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ON A SEPARATION THEOREM INVOLVING THE QUASI-RELATIVE INTERIOR

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Abstract We establish two separation theorems in which the classic interior is replaced by the quasi-relative interior.

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1. Introduction

Frequently in infinite-dimensional convex optimization problems the usual methods fail because, for instance, the interior of the positive cone in L^p,

$$C = \{u \in L^p(T, \mu) : u(t) \geqslant 0 \text{ a.e.}\},$$

is empty. For this reason, Borwein and Lewis [**2**] developed the notion of quasi-relative interior of a convex set, which is an extension of the relative interior in finite dimension.

In this paper we wish to establish two separation theorems involving the quasi-relative interior of a convex set.

Before proceeding with the discussion, we present the definitions and the properties that we need for our purposes. In the sequel, X will denote a real locally convex Hausdorff topological vector space and X^* will denote the topological dual space of all continuous linear functionals on X, whose neutral element will be denoted by θ_{X^*}, with $\bar{C}$ being the closure of C.

Given $C \subseteq X$, we define the cone generated by C as $\operatorname{cone}(C) = \{\lambda x : x \in C,\ \lambda \in \mathbb{R},\ \lambda \geqslant 0\}$.

Definition 1.1. A subset C of X is said to be a cone if $\lambda x \in C$, for all $x \in C$ and all $\lambda \geqslant 0$.

* Because of a surprising coincidence of names within our department, we have to point out that the author was born on 4 August 1968.

Definition 1.2. A convex cone C of X is said to be pointed if $C \cap (-C) = \{\theta_X\}$.

Definition 1.3. A convex cone C of X is said to be acute if $\bar{C}$ is pointed.

Definition 1.4. Let C be a convex subset of X. The quasi-relative interior of C, denoted by $\operatorname{qri} C$, is the set of those $x \in C$ for which $\overline{\operatorname{cone}}(C - x)$ is a linear subspace of X.

If C is a convex subset of X with $\operatorname{Int} C \neq \emptyset$, then $\operatorname{qri} C = \operatorname{Int} C$ [**2**]. Moreover, it is easy to note that in $\mathbb{R}^n$ the notions of relative interior and quasi-relative interior coincide.

Now, we wish to recall some useful properties concerning the quasi-relative interior of sets.

Definition 1.5. Let C be a convex subset of X. The normal cone to C at $\bar{x} \in C$ is the set

$$N_C(\bar{x}) := \{\phi \in X^* : \phi(x - \bar{x}) \leqslant 0,\ \forall x \in C\}.$$

Proposition 1.6 (Proposition 2.8 of [2]). *Let C be a convex subset of X and $\bar{x} \in C$. Then $\bar{x} \in \operatorname{qri} C$ if and only if $N_C(\bar{x})$ is a linear subspace of X^*.*

Proposition 1.7 (Proposition 2.12 of [2]). *Let C be a convex subset of X. If $\operatorname{qri} C \neq \emptyset$, then*

$$\overline{\operatorname{qri} C} = \bar{C}.$$

Proposition 1.8 (Lemma 2.9 of [2]). *Let C be a convex subset of X and suppose that $\bar{x} \in \operatorname{qri} C$ and $x \in C$. Then $(1 - \lambda)\bar{x} + \lambda x \in \operatorname{qri} C$, for all $\lambda \in [0, 1[$.*

Proposition 1.9 (Lemma 3.6 of [1]). *Let C and D be two convex subsets of X such that $\operatorname{qri} C \neq \emptyset$ and $\operatorname{qri} D \neq \emptyset$, and let $\lambda \in \mathbb{R}$. Then*

$$\operatorname{qri} C + \operatorname{qri} D \subseteq \operatorname{qri}(C + D), \tag{1.1}$$

$$\lambda \operatorname{qri} C = \operatorname{qri}(\lambda C), \tag{1.2}$$

$$\operatorname{qri}(C \times D) = \operatorname{qri} C \times \operatorname{qri} D. \tag{1.3}$$

Proposition 1.10 (Theorem 3.4 of [1]). *Let C be a convex subset of X such that $\operatorname{qri} C \neq \emptyset$, and let $\Phi \in X^*$. If $\operatorname{Int} \Phi(C) \neq \emptyset$, then*

$$\Phi(\operatorname{qri} C) = \operatorname{Int} \Phi(C).$$

Proposition 1.11. *Let C be a convex subset of X. Then*

$$\operatorname{qri} C = \operatorname{qri}(\operatorname{qri} C).$$

Proof. Obviously, $\operatorname{qri} C \supseteq \operatorname{qri}(\operatorname{qri} C)$. Let $x_0 \in \operatorname{qri} C$. We show that $\operatorname{cone}(C - x_0) = \operatorname{cone}(\operatorname{qri} C - x_0)$. For this purpose, let $z \in \operatorname{cone}(C - x_0)$; then $z = \alpha(x - x_0)$ with $x \in C$ and $\alpha \geqslant 0$. After choosing $\lambda > 1$ it is easy to observe that

$$z = \alpha\lambda\left[\left(1 - \frac{1}{\lambda}\right)x_0 + \frac{1}{\lambda}x - x_0\right].$$

