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A STRUCTURAL APPROACH TO NOETHER LATTICES

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0. In this paper we explore the extent to which embedding and isomorphism questions about a Noether lattice $\mathscr{L}$ can be reduced to questions about simpler structures associated with $\mathscr{L}$.

In § 1, we use a variation of Dilworth's congruence approach [**2**] to associate a collection of semi-local Noether lattices with a given Noether lattice $\mathscr{L}$. We show that these semi-localizations determine $\mathscr{L}$ to within isomorphism (Corollary 1.5); thus embedding and isomorphism questions about $\mathscr{L}$ are largely reduced to the semi-local case.

In § 2, we consider the influence on a semi-local Noether lattice $\mathscr{L}$ of the substructure $\partial\mathscr{L}$ consisting of all elements, all of whose associated primes are maximal. Here we find that if $\partial\mathscr{L}$ can be embedded in a semi-local Noether lattice $\mathscr{L}^*$, then $\mathscr{L}$ can be embedded in an extension $\overline{\mathscr{L}}$ of $\mathscr{L}^*$. Further, since $\partial\mathscr{L}$ splits in such a way that each component can be embedded in a localization of $\mathscr{L}$, $\mathscr{L}$ can be embedded in the direct sum of local Noether lattices, each of which is an extension of a localization of $\mathscr{L}$. It follows that embedding problems for $\mathscr{L}$ are largely dependent on the localizations of $\mathscr{L}$. The main tool of this section is that of an A-sequence [**4**]. The collection of all A-sequences in $\mathscr{L}$ is closely related to the A-adic completion of a Noetherian ring.

1. Let $\mathscr{L}$ be a Noether lattice, S a non-empty subset of $\mathscr{L}$, and $A \in \mathscr{L}$. If $A = Q_1 \wedge \ldots \wedge Q_k$ is a normal decomposition of A where Q_i is P_i-primary, we set $A_S = \bigwedge \{Q_i; P_i \leqq X, \text{ for some } X \in S\}$. Since $\{P_i; P_i \leqq X, \text{ for some } X \in S\}$ is an isolated set of primes of A, A_S is well-defined. We also note that $A_S = \bigwedge \{Q_i; P_i \leqq X, \text{ for some } X \in S\}$ is a normal decomposition of A_S, and $(A_S)_S = A_S$. We now set $I_S = I$ and $\mathscr{L}_S = \{B \in \mathscr{L}; B = B_S\}$.

LEMMA 1.1. *The operation $A \to A_S$ has the following properties:*

(1.0) $$A \leqq B \textit{ implies } A_S \leqq B_S,$$

(1.1) $$(A \wedge B)_S = (A_S \wedge B_S)_S,$$

(1.2) $$(A \vee B)_S = (A_S \vee B_S)_S,$$

(1.3) $$(A \cdot B)_S = (A_S \cdot B_S)_S,$$

(1.4) $$(A:B)_S = (A_S:B_S)_S.$$

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The proofs are straightforward modifications of the special case $S = \{D\}$, which may be found in [**2**].

By (1.0), 0_S is a least element for $\mathscr{L}_S$. Since $\mathscr{L}_S$ inherits the ascending chain condition from $\mathscr{L}$, it follows that every family of elements of $\mathscr{L}_S$ has a greatest lower bound in $\mathscr{L}_S$. Consequently, $\mathscr{L}_S$ is a complete lattice.

We denote the greatest lower and least upper bound operations in $\mathscr{L}_S$ by $\wedge_S$ and $\vee^S$, respectively. And we define the product of A and B in $\mathscr{L}_S$ by $A \cdot_S B = (AB)_S$.

LEMMA 1.2. *For elements* $A, B \in \mathscr{L}_S$,

(i) $A \wedge B = (A \wedge B)_S = A \wedge_S B$,

(ii) $(A \vee B)_S = A \vee^S B$,

(iii) $A \cdot_S (B \vee^S C) = (A \cdot_S B) \vee^S (A \cdot_S C)$,

(iv) $A:B = (A:B)_S = A:_S B$.

Proof of (i). $(A \wedge B)_S \leqq A_S$ and $(A \wedge B)_S \leqq B_S$, and so

$$(A \wedge B)_S \leqq A_S \wedge_S B_S = A \wedge_S B.$$

Furthermore, $A \wedge_S B \leqq A$ and $A \wedge_S B \leqq B$, and so

$$A \wedge_S B \leqq A \wedge B \leqq (A \wedge B)_S.$$

This establishes (i).

The remaining identities follow similarly.

Using the relations thus far developed, it is easy to see that $\mathscr{L}_S$ is a Noether lattice: every element is the finite join of elements E_S, where E is principal in $\mathscr{L}$, and elements of this type are principal in $\mathscr{L}_S$. It is also seen that for elements $Q, P \in \mathscr{L}_S$, Q is P-primary in $\mathscr{L}_S$ if, and only if, Q is P-primary in $\mathscr{L}$.

