined into $\Pi G_{[A]}$. Suppose $v \in P_\sigma$ and $A \in B(\sigma)$. Choose $B \geqslant A$ so that

$$X = \{\beta \in M(v)\colon \beta \geqslant \alpha \in M(v|_A)\} \subseteq B.$$

If $M(v|_A) \in A(\mathscr{U})$, then $X \in B(\mathscr{U})$ since the $A(\mathscr{U})$ and $B(\mathscr{U})$ are compatible and $B^{\leftarrow}(M(v|_A)) = X$. However, $X \cap (B \setminus M(v)) = \varnothing$ and since $v \in P_\sigma$, $B \setminus M(v) \in B(\mathscr{U})$. Consequently, $X \notin B(\mathscr{U})$ and so $M(v|_A) \notin A(\mathscr{U})$. Therefore, $v|_A \in Q_A$ and ι is well defined into $\Pi G_{[A]}$.

We claim that ι is one-to-one into H. Suppose $P_\sigma + v > 0$ and choose $B \supseteq M(v)$. Since $v \notin P_\sigma$, there exists $A \in B(\sigma)$ so that $M(v) \cap A \in A(\mathscr{U})$. Clearly $A \cap M(v) \subset B \vee A$ and since $A(\mathscr{U})$ and $(B \vee A)(\mathscr{U})$ are compatible, $M(v) \cap A \in A(\mathscr{U})$ implies that $A \cap M(v) \in (A \vee B)(\mathscr{U})$. Therefore,

$$(A \vee B) \setminus M(v|_{A \vee B}) \notin (A \vee B)(\mathscr{U}),$$

and so $v|_{A \vee B} \notin Q_{A \vee B}$. Therefore $\iota(P_\sigma + v)([A \vee B])$ is not zero and so ι is one-to-one. We now claim that $[A \vee B]$ is the maximum element of $S(\iota(P_\sigma + v))$ and so $\iota(P_\sigma + v) \in H$. Suppose $[C] \succ [A \vee B]$ where (without loss of generality) $C > A \vee B$. Since $M(v) \subseteq B$, $S(v|_C) \subseteq C \cap (A \vee B)$. Because $[C] \succ [A \vee B]$,

$$X = \{\gamma \in C\colon \gamma > \alpha \in A \vee B\} \in C(\mathscr{U}),$$

and because $C \setminus S(v|_C) \supset X$, $C \setminus S(v|_C) \in C(\mathscr{U})$. Therefore $v|_C \in Q_C$ and so

$$\iota(P_\sigma + v)([C]) = Q_C + 0.$$

Thus $[C] \notin S(\iota(P_\sigma + v))$.

Finally, since $\iota(V/P_\sigma) \supseteq \Sigma G_{[A]}$, H_σ is an a-extension of $\iota(V/P_\sigma)$.

REMARK. The map ι is independent of which representative of $[A]$ we choose to define the component maps, because of the nature of the isomorphisms

$$\Pi_A \mathbf{R}/Q_A \to \Pi_B \mathbf{R}/Q_B.$$

Now suppose that $\sigma \in \bar{S}(\Delta)$. Let

$$A(\sigma) = B(\sigma) \setminus \bigcup \{B(\tau)\colon \tau \in \bar{S}(\Delta) \text{ and } \tau \supset \sigma\}$$

and let

$$M_\sigma = \bigcap \{P_A : A \in B(\sigma) \setminus A(\sigma)\} = \bigcap \{P_\tau : \tau \in \bar{S}(\Delta), \tau \supset \sigma\}.$$

If there exists a smallest τ so that $\tau \supset \sigma$, then $M_\sigma = P_\tau$ and $M_\sigma \notin [P_\sigma]$. If no such τ exists, then M_σ is the largest element of the chain $[P_\sigma]$. In particular, if $A(\sigma) = \varnothing$, then $M_\sigma = P_\sigma$ and so $[P_\sigma]$ is a singleton.

Thus, the elements of $[P_\sigma]$ are in a one-to-one correspondence with $\mathscr{C}(M_\sigma/P_\sigma)$, the convex subgroups of M_σ/P_σ, except possibly for the existence of a largest element as specified above. However, M_σ/P_σ is the convex subgroup of V/P_σ which corresponds to $V(A(\sigma)/\sim, G_{[A]})$ under the a-extension of Theorem 3.8.

Thus, we now need a way of describing the convex subgroup structure of the Hahn group $V(\Gamma, G_\gamma)$ where Γ is a totally ordered set and each G_γ is an o-group with $\mathscr{C}(G_\gamma)$ its set of convex subgroups. Let

$$\mathscr{G} = \bigcup \{\{\gamma\} \times \mathscr{C}(G_\gamma) : \gamma \in \Gamma\}.$$

If $\alpha, \beta \in \Gamma$ with α covering β, we will identify $(\alpha, 0)$ with (β, G_β). Call $\mathscr{G}$ modulo this equivalence relation $\mathscr{H}$ and order it lexicographically with the first component dominating. Clearly $\mathscr{C}(V(\Gamma, G_\gamma))$ is order isomorphic to $\mathscr{H}$.

Thus, we have described $P(\Delta)$ up to the convex subgroup structure of the o-groups $G_{[A]}$. But $G_{[A]} = \Pi_A \mathbf{R}/Q_A$. Now, Q_A is a maximal ring ideal of $\Pi_A \mathbf{R}$, considered as the ring of continuous functions on the discrete space A (see Bigard, Keimel and Wolfenstein (1977), p. 179), and so $G_{[A]}$ is a real-closed η_1-field (see Gillman and Jerison (1960)). Now, we claim that $\Gamma(G_{[A]})$ (the values of $G_{[A]}$) is an η_1-set. For, if

$$P_1 \subset P_2 \subset P_3 \dots Q_3 \subset Q_2 \subset Q_1$$

are all values, choose $g_i, h_j \in G^+_{[A]}$ such that P_i is the value of g_i and Q_j is the value of h_j. Then $\{g_i\} < \{h_j\}$ and so there exists $k \in G^+_{[A]}$ with $\{g_i\} < k < \{h_j\}$. Thus, the value of k lies between the P_i's and Q_j's. But $\mathscr{C}(G_{[A]})$ is just the Dedekind–MacNeille completion of $\Gamma(G_{[A]})$, considered as a totally ordered set. Consequently, we have concluded that $\mathscr{C}(G_{[A]})$ is in each case the Dedekind–MacNeille completion of an η_1-set.

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