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By Proposition 1.8 we have

$$\left(1-\frac{1}{\lambda}\right)x_0+\frac{1}{\lambda}x\in\operatorname{qri}C$$

and then we obtain $z\in\operatorname{cone}(\operatorname{qri}C-x_0)$. Thus,

$$\overline{\operatorname{cone}}(C-x_0)=\overline{\operatorname{cone}}(\operatorname{qri}C-x_0) \tag{1.4}$$

and then $x_0\in\operatorname{qri}(\operatorname{qri}C)$. □

Before proceeding, we point out that, by (1.4), if $y_0\in X$, trivially one has

$$\operatorname{qri}C-y_0=\operatorname{qri}(\operatorname{qri}C-y_0)$$

and it is also easy to prove that

$$\operatorname{qri}C-y_0=\operatorname{qri}(C-y_0).$$

In particular, if C is an affine set, then $\operatorname{qri}C=C$.

Proposition 1.12. *Let C and D be two convex subsets of X such that $\operatorname{aff}C=\operatorname{aff}D$. Then, if $C\subseteq D$, $\operatorname{qri}C\subseteq\operatorname{qri}D$.*

Proof. Let $x_0\in\operatorname{qri}C$, then $\overline{\operatorname{cone}}(C-x_0)$ is a linear subspace of X and so $\overline{\operatorname{cone}}(C-x_0)=\overline{\operatorname{span}}(C-x_0)$. It is easy to observe that

$$\overline{\operatorname{cone}}(C-x_0)\subseteq\overline{\operatorname{cone}}(D-x_0)\subseteq\overline{\operatorname{span}}(D-x_0).$$

As $\operatorname{aff}C=\operatorname{aff}D$, one easily obtains $\overline{\operatorname{span}}(C-x_0)=\overline{\operatorname{span}}(D-x_0)$. This implies that $\overline{\operatorname{span}}(C-x_0)=\overline{\operatorname{span}}(D-x_0)$ and then $x_0\in\operatorname{qri}D$. □

Proposition 1.13. *If C is a non-trivial convex acute cone, then $\theta_X\notin\operatorname{qri}C$.*

Proof. Arguing by contradiction, let us suppose that $\theta_X\in\operatorname{qri}C$. Then $\overline{\operatorname{cone}}C$ is a linear subspace of X and then, $\bar{C}$ is also a linear subspace of X. Therefore, $\bar{C}\cap(-\bar{C})=\bar{C}$ and this contradicts the fact that C is acute and non-trivial. □

2. Separation theorems

Before proceeding, we point out that, generally, separation between sets can be hard in the infinite-dimensional case working only with the quasi-relative interior. We show two examples.

Example 2.1. Let X be an infinite-dimensional normed vector space and let $\varphi:X\to\mathbb{R}$ be a non-continuous linear functional. Consider the affine set $S:=\{x\in X:\varphi(x)=1\}$. In this case $\operatorname{qri}S=S$ and $\theta_X\notin\operatorname{qri}S$. Anyway θ_X cannot be separated from S; in fact, if there exists $g\in X^*$ such that $g(x)\leqslant 0$ for each $x\in S$, then $g(x)\leqslant 0$ for each $x\in\bar{S}=X$, and so $g=\theta_{X^*}$.

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Example 2.2. Let X be an infinite-dimensional normed vector space and let $V \neq X$ be a dense linear subspace. Let $x_0 \notin V = \operatorname{qri} V$. Also in this case x_0 cannot be separated from V; in fact, if there exists $g \in X^*$ such that $g(x) \leqslant g(x_0)$ for each $x \in V$, then $g(x) \leqslant g(x_0)$ for each $x \in \bar{V} = X$, and so $g = \theta_{X^*}$.

Before proving the main results, we need to establish the following two propositions.