We note that if $A \in \mathscr{L}$ and $S \subseteq \mathscr{L}$, then there is a finite subset T of $\mathscr{L}$ such that $A_T = A_S$. This is so because every prime of A_S is a prime of A, and $(A_S)_T = A_T$. Also, if $A_T = A_S$ and $T \subseteq U \subseteq S$, then $A_U = A_S$. Hence, if F is any finite subset of $\mathscr{L}$, then S has a finite subset T such that $A_S = A_T$ for all $A \in F$. As a consequence, we have the following lemma.

LEMMA 1.3. *Let A and B be elements of $\mathscr{L}$ and $S \subseteq \mathscr{L}$. Let K be the set of primes associated with any of the elements A_S, B_S, $(A \wedge B)_S$, $(A \vee B)_S$, $(AB)_S$. If T is any subset of S such that each element of K is contained in an element of T, then*

(i) $A_S, B_S \in \mathscr{L}_T$,

(ii) $A_S \wedge_S B_S = A_S \wedge_T B_S$,

(iii) $A_S \vee^S B_S = A_S \vee^T B_S$,

(iv) $A_S \cdot_S B_S = A_S \cdot_T B_S$,

(v) $A_S :_S B_S = A_S :_T B_S$,

Proof. Since each prime of A_S, B_S, $(A \vee B)_S$, $(A \wedge B)_S$, and $(AB)_S$ is

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contained in an element of T, we have that $A_S = A_T$, $B_S = B_T$, $(A \wedge B)_S$, $(A \vee B)_S$, and $(AB)_S$ are elements of $\mathscr{L}_T$. Then, for example,

$$A_S \vee^S B_S = (A_S \vee B_S)_S = (A \vee B)_S = (A \vee B)_T$$
$$= (A_T \vee B_T)_T = A_T \vee^T B_T = A_S \vee^T B_S$$

(Lemma 1.2). The rest of the lemma follows similarly.

We are now in a position to prove the following.

THEOREM 1.4. *Let* $\mathscr{L}$ *and* $\mathscr{L}_*$ *be Noether lattices,* $S \subseteq \mathscr{L}$, *and* ψ *a map of* S *into* $\mathscr{L}_*$. *Assume that, for every finite subset* T *of* S, *there is given a multiplicative lattice morphism* φ_T *of* $\mathscr{L}_T$ *into* $\mathscr{L}_{*\psi(T)}$ *in such a way that* $T_1 \subseteq T_2$ *implies* $\varphi_{T_1} \leqq \varphi_{T_2}$. *Then there is a unique morphism* φ_S *of* $\mathscr{L}_S$ *into* $\mathscr{L}_{*\psi(S)}$ *such that* $\varphi_T \leqq \varphi_S$ *for every finite subset* T *of* S. *Furthermore,*

(i) φ_S *is onto if each map* φ_T *is onto,*
(ii) φ_S *is one-to-one if each map* φ_T *is one-to-one,*
(iii) φ_S *preserves residuals if each map* φ_T *does,*
(iv) φ_S *takes primaries to primaries, primes to primes, and principal elements to principal elements if each map* φ_T *does.*

Proof. Let S_f be the collection of finite subsets of S. Then $\mathscr{L}_S = \bigcup_{T \in S_f} \mathscr{L}_T$, and so the uniqueness of φ_S is immediate. Also, if $A \in \mathscr{L}_{T_1} \cap \mathscr{L}_{T_2}$ and if $T = T_1 \cup T_2$, then $\mathscr{L}_{T_1} \cup \mathscr{L}_{T_2} \subseteq \mathscr{L}_T$ and $\varphi_{T_1}(A) = \varphi_T(A) = \varphi_{T_2}(A)$. Hence we can define φ_S on $\mathscr{L}_S$ by $\varphi_S(A) = \varphi_T(A)$ if $A \in \mathscr{L}_T$ or, equivalently, $\varphi_S(A) = \bigwedge_{T \in S_f} \varphi_T(A_T)$. Then, given $A, B \in \mathscr{L}_S$, there is only a finite number of primes associated with A, B, $(A \vee B)_S$, and $(AB)_S$, and so we can choose a finite subset T_1 of S such that each prime of A, B, $(A \vee B)_S$, and $(AB)_S$ is contained in an element of T_1. Similarly, we can choose a finite subset T_2 of S so that each prime of $\varphi_S(A)_{\psi(S)}, \varphi_S(B)_{\psi(S)}$, $(\varphi_S(A) \vee \varphi_S(B))_{\psi(S)}$, and $(\varphi_S(A)\varphi_S(B))_{\psi(S)}$ is contained in an element of $\psi(T_2)$. Set $T = T_1 \cup T_2$. Then by Lemma 1.3, $\varphi_S(A \vee^S B) = \varphi_T(A \vee^T B) = \varphi_T(A) \vee^{\psi(T)} \varphi_T(B) = \varphi_S(A) \vee^{\psi(T)} \varphi_S(B)$, and similarly for $A \wedge_S B$, $A \cdot {}_S B$. Hence φ_S is a morphism of $\mathscr{L}_S$ into $\mathscr{L}_{*\psi(S)}$. It is immediate that φ_S is one-to-one if each map φ_T is one-to-one, and also that φ_S is onto if each φ_T is onto. If each φ_T preserves residuals (i.e., $\varphi_T(A :_T B) = \varphi_T(A) :_{\psi(T)} \varphi_T(B)$), then φ_S preserves residuals by Lemma 1.3. Since the primaries and primes of $\mathscr{L}_S$ are the primaries and primes of $\mathscr{L}$ which are elements of $\mathscr{L}_S$, it is clear that φ_S preserves primes and primaries if each φ_T does.

