Bull. Aust. Math. Soc. **94** (2016), 65–69
doi:10.1017/S000497271500146X

AN OPEN MAPPING THEOREM

SAAK S. GABRIYELYAN and SIDNEY A. MORRIS✉

(Received 5 August 2015; accepted 25 October 2015; first published online 8 January 2016)

Abstract

It is proved that any surjective morphism $f : \mathbb{Z}^\kappa \to K$ onto a locally compact group K is open for every cardinal κ. This answers a question posed by Hofmann and the second author.

2010 *Mathematics subject classification*: primary 22A05; secondary 46A30.

Keywords and phrases: pro-Lie group, locally compact abelian group, open mapping theorem.

1. Introduction

In this paper we assume that all topological groups are Hausdorff and abelian. In the literature it is common to ask whether a surjective continuous homomorphism $f : G \to K$ of a topological group G onto a topological group K is an open mapping. Positive results in this direction are known as 'open mapping theorems' in the literature in functional analysis and topological algebra (see, for example, [2, Theorem 2.25] for Banach spaces, [8] for Polish groups and [5, Theorem 9.60] and [4] for pro-Lie groups). Most results of this type impose a countability condition on G. Indeed, if K is any countable nondiscrete group or an infinite compact one and $G := K_d$ is the group K endowed with the discrete topology, then the identity map $i : G \to K$ is not open. Noting that for every uncountable cardinal κ the totally disconnected abelian group $G = \mathbb{Z}^\kappa$ is neither a Polish group nor a locally compact group, Hofmann and the second author posed the following question: *Is a surjective morphism $f : \mathbb{Z}^\kappa \to K$ onto a compact group open for every cardinal κ?* (See [7, Question 5].) We answer this question in the affirmative.

We will use the following notation and terminology. For a topological group K, we denote by K_0 the connected component of the identity. A topological group K is called *almost connected* [5] if the quotient group K/K_0 is compact. A topological group G is called a *pro-Lie group* [5] if it is a closed subgroup of a product of finite-dimensional Lie groups. So, the group $\mathbb{Z}^\kappa$ is a non-almost-connected pro-Lie group. Every compact group is an almost connected pro-Lie group.

We denote by $\mathcal{CDA}$ the class of all abelian groups G with a subgroup topology such that for every open subgroup H of G, the quotient group G/H is countable. Note that the class $\mathcal{CDA}$ is closed under taking Hausdorff quotient groups and arbitrary products (with the Tychonoff topology).

2. Results

The first lemma is an immediate consequence of [5, Proposition 5.43].

Lemma 2.1. *Every nontotally disconnected abelian pro-Lie group K has the circle group, $\mathbb{T}$, as a quotient group.*

Lemma 2.2. *Let $G \in \mathcal{CDA}$. If K is a pro-Lie group and there is a surjective continuous homomorphism $f : G \to K$ onto K, then K also belongs to the class $\mathcal{CDA}$.*

Proof. First we show that K is totally disconnected. Suppose, for a contradiction, that K is not totally disconnected. Then, by Lemma 2.1, there is a continuous homomorphism $\bar{f}$ from G onto $\mathbb{T}$. Let U be an arbitrary neighbourhood of the identity of the circle group $\mathbb{T}$ not containing any nonsingleton subgroup. As $\bar{f}^{-1}(U)$ is an open neighbourhood of zero of G, $\bar{f}^{-1}(U)$ contains an open subgroup H of G such that G/H is countable. Since $\bar{f}(H) \subseteq U$ and U does not contain nondegenerate subgroups, we have $\bar{f}(H) = \{0\}$ in $\mathbb{T}$. Hence, $\mathbb{T}$, being algebraically isomorphic to $G/\ker(\bar{f})$, is an algebraic homomorphic image of G/H and therefore is countable, which is a contradiction. This contradiction shows that the supposition is false, and therefore K is totally disconnected.