Proposition 2.3. *Let C be a convex subset of X such that $\operatorname{qri} C \neq \emptyset$ and $x_0 \in X$ such that $\overline{\operatorname{cone}}[\operatorname{qri} C - x_0]$ is not a linear subspace of X. Then $\exists g \in X^* \setminus \{\theta_{X^*}\}$ such that $g(x) \leqslant g(x_0)$ for all $x \in C$.*

Proof. First, if $x_0 \in C$, $x_0 \in C \setminus \operatorname{qri} C$. Hence, Proposition 1.6 ensures that $N_C(x_0)$ is not a linear subspace of X^*, which means that $N_C(x_0) \neq \{\theta_{X^*}\}$. Then $\exists g \in N_C(x_0)$ such that $g \neq \theta_{X^*}$; this ensures that $g(x) \leqslant g(x_0)$ for all $x \in C$.

Instead, if $x_0 \in X \setminus C$, we take $A = C - x_0$ and $B = \operatorname{conv}[\operatorname{qri} A \cup \{\theta_X\}]$. It is easy to prove that $\overline{\operatorname{cone}} B = \overline{\operatorname{cone}}[\operatorname{qri} C - x_0]$. This ensures that $\theta_X \in B \setminus \operatorname{qri} B$ and for the previous case we find that $\exists g \in X^* \setminus \{\theta_{X^*}\}$ such that $g(x) \leqslant 0$ for all $x \in B$ and then $g(x) \leqslant g(x_0)$ for all $x \in C$. □

Proposition 2.4. *Let C be a convex subset of X such that $\operatorname{qri} C \neq \emptyset$ and $x_0 \in X$ such that $\operatorname{cone}[\operatorname{qri} C - x_0]$ is acute. Then $\exists g \in X^* \setminus \{\theta_{X^*}\}$ such that $g(x) \leqslant g(x_0)$ for all $x \in C$.*

Proof. First, if $C = \{x_0\}$, then the conclusion holds, taking as g any non-zero continuous linear functional. If $C \neq \{x_0\}$, it is easy to observe that Proposition 1.8 ensures that $\operatorname{qri} C \neq \{x_0\}$ and then the set $V = \operatorname{cone}[\operatorname{qri} C - x_0]$ is a non-trivial acute cone. Obviously, $\theta_X \in V$ and, by Proposition 1.13, $\theta_X \notin \operatorname{qri} V$. Therefore, $\overline{\operatorname{cone}}[\operatorname{qri} C - x_0]$ is not a linear subspace of X and the conclusion follows by Proposition 2.3. □

Now we are able to prove our main result.

Theorem 2.5. *Let S and T be non-empty convex subsets of X with $\operatorname{qri} S \neq \emptyset$, $\operatorname{qri} T \neq \emptyset$ and such that $\overline{\operatorname{cone}}(\operatorname{qri} S - \operatorname{qri} T)$ is not a linear subspace of X. Then there exists $\Phi \in X^* \setminus \{\theta_{X^*}\}$ such that $\Phi(s) \leqslant \Phi(t)$ for all $s \in S$, $t \in T$.*

Proof. Let us consider the convex set $\operatorname{qri} S - \operatorname{qri} T$. By Proposition 1.11 and (1.1), one has

$$\operatorname{qri} S - \operatorname{qri} T = \operatorname{qri}(\operatorname{qri} S) - \operatorname{qri}(\operatorname{qri} T) \subseteq \operatorname{qri}(\operatorname{qri} S - \operatorname{qri} T) \subseteq \operatorname{qri} S - \operatorname{qri} T$$

and then $\operatorname{qri}(\operatorname{qri} S - \operatorname{qri} T) \neq \emptyset$. Since $\overline{\operatorname{cone}}[\operatorname{qri}(\operatorname{qri} S - \operatorname{qri} T)]$ is not a linear subspace of X, by Proposition 2.3, taking $x_0 = \theta_X$, there exists $\Phi \in X^* \setminus \{\theta_{X^*}\}$ such that $\Phi(z) \leqslant 0$ for all $z \in \operatorname{qri} S - \operatorname{qri} T$.

It is easy to observe that the previous fact implies that

$$\sup_{\operatorname{qri} S} \Phi \leqslant \inf_{\operatorname{qri} T} \Phi. \tag{2.1}$$

Now we note that

$$\operatorname{qri} S \subseteq S \subseteq \bar{S} = \overline{\operatorname{qri} S},$$
$$\operatorname{qri} T \subseteq T \subseteq \bar{T} = \overline{\operatorname{qri} T},$$

where we have also made use of Proposition 1.7. So, by a general property of the continuous functions, one has $\sup_{\operatorname{qri} S} \Phi = \sup_S \Phi$, and $\inf_{\operatorname{qri} T} \Phi = \inf_T \Phi$. Therefore, (2.1) ensures that

$$\sup_S \Phi \leqslant \inf_T \Phi.$$

Then Φ is the continuous linear functional that separates S and T. □

Remark 2.6. We observe that, by Proposition 2.4, the previous result continues to hold if we replace the condition that $\overline{\operatorname{cone}}(\operatorname{qri} S - \operatorname{qri} T)$ is not a linear subspace of X with the condition that $\operatorname{cone}(\operatorname{qri} S - \operatorname{qri} T)$ is acute.