Now, assume that each φ_T preserves principal elements. Let E be principal in $\mathscr{L}_S$. Then E_T is principal in $\mathscr{L}_T = (\mathscr{L}_S)_T$, and thus $\varphi_T(E_T)$ is principal in $\mathscr{L}_{*\psi(T)}$. From this we conclude that $\varphi_S(E)$ is principal in $\mathscr{L}_{*\psi(S)}$ (Lemma 1.3).

Let $\mathscr{M}(\mathscr{L})$ denote the set of all maximal elements of $\mathscr{L}$.

COROLLARY 1.5. *Let* $\mathscr{L}$ *and* $\mathscr{L}_*$ *be Noether lattices and* ψ *a map of* $\mathscr{M}(\mathscr{L})$ *onto* $\mathscr{M}(\mathscr{L}_*)$. *Assume that for each finite subset* S *of* $\mathscr{M}(\mathscr{L})$ *there is a morphism*

φ_S of $\mathscr{L}_S$ into $\mathscr{L}_{*\psi(S)}$ in such a way that $S_1 \subseteq S_2$ implies $\varphi_{S_1} \leqq \varphi_{S_2}$. Then there is a unique morphism φ of $\mathscr{L}$ into $\mathscr{L}_*$ such that $\varphi_S \leqq \varphi$ for all $S \in \mathscr{M}(\mathscr{L})_f$. As in Theorem 1.4, *$\varphi$ inherits the special properties of the maps φ_S. In particular, φ is a Noether lattice embedding (in the sense of* [**1**]) *if each of the maps φ_S is.*

We note that for $S \in \mathscr{M}(\mathscr{L})_f$, $\mathscr{L}_S$ is a semi-local Noether lattice. Hence, a Noether lattice is determined by its semi-localizations.

2. We are now interested primarily in semi-local Noether lattices. For such a Noether lattice $\mathscr{L}$, we let $\partial\mathscr{L}$ denote the subset consisting of I and all elements A such that every associated prime is a maximal element. We use $\partial\mathscr{L}^0$ to denote $\partial\mathscr{L} \cup \{0\}$. Then $\partial\mathscr{L}^0$ is a complete, modular, multiplicative lattice. In this section, we use $\partial\mathscr{L}$ to reduce the embedding problem for a semi-local Noether lattice to the local case. Before we begin, however, we require some definitions.

(2.0). If $\{B_i\}$ is any sequence of elements of $\mathscr{L}$ and $A \in \mathscr{L}$, then $\{B_i\}$ is an A-sequence if, given $n \geqq 1$, it follows that $B_i \vee A^n$ is constant for large i.

(2.1). An A-sequence $\{B_i\}$ is a regular A-sequence if, given n, it follows that $B_i \vee A^n$ is constant for all $i \geqq n$.

(2.2). An A-sequence $\{B_i\}$ is a completely regular A-sequence if $B_{n+1} \vee A^n = B_n$ for all $n \geqq 1$.

(2.3). $\mathscr{L}$ is A-complete if, given any completely regular A-sequence $\{B_i\}$, it follows that $B_n = (\bigwedge_i B_i) \vee A^n$ for all $n \geqq 1$.

If $\{B_i\}$ is any A-sequence and if $C_i = \bigwedge_j(B_j \vee A^i)$, then $\{C_i\}$ is a completely regular A-sequence. This follows since if $B_j \vee A^i$ and $B_j \vee A^{i+1}$ are constant for $j \geqq k$, then $C_i = B_j \vee A^i = (B_j \vee A^{i+1}) \vee A^i = C_{i+1} \vee A^i$.

We note that if $\bigwedge_i(B \vee A^i) = B$ for all $B \in \mathscr{L}$, then a sequence $\{B_i\}$ of elements of $\mathscr{L}$ is an A-sequence if, and only if, $\{B_i\}$ is a Cauchy sequence relative to the metric: $d(D, C) = 1/2^n$ if $D \vee A^n = C \vee A^n$ and $D \vee A^{n+1} \neq C \vee A^{n+1}$.

