Canad. Math. Bull. Vol. **40** (3), 1997 pp. 257–265

A CHARACTERIZATION OF REAL HYPERSURFACES IN COMPLEX SPACE FORMS IN TERMS OF THE RICCI TENSOR

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ABSTRACT. We study real hypersurfaces of a complex space form $M_n(c)$, $c \neq 0$ under certain conditions of the Ricci tensor on the orthogonal distribution T_o.

1. **Introduction.** Let $M_n(c)$ denote an n-dimensional complex space form with constant holomorphic sectional curvature c. It is well known that a complete and simply connected complex space form consists of a complex projective space $P_n(\mathbf{C})$, a complex Euclidean space $\mathbf{C}^n$, or a complex hyperbolic space $H_n(\mathbf{C})$, according as $c > 0$, $c = 0$ or $c < 0$. In this paper we consider real hypersurfaces M of $M_n(c)$, $c \neq 0$, namely of $P_n(\mathbf{C})$ or $H_n(\mathbf{C})$.

Now, let M be a real hypersurface of an n-dimensional complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (φ, ξ, η, g) induced from the complex structure J of $P_n(\mathbf{C})$ or $H_n(\mathbf{C})$.

The study of real hypersurfaces of $P_n(\mathbf{C})$ was initiated by Takagi [19], who proved that all homogeneous hypersurfaces of $P_n(\mathbf{C})$ could be divided into six types which are said to be of type A_1, A_2, B, C, D, and E. Many results for real hypersurfaces of complex projective space have been obtained by Cecil and Ryan [3], Kimura [8], Kon [13], S. Maeda [14], [15], Okumura [18] and so on (for more details see [14]). On the other hand, real hypersurfaces of $H_n(\mathbf{C})$ have also been investigated by many authors, from different points of view (*cf.* [1], [2], [4], [5], [16], [17], *etc.*). In particular Berndt [1], [2] showed recently that all real hypersurfaces with constant principal curvatures of complex hyperbolic space $H_n(\mathbf{C})$ are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in $H_n(\mathbf{C})$ they are said to be of *type* A_0, A_1, A_2 and B.

M. Kimura and S. Maeda [11], [12] investigated the condition

$$(\nabla_X S)Y = \mu\big(g(\varphi X, Y)\xi + \eta(Y)\varphi X\big) \tag{1.1}$$

where S is the Ricci tensor, μ is a non-zero constant for any tangent vector fields X and Y of M in $P_n(\mathbf{C})$. They used it to find a lower bound of $\|\nabla S\|$. Also T. Taniguchi [20] extended the results of M. Kimura and S. Maeda to real hypersurfaces in $H_n(\mathbf{C})$.

Research supported in part by E.E.C. contract CHRX-CT94-0661 and by G.G.S.R.T.
Received by the editors April 19, 1996.
AMS subject classification: Primary: 53C40; secondary: 53C15.

On the other hand, the condition

$$g\big((S\varphi - \varphi S)X, Y\big) = 0 \tag{1.2}$$

for any tangent vector fields X and Y of M was considered by M. Kimura [9], [10] for $c > 0$ and U.-H. Ki and Y. J. Suh [6] for $c < 0$.

Now, let us define a distribution T_o by $T_o = \{X \in T_xM | X \perp \xi_x\}$ of a real hypersurface M of $M_n(c)$, $c \neq 0$, which is orthogonal to the structure vector field ξ and holomorphic with respect to the structure tensor φ. If we restrict the properties (1.1) and (1.2) to the orthogonal distribution T_0, then for any vector fields X and Y in T_0 the Ricci tensor S of M satisfies the following conditions

$$(\nabla_X S)Y = \mu g(\varphi X, Y)\xi \tag{1.3}$$

and

$$(S\varphi - \varphi S)X = \theta(X)\xi \tag{1.4}$$

for a 1-form θ defined on T_o, where μ is a constant. Thus the above conditions (1.3) and (1.4) are weaker than the conditions (1.1) and (1.2), respectively and it is natural to study real hypersurfaces of $M_n(c)$, $c \neq 0$, under these conditions.