Remark 2.7. Now we want to observe that it is not generally true that, if there exists $\Phi \in X^* \setminus \{\theta_{X^*}\}$ separating S and T, then $\overline{\operatorname{cone}}(\operatorname{qri} S - \operatorname{qri} T)$ is not a linear subspace of X (or $\operatorname{cone}(\operatorname{qri} S - \operatorname{qri} T)$ is acute). To show this, we can consider the following simple example.

Let $X = \mathbb{R}^2$, $S = \{(x, y) \in \mathbb{R}^2 : 2x + 3y \geqslant 0\}$ and $T = \{(0, 0)\}$. Obviously, S and T are convex and $\operatorname{qri} T = \{(0, 0)\}$. Moreover, the continuous linear functional $\Phi(x, y) = 2x + 3y$ for all $(x, y) \in \mathbb{R}^2$ separates S and T, but in this case $\overline{\operatorname{cone}}(\operatorname{qri} S - \operatorname{qri} T) = S$ is not a linear subspace of $\mathbb{R}^2$ (and $\operatorname{cone}(\operatorname{qri} S - \operatorname{qri} T) = S$ is not acute).

We note that the sets in Examples 2.1 and 2.2 do not satisfy the hypotheses of Theorem 2.5. In fact the sets $\overline{\operatorname{cone}}(S)$ in Example 2.1 and $\overline{\operatorname{cone}}(V - x_0)$ in Example 2.2 coincide with the entire space X. Moreover, the sets $\operatorname{cone}(S)$ and $\operatorname{cone}(V - x_0)$ are pointed but not acute (and so the hypothesis that the cone is acute cannot be weakened by the hypothesis that the cone is pointed).

Now we wish to state a strict separation theorem.

Theorem 2.8. *Let S and T be non-empty disjoint convex subsets of X such that $\operatorname{qri} S \neq \emptyset$ and $\operatorname{qri} T \neq \emptyset$. Suppose that there exists a convex set $V \subseteq X$ such that $\overline{V - V} = X$, $\theta_X \in \operatorname{qri} V$, and $\overline{\operatorname{cone}}(\operatorname{qri}(S - T) - \operatorname{qri} V)$ is not a linear subspace of X. Then there exists $\Phi \in X^* \setminus \{\theta_{X^*}\}$ such that $\sup_S \Phi < \inf_T \Phi$.*

Proof. We apply Theorem 2.5 to the sets $S - T$ and V. In particular, by (1.1) and (1.2), we obtain

$$\operatorname{qri} S - \operatorname{qri} T \subseteq \operatorname{qri}(S - T)$$

and then $\operatorname{qri}(S - T) \neq \emptyset$. Moreover, by hypothesis, $\overline{\operatorname{cone}}(\operatorname{qri}(S - T) - \operatorname{qri} V)$ is not a linear subspace of X. Therefore, there exists $\Phi \in X^* \setminus \{\theta_{X^*}\}$ such that $\Phi(x - y) \leqslant \Phi(v)$ for each $x \in S$, $y \in T$, $v \in V$. Certainly, we can find $\bar{v} \in V$ such that $\Phi(\bar{v}) \neq 0$. In fact if $\Phi(V) = \{0\}$, we obtain $\Phi(\overline{V - V}) = \{0\}$, that is $\Phi = \theta_{X^*}$. This ensures that $\Phi(V)$

is a real non-degenerate interval and consequently $\mathrm{Int}\,\Phi(V) \neq \emptyset$. By Proposition 1.10, $0 \in \mathrm{Int}\,\Phi(V)$, and hence there exists $\tilde{v} \in V$ such that $\Phi(\tilde{v}) < 0$. Therefore,

$$\sup_S \Phi - \inf_T \Phi \leqslant \Phi(\tilde{v}) < 0,$$

and this completes the proof. □

Remark 2.9. Also in this case, we observe that Theorem 2.8 continues to hold if we replace the condition that $\overline{\mathrm{cone}}(\mathrm{qri}(S - T) - \mathrm{qri}\,V)$ is not a linear subspace of X with the condition that $\mathrm{cone}(\mathrm{qri}(S - T) - \mathrm{qri}\,V)$ is acute.

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