LEMMA 2.1. *Let A, B, and C be elements of $\mathscr{L}$. Then there is a positive integer k such that $A \wedge (B \vee C^n) \leqq (A \wedge B) \vee AC^{n-k}$ for all $n \geqq k$.*

Proof. By the Artin-Rees Lemma for Noether lattices [**3**],

$$(A \vee B) \wedge (B \vee C^n) \leqq [(A \vee B) \wedge (B \vee C^k)](B \vee C^{n-k}) \vee B,$$

for some k and for all $n \geqq k$. Then

$$A \wedge (B \vee C^n) \leqq (A \vee B) \wedge (B \vee C^n)$$
$$\leqq ((A \wedge (B \vee C^k)) \vee B)(B \vee C^{n-k}) \vee B = (A \wedge (B \vee C^k))C^{n-k} \vee B,$$

and so

$$A \wedge (B \vee C^n) \leqq A \wedge ((A \wedge (B \vee C^k))C^{n-k} \vee B)$$
$$\leqq (A \wedge (B \vee C^k))C^{n-k} \vee (A \wedge B) \leqq (A \wedge B) \vee AC^{n-k}.$$

COROLLARY 2.2. *Let $A_1, \ldots, A_s$ and C be elements of $\mathscr{L}$. Then for some k and for all $n \geqq k$,*

$$\bigwedge_{i=1}^{s} (A_i \vee C^n) \leqq \left(\bigwedge_{i=1}^{s} A_i\right) \vee C^{n-k}.$$

Proof. By induction, we can assume that

$$\bigwedge_{i=1}^{s-1} (A_i \vee C^n) \leqq \left(\bigwedge_{i=1}^{s-1} A_i\right) \vee C^{n-k_1} \qquad \text{for all } n \geqq k_1.$$

By Lemma 2.1, we can choose k_2 such that

$$\begin{aligned}\left(\left(\bigwedge_{i=1}^{s-1} A_i\right) \vee C^{n-k_1}\right) \wedge (A_s \vee C^{n-k_1}) &= \left(\left(\bigwedge_{i=1}^{s-1} A_i\right) \wedge (A_s \vee C^{n-k_1})\right) \vee C^{n-k_1}\\ &\leqq \left(\bigwedge_{i=1}^{s} A_i\right) \vee C^{n-k_1-k_2} \vee C^{n-k_1}\\ &= \left(\bigwedge_{i=1}^{s} A_i\right) \vee C^{n-k_1-k_2}\end{aligned}$$

for all $n \geqq k_1 + k_2$. Thus

$$\begin{aligned}\bigwedge_{i=1}^{s} (A_i \vee C^n) &\leqq \left(\left(\bigwedge_{i=1}^{s-1} A_i\right) \vee C^{n-k_1}\right) \wedge (A_s \vee C^n)\\ &\leqq \left(\left(\bigwedge_{i=1}^{s-1} A_i\right) \vee C^{n-k_1}\right) \wedge (A_s \vee C^{n-k_1})\\ &\leqq \left(\bigwedge_{i=1}^{s} A_i\right) \vee C^{n-k},\end{aligned}$$

for all $n \geqq k = k_1 + k_2$.

COROLLARY 2.3. *Let A, B, and C be elements of $\mathscr{L}$. Then, for some k and all $n \geqq k$,*

$$(A \vee C^n):(B \vee C^n) \leqq (A:B) \vee C^{n-k}.$$

Proof. If B is principal, we choose k such that

$$(A \vee C^n) \wedge B \leqq (A \wedge B) \vee BC^{n-k}$$

for all $n \geqq k$ (Lemma 2.1). Then

$$((A \vee C^n):B)B = (A \vee C^n) \wedge B \leqq (A \wedge B) \vee BC^{n-k} = ((A:B) \vee C^{n-k})B$$

and hence $(A \vee C^n):(B \vee C^n) \leqq (A \vee C^n):B \leqq (A:B) \vee C^{n-k}$, for all $n \geqq k$.

If B is arbitrary, we write B as the join $B = B_1 \vee \ldots \vee B_s$ of principal elements. Then

$$\begin{aligned}(A \vee C^n):(B \vee C^n) = (A \vee C^n):B &= (A \vee C^n):(B_1 \vee \ldots \vee B_s)\\ &= \bigwedge_{i=1}^{s} (A \vee C^n):B_i \leqq \bigwedge_{i=1}^{s} ((A:B_i) \vee C^{n-k_i}),\end{aligned}$$

where k_i is chosen for B_i as above. Let $k' = \max\{k_1, \ldots, k_s\}$. Then

$$\bigwedge_{i=1}^{s} ((A:B_i) \vee C^{n-k_i}) \leqq \bigwedge_{i=1}^{s} ((A:B_i) \vee C^{n-k'}) \leqq \left(\bigwedge_{i=1}^{s} (A:B_i)\right) \vee C^{n-k'-k''},$$

for some k'' and all $n \geqq k' + k''$. Since $\bigwedge_{i=1}^{s}(A:B_i) = A:(\bigvee_{i=1}^{s}B_i) = A:B$, we have $(A \vee C^n):(B \vee C^n) \leqq (A:B) \vee C^{n-k}$, for all $n \geqq k = k' + k''$.