Being totally disconnected, the pro-Lie group K is prodiscrete by [5, Corollary 4.23]. So, K has a subgroup topology. It remains to show that for every open subgroup H of K the quotient group K/H is countable. This follows from the facts that $G/f^{-1}(H)$ is countable and f is surjective. □

Lemma 2.3. *Let K be an almost connected abelian pro-Lie group which is either totally disconnected or a torsion group. Then K is compact.*

Proof. By [5, Theorem 5.20], a pro-Lie group is abelian almost connected if and only if it is isomorphic to $\mathbb{R}^\kappa \times C$ for some cardinal κ and a compact abelian group C. By our assumption on K, we obtain $\kappa = 0$ and hence K is compact. □

For every $m > 1$ and cardinal number κ, the group $\mathbb{Z}^\kappa / m\mathbb{Z}^\kappa = \mathbb{Z}(m)^\kappa$ is compact. Being motivated by this fact, we denote by $\mathcal{CDA}_k$ the class of all groups $G \in \mathcal{CDA}$ for which G/G_m is compact, for every natural number $m > 1$, where $G_m := \mathrm{cl}_G(mG)$, the closure in G of mG. So, $\mathbb{Z}^\kappa \in \mathcal{CDA}_k$. Note that the group G/G_m has exponent $\leq m$ for every $m > 1$. We note also that the class $\mathcal{CDA}_k$ is closed under taking Hausdorff quotient groups and arbitrary products.

Lemma 2.4. *Let $G \in \mathcal{CDA}_k$. If $f : G \to K$ is a surjective continuous homomorphism onto an almost connected torsion pro-Lie group K, then f is an open mapping.*

Proof. By Lemma 2.3, we shall assume that K is a compact abelian group.

https://doi.org/10.1017/S000497271500146X Published online by Cambridge University Press

Since K is torsion, there is an $m \in \mathbb{N}$ such that $mK = 0$ by [3, Theorem 25.9]. Then the closed subgroup G_m of G is contained in the kernel, $\ker(f)$, of f. So, f induces an injective continuous homomorphism $\tilde{f}$ from $G/\ker(f)$, which is isomorphic to $(G/G_m)/(\ker(f)/G_m)$, onto K. As G/G_m is a compact group, we obtain that $G/\ker(f)$ is also compact. Hence, $\tilde{f}$ is a topological group isomorphism of the compact group $G/\ker(f)$ onto K. Since the projection $\pi : G \to G/\ker(f)$ is an open mapping, we see that $f = \tilde{f} \circ \pi$ is also an open mapping, as required. □

Recall that an abelian group G is called *algebraically compact* if G is a direct summand of an abelian group which admits a compact group topology (see [1, Corollary]). To prove Theorem 2.6, we need the following lemma, which is an immediate corollary of [9, Theorem 6.4].

Lemma 2.5. *The group $\mathbb{Z}$ is not algebraically compact.*

Now we prove our main result.

Theorem 2.6. *Let K be a pro-Lie group which has an open almost connected subgroup H. For every cardinal κ, any surjective continuous homomorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping.*

Proof. Without loss of generality, we shall assume that the group H is infinite and hence the cardinal κ is also infinite. We split the proof into two steps.

Step 1. Assume that K is an almost connected pro-Lie group. By Lemmas 2.2 and 2.3, we can assume also that K is compact. It is enough to prove that the image $S := f(U)$ of an open subgroup $U = \{0_i\} \times \mathbb{Z}^{\kappa\setminus\{i\}}$ of $\mathbb{Z}^\kappa$ is open in K for every $i \in \kappa$.

Set $e := f(1_i) \in K$ and let $\langle e \rangle$ be the cyclic subgroup of K generated by e. Note that, by hypothesis, $K = \langle e \rangle + S$. We have to show that S is open.

We claim that there is an $m \in \mathbb{N}$ such that $me \in S$. Suppose that this is not the case; then we obtain that $\langle e \rangle \cap S = \{0\}$ and hence the subgroup $\langle e \rangle \cong \mathbb{Z}$ is a direct (algebraic) summand of the compact group K. So, $\mathbb{Z}$ is an algebraically compact group, which is false since it contradicts Lemma 2.5.

So, let $m \in \mathbb{N}$ be such that $me \in S$. Then $mK \subset S$. Let $\pi : K \to K/mK$ be the quotient map. Since K/mK is torsion, Lemma 2.4 implies that the map $\bar{f} := \pi \circ f$ is open. So, $\bar{f}(U)$ is open in K/mK. Hence, the subgroup $f(U) = S = \pi^{-1}(\bar{f}(U))$ is open in K. Thus, f is an open mapping.