We show the following

THEOREM. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 3$. If it satisfies (1.3) and (1.4) for any vector fields X and Y in T_0, then M is locally congruent to one of the following:*

(1) In case $M_n(c) = P_n(\mathbf{C})$

- *(a) a homogeneous real hypersurface which lies on a tube of radius r over a totally geodesic $P_k(\mathbf{C})(1 \leq k \leq n-1)$, where $0 < r < \frac{\pi}{2}$,*
- *(b) a homogeneous real hypersurface which lies on a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \pi/4$ and $\cot^2 2r = n-2$,*
- *(c) a homogeneous real hypersurface which lies on a tube of radius r over $P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, where $0 < r < \pi/4$, $\cot^2 2r = 1/(n-2)$, and $n (\geq 5)$ is odd,*
- *(d) a homogeneous real hypersurface which lies on a tube of radius r over a complex Grassmann $G_{2,5}(\mathbf{C})$, where $0 < r < \pi/4$, $\cot^2 2r = 3/5$ and $n = 9$,*
- *(e) a homogeneous real hypersurface which lies on a tube of radius r over a Hermitian symmetric space $\mathrm{SO}(10)/U(5)$, where $0 < r < \pi/4$, $\cot^2 2r = 5/9$ and $n = 15$,*
- *(f) a nonhomogeneous real hypersurface which lies on a tube of radius r over a k-dimensional Kaehler submanifold $\tilde{N}$ on which the rank of each shape operator is not greater than 2 with nonzero principal curvatures not equal to $\pm\sqrt{(2k-1)/(2n-2k-1)}$ and $\cot^2 r = (2k-1)/(2n-2k-1)$, where $k = 1, \ldots, n-1$.*

https://doi.org/10.4153/CMB-1997-031-5 Published online by Cambridge University Press

(2) In case $M_n(c) = H_n(\mathbf{C})$

(A_0) a horosphere in $H_n(\mathbf{C})$, i.e. a Montiel tube,

(A_1) a tube of a totally geodesic hyperplane $H_k(\mathbf{C})(k = 0$ or $n-1)$,

(A_2) a tube of a totally geodesic $H_k(\mathbf{C})(1 \leq k \leq n-2)$.

REMARK. Real hypersurfaces of the complex space forms $M_n(c)$, $c \neq 0$, under the conditions $(\nabla_X A)Y = -\frac{c}{4}g(\varphi X, Y)\xi$ and $(A\varphi - \varphi A)X = \theta(X)\xi$, for any vector fields $X, Y \in T_0$, where A is the shape operator, have been investigated by U.-H. Ki and Y. J. Suh in [7].

2. **Preliminaries.** Let M be a real hypersurface of an $n(\geq 3)$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(c \neq 0)$ and let N be a unit normal vector field on a neighborhood of a point x in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood of x in M, the transformations of X and N under J can be represented as

$$JX = \varphi X + \eta(X)N, \quad JN = -\xi,$$

where φ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor on $M_n(c)$. The set of tensors (φ, ξ, η, g) is an almost contact metric structure on M:

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \ \varphi\xi = 0. \tag{2.1}$$

Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \varphi AX \tag{2.2}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A is the shape operator of M. Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ &- 2g(\varphi X, Y)\varphi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.3}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi\}$$

By (2.1), (2.2) and (2.3) we get

$$SX = \frac{c}{4}\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X \tag{2.4}$$

$$\begin{aligned} (\nabla_X S)Y = \frac{c}{4}\{&-3g(\varphi AX, Y)\xi - 3\eta(Y)\varphi AX\} + (Xh)AY \\ &+ (hI - A)(\nabla_X A)Y - (\nabla_X A)AY \end{aligned} \tag{2.5}$$

where $h = traceA$, S is the Ricci tensor of type (1,1) on M, and I is the identity map.

Now we recall without proof the following propositions, which were proved by M. Kimura [9], [10] and U.-H. Ki and Y. J. Suh [6], in the case $c > 0$ and $c < 0$, respectively.

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PROPOSITION A [9], [10]. *Let M be a real hypersurface of $P_n(\mathbf{C})(n \geq 3)$. Then the Ricci tensor of M commutes with the almost contact structure φ of M induced from $P_n(\mathbf{C})$, ξ is principal and the focal map has constant rank on M if and only if M is locally congruent to one of the following :*

(a) a homogeneous real hypersurface which lies on a tube of radius r over a totally geodesic $P_k(\mathbf{C})(1 \leq k \leq n-1)$, where $0 < r < \frac{\pi}{2}$,

(b) a homogeneous real hypersurface which lies on a tube of radius r over a complex quadric Q_{n-1}, where $0 < r < \pi/4$ and $\cot^2 2r = n-2$,