Now, let $\partial_C(\mathscr{L}) = \{A \in \mathscr{L}; A \geqq C^n$, for some $n\}$, so that $\partial_C(\mathscr{L})$ is a sub-multiplicative lattice of $\mathscr{L}$. Let $\mathscr{J}(\mathscr{L})$ denote the greatest lower bound of the collection of maximal elements of $\mathscr{L}$.

THEOREM 2.4. *Let C and C_* be elements of Noether lattices $\mathscr{L}$ and $\mathscr{L}_*$, respectively, and $\partial\varphi$ a morphism of $\partial_C(\mathscr{L})$ into $\partial_{C_*}(\mathscr{L}_*)$ such that $\partial\varphi(C) = C_*$. If $\mathscr{L}_*$ is C_*-complete and $C_* \leqq \mathscr{J}(\mathscr{L}_*)$, then $\partial\varphi$ extends uniquely to a morphism φ of $\mathscr{L}$ into $\mathscr{L}_*$. Furthermore:*

(i) *φ preserves residuals if $\partial\varphi$ preserves residuals;*

(ii) *φ is one-to-one if $\partial\varphi$ is one-to-one and $C \leqq \mathscr{J}(\mathscr{L})$;*

(iii) *If $\partial\varphi$ maps $\partial_C(\mathscr{L})$ onto $\partial_{C_*}(\mathscr{L}_*)$, $\mathscr{L}$ is C-complete, and either $\partial\varphi$ is one-to-one or $\mathscr{L}/C$ is finite-dimensional, then φ maps $\mathscr{L}$ onto $\mathscr{L}_*$.*

Proof. Set $\varphi(A) = \bigwedge_n \partial\varphi(A \vee C^n)$. Then

$$\partial\varphi(A \vee C^{n+1}) \vee C_*^n = \partial\varphi(A \vee C^{n+1}) \vee \partial\varphi(C^n) = \partial\varphi(A \vee C^n),$$

and so $\{\partial\varphi(A \vee C^n)\}$ is a completely regular C_*-sequence in $\mathscr{L}_*$. Since $\mathscr{L}_*$ is C_*-complete, it follows that

$$\varphi(A) \vee C_*^n = \bigwedge_i \partial\varphi(A \vee C^i) \vee C_*^n = \partial\varphi(A \vee C^n)$$

for all n. Then

$$\varphi(A) \vee \varphi(B) \vee C_*^n = \partial\varphi(A \vee C^n) \vee \partial\varphi(B \vee C^n) = \partial\varphi(A \vee B \vee C^n) = \varphi(A \vee B) \vee C_*^n,$$

for all n. Hence, by the intersection theorem and the relation $C_* \leqq \mathscr{J}(\mathscr{L}_*)$, it follows that

$$\varphi(A) \vee \varphi(B) = \bigwedge_n (\varphi(A) \vee \varphi(B) \vee C_*^n) = \bigwedge_n (\varphi(A \vee B) \vee C_*^n) = \varphi(A \vee B).$$

Similarly,

$$(\varphi(A)\varphi(B)) \vee C_*^n = ((\varphi(A) \vee C_*^n)(\varphi(B) \vee C_*^n)) \vee C_*^n = (\partial\varphi(A \vee C^n)\partial\varphi(B \vee C^n)) \vee \partial\varphi(C^n) = \partial\varphi(((A \vee C^n)(B \vee C^n)) \vee C^n) = \partial\varphi(AB \vee C^n) = \varphi(AB) \vee C_*^n,$$

for all n, and thus $\varphi(A)\varphi(B) = \varphi(AB)$.

To see that φ preserves meets, we use Corollary 2.2 to choose k so that $(A \vee C^n) \wedge (B \vee C^n) \leqq (A \wedge B) \vee C^{n-k}$, for all $n \geqq k$. Then

$$\varphi(A) \wedge \varphi(B) \leqq (\varphi(A) \vee C_*^n) \wedge (\varphi(B) \vee C_*^n) = \partial\varphi(A \vee C^n) \wedge \partial\varphi(B \vee C^n)$$
$$= \partial\varphi((A \vee C^n) \wedge (B \vee C^n)) \leqq \partial\varphi((A \wedge B) \vee C^{n-k}) = \varphi(A \wedge B) \vee C_*^{n-k},$$

for all $n \geqq k$. Thus $\varphi(A) \wedge \varphi(B) \leqq \bigwedge_n(\varphi(A \wedge B) \vee C_*^{n-k}) = \varphi(A \wedge B)$. Since φ is clearly isotone, it follows that $\varphi(A \wedge B) = \varphi(A) \wedge \varphi(B)$.