Step 2. Assume that K contains an open almost connected subgroup H. Since the subgroup $X := f^{-1}(H)$ of $\mathbb{Z}^\kappa$ is open, we can find a finite subset $F = \{i_1, \ldots, i_n\}$ of κ such that X contains the open subgroup $Y := \mathbb{Z}^{\kappa\setminus F}$. Since X/Y is a subgroup of $\mathbb{Z}^n = \mathbb{Z}^\kappa/Y$, there is a $k \in \mathbb{N}$ such that $X/Y = \mathbb{Z}^k$ by [3, Theorem A 26].

As the projection π_Y of X onto Y is continuous and $\pi_Y(y) = y$, for every $y \in Y$, we obtain that $X = X/Y \times Y$; see [3, Proposition 6.22]. So, X is topologically isomorphic to $\mathbb{Z}^k \times Y$. Hence, the restriction map $p := f|_X$ from X onto H is open by Step 1. As H is open, we see that f is also an open mapping, as required. □

https://doi.org/10.1017/S000497271500146X Published online by Cambridge University Press

The principal structure theorem for locally compact abelian groups [10, Theorem 25] says that every locally compact abelian group K has an open subgroup H which is topologically isomorphic to $\mathbb{R}^n \times C$, where C is a compact abelian group and n is a nonnegative integer. So, H is an almost connected pro-Lie group. So, as an immediate consequence of Theorem 2.6, we obtain Corollary 2.7, which provides a positive answer to [7, Question 5].

Corollary 2.7. *Let K be a locally compact abelian group. For every cardinal κ, any surjective continuous homomorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping. In particular, this is the case if K is compact.*

Indeed, since a pro-Lie group K with the property that K/K_0 is locally compact has an open subgroup which is an almost connected pro-Lie group by [6, Corollary 8.12], we obtain a stronger result, as follows.

Corollary 2.8. *Let K be an abelian pro-Lie group K with the property that K/K_0 is locally compact. Then, for every cardinal κ, any surjective continuous homomorphism $f : \mathbb{Z}^\kappa \to K$ is an open mapping.*

We conclude with an open question.

Question 2.9. Is every surjective continuous homomorphism from $\mathbb{Z}^\kappa$ onto a pro-Lie group K open?

Acknowledgements

The authors thank the referee for his corrections and pertinent remarks. The second author thanks the Ben Gurion University of the Negev for hospitality during which the research for this paper was done.

References

[1] S. Balcerzyk, 'On the algebraically compact groups of I. Kaplansky', *Fund. Math.* **44** (1957), 91–93.

[2] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelan and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis* (Springer, New York, 2010).

[3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, 2nd edn, Vol. I (Springer, Berlin, 1979).

[4] K. H. Hofmann and S. A. Morris, 'An open mapping theorem for pro-lie groups', *J. Aust. Math. Soc.* **83** (2007), 55–77.

[5] K. H. Hofmann and S. A. Morris, *The Lie Theory of Connected Pro-Lie Groups* (EMS Publishing House, Zürich, 2007).

[6] K. H. Hofmann and S. A. Morris, 'The structure of almost connected pro-Lie groups', *J. Lie Theory* **21** (2011), 347–383.

[7] K. H. Hofmann and S. A. Morris, 'Pro-Lie groups: a survey with open problems', *Axioms* **4** (2015), 294–312.

[8] Sh. Koshi and M. Takesaki, 'An open mapping theorem on homogeneous spaces', *J. Aust. Math. Soc.* **53** (1992), 51–54.

[9] M. W. Legg and E. A. Walker, 'An algebraic treatment of algebraically compact groups', *Rocky Mountain J. Math.* **5** (1975), 291–299.

[10] S. A. Morris, *Pontryagin Duality and the Structure of Locally Compact Abelian Groups* (Cambridge University Press, Cambridge, 1977).

SAAK S. GABRIYELYAN, Department of Mathematics,
Ben-Gurion University of the Negev, PO Box 653, Beer-Sheva, Israel
e-mail: saak@math.bgu.ac.il

SIDNEY A. MORRIS, Faculty of Science and Technology,
Federation University Australia, PO Box 663, Ballarat,
Victoria 3353, Australia
and
Department of Mathematics and Statistics, La Trobe University,
Melbourne, Victoria 3086, Australia
e-mail: morris.sidney@gmail.com

https://doi.org/10.1017/S000497271500146X Published online by Cambridge University Press