(c) a homogeneous real hypersurface which lies on a tube of radius r over $P_1(\mathbf{C}) \times P_{(n-1)/2}(\mathbf{C})$, where $0 < r < \pi/4$, $\cot^2 2r = 1/(n-2)$, and $n(\geq 5)$ is odd,

(d) a homogeneous real hypersurface which lies on a tube of radius r over a complex Grassmann $G_{2,5}(\mathbf{C})$, where $0 < r < \pi/4$, $\cot^2 2r = 3/5$ and $n = 9$,

(e) a homogeneous real hypersurface which lies on a tube of radius r over a Hermitian symmetric space $\mathrm{SO}(10)/U(5)$, where $0 < r < \pi/4$, $\cot^2 2r = 5/9$ and $n = 15$,

(f) a nonhomogeneous real hypersurface which lies on a tube of radius r over a k-dimensional Kaehler submanifold $\tilde{N}$ on which the rank of each shape operator is not greater than 2 with nonzero principal curvatures not equal to $\pm\sqrt{(2k-1)/(2n-2k-1)}$ and $\cot^2 r = (2k-1)/(2n-2k-1)$, where $k = 1, \ldots, n-1$.

PROPOSITION B [6]. *Let M be a real hypersurface of $H_n(\mathbf{C})(n \geq 3)$. Then the Ricci tensor of M commutes with the almost contact structure φ of M induced from $H_n(\mathbf{C})$ if and only if M is locally congruent to one of the following:*

(A_0) a horosphere in $H_n(\mathbf{C})$, i.e. a Montiel tube,

(A_1) a tube of a totally geodesic hyperplane $H_k(\mathbf{C})(k = 0$ or $n-1)$,

(A_2) a tube of a totally geodesic $H_k(\mathbf{C})(1 \leq k \leq n-2)$.

3. **Certain lemmas.** For our purpose we need the next two lemmas obtained from the restricted conditions (1.3) and (1.4).

LEMMA 3.1. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M satisfies (1.3) and (1.4), then we have*

$$\theta(Y)g(AX,\varphi Z) + \theta(\varphi Y)g(AX,Z) + \theta(Z)g(AX,\varphi Y) + \theta(\varphi Z)g(AX,Y) = 0 \tag{3.1}$$

for any $X, Y, Z \in T_0$.

PROOF. For vector fields X, Y and Z orthogonal to ξ, the condition (1.4) implies that $g\big((S\varphi - \varphi S)Y, Z\big) = 0$. Differentiating this equation covariantly in the direction of X, we get

$$\begin{aligned} &g\big((\nabla_X S)Y, \varphi Z\big) + g\big((\nabla_X S)Z, \varphi Y\big) + g\big((S\varphi - \varphi S)Y, \nabla_X Z\big) \\ &\qquad + g\big((\nabla_X \varphi)Y, SZ\big) + g\big((\nabla_X \varphi)Z, SY\big) + g\big((S\varphi - \varphi S)Z, \nabla_X Y\big) = 0 \end{aligned} \tag{3.2}$$

By using (2.2) and (1.4) we get $g(\nabla_X Y, \xi) = -g(\varphi AX, Y)$ and $\theta(X) = g(S\xi, \varphi X)$. Now using (1.4) we obtain

$$g\big((S\varphi - \varphi S)Y, \nabla_X Z\big) = g\big((S\varphi - \varphi S)Y, \xi\big) g(\nabla_X Z, \xi) = g(AX, \varphi Z)\theta(Y).$$

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Finally from this, (2.2), (1.3) and (3.2) we obtain (3.1).

In the study of real hypersurfaces of $M_n(c)$, $c \neq 0$, it is a crucial condition that the structure vector ξ is principal. In fact in proofs of many known results it seems that the most difficult part is to show that ξ is principal under a certain condition. For this reason, the next lemma is important.

LEMMA 3.2. *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M satisfies (1.3) and (1.4), then the structure vector field ξ is principal.*

PROOF. In order to prove this lemma, let us suppose that there is a point where ξ is not principal. Then there exists a neighborhood $\mathcal{U}$ of this point, on which we can define a unit vector field U orthogonal to ξ in such a way that

$$A\xi = \alpha\xi + \beta U \tag{3.3}$$

where β denotes the length of vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point x in $\mathcal{U}$. Hereafter, unless otherwise stated, let us continue our discussion on this neighborhood $\mathcal{U}$. Let $V = \nabla_\xi \xi$. Then, from this together with (2.2) and (3.3) it follows $V = \varphi A\xi = \beta\varphi U$ and $\eta(V) = 0$.