To see that φ preserves residuals if $\partial\varphi$ does, we use Corollary 2.3 to choose k so that $(A \vee C^n):(B \vee C^n) \leqq (A:B) \vee C^{n-k}$, for all $n \geqq k$. Then

$$(\varphi(A):\varphi(B)) \leqq (\varphi(A) \vee C_*^n):(\varphi(B) \vee C_*^n) = \partial\varphi(A \vee C^n):\partial\varphi(B \vee C^n)$$
$$= \partial\varphi((A \vee C^n):(B \vee C^n)) \leqq \partial\varphi((A:B) \vee C^{n-k}) = \varphi(A:B) \vee C_*^{n-k},$$

for all $n \geqq k$. Thus $\varphi(A):\varphi(B) \leqq \bigwedge_n(\varphi(A:B) \vee C_*^{n-k}) = \varphi(A:B)$. Also, since φ is isotone, we have $\varphi(A:B)\varphi(B) = \varphi((A:B)B) \leqq \varphi(A)$, so that $\varphi(A:B) \leqq \varphi(A):\varphi(B)$. Hence $\varphi(A:B) = \varphi(A):\varphi(B)$, if $\partial\varphi$ preserves residuals.

Assume now that $\partial\varphi$ is one-to-one and that $C \leqq \mathscr{J}(\mathscr{L})$. Then $\varphi(A) = \varphi(B)$ implies $\partial\varphi(A \vee C^n) = \varphi(A) \vee C_*^n = \varphi(B) \vee C_*^n = \partial\varphi(B \vee C^n)$, for all n. Hence $A \vee C^n = B \vee C^n$, for all n, and therefore

$$A = \bigwedge_n (A \vee C^n) = \bigwedge_n (B \vee C^n) = B.$$

We now assume that $\mathscr{L}$ is C-complete and that $\partial\varphi$ maps $\partial_C(\mathscr{L})$ onto $\partial_{C_*}(\mathscr{L}_*)$. If $\partial\varphi$ is one-to-one and $D_* \in \mathscr{L}_*$, then for each i there is a unique $D_i \geqq C^i$ such that $\partial\varphi(D_i) = D_* \vee C_*^i$. Further, since

$$\partial\varphi(D_{i+1} \vee C^i) = (D_* \vee C_*^{i+1}) \vee C_*^i = \partial\varphi(D_i),$$

we have that $\{D_i\}$ is a completely regular C-sequence in $\mathscr{L}$. If $D = \bigwedge_i D_i$, then $D \vee C^i = D_i$ for all i, and so

$$\varphi(D) \vee C_*^i = \varphi(D \vee C^i) = \varphi(D_i) = D_* \vee C_*^i,$$

for all i, and therefore $\varphi(D) = \bigwedge_i(\varphi(D) \vee C_*^i) = \bigwedge_i(D_* \vee C_*^i) = D_*$. On the other hand, if $\mathscr{L}/C$ is finite-dimensional, then $\mathscr{L}/C^i$ is finite-dimensional for all i [**3**]. In this case, if $D_* \in \mathscr{L}_*$ we choose $D_i' \geqq C^i$ for each i such that $\varphi(D_i') = D_* \vee C_*^i$, but of course D_i' need not be uniquely determined. Set $\bar{D}_i = \bigwedge_{j \leqq i} D_j'$. Then $\{\bar{D}_i\}$ is a decreasing sequence in $\mathscr{L}$ such that

$$\varphi(\bar{D}_i) = \varphi\left(\bigwedge_{j \leqq i} D_j'\right) = \bigwedge_{j \leqq i} \varphi(D_j') = \bigwedge_{j \leqq i} (D_* \vee C_*^j) = D_* \vee C_*^i.$$

Then by the descending chain condition in $\mathscr{L}/C^i$, it follows that $\{\bar{D}_i\}$ is a C-sequence. Set $D_i = \bigwedge_j(\bar{D}_j \vee C^i)$, and $D = \bigwedge_i D_i$. Then $\{D_i\}$ is a completely regular C-sequence with $\varphi(D_i) = D_* \vee C_*^i$, for all i. Therefore, $D_* \leqq D_* \vee C_*^i = \varphi(D_i) \leqq \varphi(D \vee C^i) = \varphi(D) \vee C_*^i$. Hence

$$D_* \leqq \bigwedge_n (\varphi(D) \vee C_*^n) = \varphi(D) \leqq \bigwedge_n \varphi(D_n) = \bigwedge_n (D_* \vee C^n) = D_*.$$

If $\mathscr{L}$ is a semi-local Noether lattice, then $\mathscr{L}/\mathscr{J}(\mathscr{L})$ is finite-dimensional. Also, in this case, $\partial_{\mathscr{J}}(\mathscr{L}) = \partial(\mathscr{L})$. Hence we have the following result.

COROLLARY 2.5. *Let $\mathscr{L}$ be a semi-local Noether lattice and $\mathscr{L}_*$ an arbitrary Noether lattice. Let $\partial\varphi$ be a morphism of $\partial\mathscr{L}$ into $\mathscr{L}_*$ with $\partial\varphi(\mathscr{J}(\mathscr{L})) = C_* \leqq \mathscr{J}(\mathscr{L}_*)$. If $\mathscr{L}_*$ is C_*-complete, then $\partial\varphi$ extends to a morphism φ of $\mathscr{L}$ into $\mathscr{L}_*$. Further, φ maps $\mathscr{L}$ onto $\mathscr{L}_*$ if $\partial\varphi$ maps $\partial\mathscr{L}$ onto $\partial\mathscr{L}_*$ and $\mathscr{L}$ is $\mathscr{J}$-complete. And φ is one-to-one if $\partial\varphi$ is one-to-one.*

Proof. If $\partial\varphi(\mathscr{J}(\mathscr{L})) = C_*$, then $A \geqq \mathscr{J}^n$ implies $\partial\varphi(A) \geqq \partial\varphi(\mathscr{J}^n) = (\partial\varphi(\mathscr{J}))^n = C_*^n$, and thus $\partial\varphi$ is a morphism of $\partial_{\mathscr{J}}\mathscr{L}$ into $\partial_{C_*}(\mathscr{L})$ with $\partial\varphi(\mathscr{J}) = C_*$.

Hence, a semi-local Noether lattice $\mathscr{L}$ which is $\mathscr{J}(\mathscr{L})$-complete is determined by $\partial\mathscr{L}$. It will be shown later that a semi-local Noether lattice $\mathscr{L}$ is embeddable in a semi-local Noether lattice $\mathscr{L}^*$ which is $\mathscr{J}(\mathscr{L}^*)$-complete and has the property that $\partial\mathscr{L} \cong \partial\mathscr{L}^*$. In fact, if $\mathscr{L}$ is the lattice of ideals of a Noetherian ring R, then $\mathscr{L}^*$ is the lattice of ideals of the completion R^* of R in the Jacobson radical topology. For the present, however, we are interested in the structure of semi-local Noether lattices $\mathscr{L}$ which are $\mathscr{J}(\mathscr{L})$-complete.

LEMMA 2.6. *Let $\mathscr{L}$ be a semi-local Noether lattice which is $\mathscr{J}(\mathscr{L})$-complete, and let P be a maximal prime of $\mathscr{L}$. Then $\mathscr{L}$ is P-complete.*

Proof. Let $\{A_i\}$ be a completely regular P-sequence in $\mathscr{L}$. Then $\{A_i\}$ is decreasing, and so by the descending chain condition in $\mathscr{L}/\mathscr{J}(\mathscr{L})^n$, $\{A_i\}$ is a $\mathscr{J}(\mathscr{L})$-sequence. For each i, set $B_i = \bigwedge_j(A_j \vee \mathscr{J}(\mathscr{L})^i)$. Then $A_j \leqq B_i \leqq A_j \vee \mathscr{J}(\mathscr{L})^i \leqq A_i$, for large j, and thus $\bigwedge_i B_i = \bigwedge_i A_i$. Since $\mathscr{L}$ is $\mathscr{J}(\mathscr{L})$-complete, the result follows.

COROLLARY 2.7. *Let $\mathscr{L}$ be a semi-local Noether lattice which is $\mathscr{J}(\mathscr{L})$-complete. If P is any maximal prime of $\mathscr{L}$, then $\mathscr{L}_P = \mathscr{L}_{\{P\}}$ is P-complete.*

Proof. Let $\{A_i\}$ be any completely regular P-sequence in $\mathscr{L}_P$. Then $\{A_i\}$ is a completely regular P-sequence in $\mathscr{L}$, and so for each n, $A_n = (\bigwedge_i A_i) \vee P^n$. Hence $A_n = (A_n)_P = ((\bigwedge_i A_i) \vee P^n)_P = (\bigwedge_i A_i) \vee^P P^n$. It follows that $\mathscr{L}_P$ is P-complete.

THEOREM 2.8. *Let $\mathscr{L}$ be a semi-local Noether lattice which is $\mathscr{J}(\mathscr{L})$-complete. Let $P_1, \ldots, P_k$ be the maximal primes of $\mathscr{L}$. Then $\mathscr{L}$ is the direct sum of the local Noether lattices $\mathscr{L}_i = \mathscr{L}_{P_i}$.*

Proof. Let A be any element of $\partial\mathscr{L}$. Then A has a decomposition $A = Q_1 \wedge \ldots \wedge Q_k$ where, for each i, either Q_i is P_i-primary or $Q_i = I$. Since each of the primes of A is maximal, it follows that the decomposition is unique and that $A = Q_1 \wedge \ldots \wedge Q_k = Q_1 \ldots Q_k$. Consequently, the map $(Q_1, \ldots, Q_k) \rightarrow Q_1 \wedge \ldots \wedge Q_k$ of $\partial\mathscr{L}_1 \oplus \ldots \oplus \partial\mathscr{L}_k$ to $\partial\mathscr{L}$ is a multiplicative lattice isomorphism.