Putting $X = Y = V$, $Z = \varphi V$ or $X = V$, $Y = Z = \varphi V$ in (3.1) we get

$$\begin{aligned} &\theta(V)g(AV, V) - \theta(\varphi V)g(AV, \varphi V) = 0 \\ &\theta(\varphi V)g(AV, V) + \theta(V)g(AV, \varphi V) = 0 \end{aligned} \tag{3.4}$$

We distinguish two cases: (I) $g(AV, V) \neq 0$ and (II) $g(AV, V) = 0$

(I) Let $g(AV, V) \neq 0$.

From (3.4) we get $\theta(V) = \theta(\varphi V) = 0$. Now putting $Z = V$ or $Z = \varphi V$ in (3.1) we obtain

$$\begin{aligned} &\theta(\varphi Y)AV + \theta(Y)A\varphi V = -\beta^2\theta(Y)\xi \\ &\theta(Y)AV - \theta(\varphi Y)A\varphi V = \beta^2\theta(\varphi Y)\xi \end{aligned} \tag{3.5}$$

Therefore $\big(\theta(Y)^2 + \theta(\varphi Y)^2\big)AV = 0$ and since $AV \neq 0$ we have $\theta(Y) = 0$, namely $(S\varphi - \varphi S)Y = 0$ for any vector field $Y \in T_0$.

Now, from $\theta(X) = 0$, (1.4) and (2.4), we obtain $h\eta(AX) - \eta(A^2X) = 0$ for any $X \in T_0$. This, by using (3.3), implies

$$AU = (h - \alpha)U + \beta\xi \tag{3.6}$$

This and (3.3) give $A^2\xi = (\alpha^2 + \beta^2)\xi + h\beta U$. Consequently, from (2.4) we take $S\xi = k\xi$ with $k = \frac{c}{2}(n-1) + \alpha h - \alpha^2 - \beta^2$. Thus $(S\varphi - \varphi S)\xi = 0$ and finally from (1.3) we have

$$S\varphi = \varphi S. \tag{3.7}$$

Now, from (1.3) and (2.5) we get

$$\begin{aligned} -\frac{3}{4}cg(\varphi AX, Y) + (Xh)g(AY, \xi) + hg\big((\nabla_X A)Y, \xi\big) \\ - g\big(A(\nabla_X A)Y, \xi\big) - g\big((\nabla_X A)AY, \xi\big) = \mu g(\varphi X, Y). \end{aligned} \tag{3.8}$$

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This, for $Y = U$, gives

$$(3.9)\quad \begin{aligned} -\frac{3}{4}cg(\varphi AX, U) - g\big((\nabla_X A)U, \alpha\xi + \beta U\big) \\ + (Xh)\beta + hg\big(\nabla_X(\alpha\xi + \beta U) - A\varphi AX, U\big) \\ - g\big(\nabla_X(\alpha\xi + \beta U) - A\varphi AX, (h-\alpha)U + \beta\xi\big) \\ = \mu g(\varphi X, U). \end{aligned}$$

By using (2.2) and (3.6) we obtain from (3.9)

$$\frac{3}{4}cg(A\varphi U, X) = -\mu g(\varphi U, X), \text{ for any } X \in T_0.$$

Also by using (3.3) we take $\eta(A\varphi U) = g(\varphi U, A\xi) = 0$. Thus

$$(3.10)\quad A\varphi U = -\frac{4\mu}{3c}\varphi U.$$

Now, from (2.4) by using (3.6) we calculate $SU = \rho U$, with $\rho = \frac{c}{4}(2n+1) + \alpha h - \alpha^2 - \beta^2$. Now (3.7) implies $S\varphi U = \rho\varphi U$. Differentiating this equation covariantly in the direction of U, we get

$$(\nabla_U S)\varphi U + S(\nabla_U \varphi)U + S\varphi\nabla_U U = (U\rho)\varphi U + \rho(\nabla_U\varphi)U + \rho\varphi\nabla_U U.$$

Taking the inner product of this with ξ and using (1.3) we get $\mu = -\frac{3}{4}c(h-\alpha)$. Now from $S\varphi U = \rho\varphi U$, (2.4) and (3.10) we obtain $\beta = 0$, which is a contradiction.