The maximal primes of $\mathscr{L}^* = \mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_k$ are the elements $(I, \ldots, P_i, \ldots, I)$, thus $\partial\mathscr{L}^* = \partial\mathscr{L}_1 \oplus \ldots \oplus \partial\mathscr{L}_k$ and $\partial\mathscr{L}^* \cong \partial\mathscr{L}$. Also, each component $\mathscr{L}_i$ of $\mathscr{L}^*$ is P_i-complete (Lemma 2.7), and hence $\mathscr{L}^*$ is $\mathscr{J}$-complete. It follows (Corollary 2.5) that $\mathscr{L} \cong \mathscr{L}^* = \mathscr{L}_1 \oplus \ldots \oplus \mathscr{L}_k$.

THEOREM 2.9. *Let $\mathscr{L}$ be a semi-local Noether lattice. Then $\mathscr{L}$ is a sublattice of a Noether lattice $\mathscr{L}_*$ which is semi-local and $\mathscr{J}(\mathscr{L}_*)$-complete, and has the property that $\partial\mathscr{L} \cong \partial\mathscr{L}_*$.*

Proof. Let $P_1, \ldots, P_k$ be the maximal elements of $\mathscr{L}$, and set $\mathscr{L}_i = \mathscr{L}_{P_i}$, $i = 1, \ldots, k$. In [**4**] it was shown that any local Noether lattice $(\mathscr{L}_i, P_i)$ can be embedded in a local Noether lattice $(\mathscr{L}_i^*, P_i^*)$ which is P_i^*-complete in such a way that $\partial\mathscr{L}_i \cong \partial\mathscr{L}_i^*$. We use that result and set

$$\mathscr{L}_* = \mathscr{L}_1^* \oplus \ldots \oplus \mathscr{L}_k^*.$$

It follows that $\mathscr{L}_*$ is $\mathscr{J}(\mathscr{L}_*)$-complete. Also, since

$$\partial\mathscr{L} \cong \partial\mathscr{L}_{P_1} \oplus \ldots \oplus \partial\mathscr{L}_{P_k} \cong \partial\mathscr{L}_1^* \oplus \ldots \oplus \partial\mathscr{L}_k^*,$$

it follows from Corollary 2.5 that $\mathscr{L}$ is embedded in $\mathscr{L}_*$ in the desired way.

THEOREM 2.10. *Let $(\mathscr{L}, P_1, \ldots, P_k)$ and $(\mathscr{L}^*, P_1^*, \ldots, P_k^*, \ldots, P_n^*)$ be semi-local Noether lattices. Assume that, for each $i = 1, \ldots, k$ there is a morphism φ_i of $\mathscr{L}_{P_i}$ into $\mathscr{L}^*_{P_i^*}$. If $\mathscr{L}^*$ is $\mathscr{J}(\mathscr{L}^*)$-complete, then there is a morphism φ of $\mathscr{L}$ into $\mathscr{L}^*$. Further, φ is one-to-one if each φ_i is one-to-one.*

Proof. In this case, there is a natural morphism $\partial\varphi$ of

$$\partial\mathscr{L} = \partial\mathscr{L}_{P_1} \oplus \ldots \oplus \partial\mathscr{L}_{P_k}$$

into $\partial\mathscr{L}^* = \partial\mathscr{L}^*_{P_1^*} \oplus \ldots \oplus \partial\mathscr{L}^*_{P_k^*} \oplus \ldots \oplus \partial\mathscr{L}^*_{P_n^*}$ defined by

$$\partial\varphi(A_1, \ldots, A_k) = (\varphi_i(A_i), \ldots, \varphi_k(A_k), I, I, \ldots, I).$$

The result follows from Corollary 2.5.

REFERENCES

1. K. P. Bogart, *Structure theorems for regular local Noether lattices*, Michigan Math. J. *15* (1968), 167–176.
2. R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math. *12* (1962), 481–498.
3. E. W. Johnson, *A-transforms and Hilbert functions in local lattices*, Trans. Amer. Math. Soc. *137* (1969), 125–140.
4. E. W. Johnson and J. A. Johnson, *M-primary elements of a local Noether lattice*, Can. J. Math. *22* (1970), 327–331.
5. O. Zariski and P. Samuel, *Commutative algebra*, Vol. II, The University Series in Higher Mathematics (Van Nostrand, Princeton, N.J.–Toronto–London–New York, 1960).

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