(II) Let $g(AV, V) = 0$

In this case we have from (3.4)

$$(3.11)\quad \theta(\varphi V)g(AV, \varphi V) = 0, \quad \theta(V)g(AV, \varphi V) = 0$$

Next, putting $Y = V$, $X = Z = \varphi V$ or $Y = Z = V$, $X = \varphi V$ in (3.1) we get

$$(3.12)\quad \theta(\varphi V)g(A\varphi V, \varphi V) = 0, \quad \theta(V)g(A\varphi V, \varphi V) = 0$$

We will prove that $\theta(V)^2 + \theta(\varphi V)^2 = 0$.

Assume, for the moment, that $\theta(V)^2 + \theta(\varphi V)^2 \neq 0$. Then from (3.11) and (3.12) we have $g(AV, \varphi V) = 0$ and $g(A\varphi V, \varphi V) = 0$. Now putting in (3.1) $Z = V$ we get

$$\theta(\varphi V)AY + \theta(\varphi Y)AV + \theta(V)A\varphi Y + \theta(Y)A\varphi V = \sigma\xi$$

Taking the inner product of this by V we obtain

$$(3.13)\quad \theta(\varphi V)AV - \theta(V)\varphi AV = 0.$$

If we suppose for the moment that $\theta(\varphi V) \neq 0$, then $AV = \lambda\varphi AV$ with $\lambda = \theta(V)/\theta(\varphi V)$. Thus, by using (1.4) and (2.4) we have $\theta(\varphi V) = \eta(A^2V) - h\eta(AV) = \eta(A^2V) = g\big(A(\lambda\varphi AV), \xi\big) = \lambda g(\varphi AV, \alpha\xi + \beta U) = -\lambda g(AV, V) = 0$, which is a contradiction.

Thus $\theta(\varphi V) = 0$ and so $\theta(V) \neq 0$. Now we have from (3.13) $\varphi AV = 0$, which implies $AV = 0$.

Next, putting $Y = Z = V$ in (3.1) we obtain $A\varphi V \parallel \xi$. Thus

$$A\varphi V = g(A\varphi V, \xi)\xi = g(\varphi V, A\xi)\xi = -\beta^2 \xi.$$

Using this relation, and putting $Z = \varphi V$ in (3.1) we obtain $AX \parallel \xi$ for any $X \in T_0$.

Thus $g(AX, \xi)\xi = g(X, A\xi)\xi = \beta g(X, U)\xi$ for any $X \in T_0$, which means that

$$AY = 0, \quad AU = \beta\xi$$

for any $Y \in T_0$ orthogonal to U.

Hence, from (2.4) we take $S\xi = \kappa\xi + \beta(h-\alpha)U$ where $\kappa = \frac{c}{2}(n-1) + \alpha h - \alpha^2 - \beta^2$.

Therefore, from (1.4) we find that $\theta(U) = g\big((S\varphi - \varphi S)U, \xi\big) = 0$, which implies

$$S\varphi U = \varphi SU. \tag{3.14}$$

Now from (2.4) we calculate

$$SU = (\frac{c}{4}(2n+1) - \beta^2)U + \beta(h-\alpha)\xi, \quad S\varphi U = \frac{c}{4}(2n+1)\varphi U$$

Combining these, with (3.14), we take $\beta = 0$. This makes a contradiction and the assertion $\theta(V)^2 + \theta(\varphi V)^2 \neq 0$ is false.

Now we have $\theta(V) = 0$ and $\theta(\varphi V) = 0$. From these we obtain $g(AU, U) = h - \alpha$, $g(AV, U) = 0$. Putting $Z = V$ in (3.1), we find $\theta(\varphi Y)AV + \theta(Y)A\varphi V = \nu\xi$, which gives $\theta(Y)g(A\varphi V, U) = 0$, or $(h-\alpha)\theta(Y) = 0$.

We will prove that the assertion $h - \alpha \neq 0$ is false.

Assume, for the moment, that $h - \alpha \neq 0$. Thus

$$\theta(Y) = 0, \quad \forall Y \in T_0. \tag{3.15}$$

This implies that $AU = (h-\alpha)U + \beta\xi$ and from (3.3) we get $A^2\xi = (\alpha^2 + \beta^2)\xi + \beta hU$. Hence the equation (2.4) gives $S\xi = \kappa\xi$, with $\kappa = \frac{c}{2}(n-1) + \alpha h - \alpha^2 - \beta^2$. This and (3.15), by using (2.4), imply (3.7).

On the other hand, using (2.5) and (1.3), for $Y = U$, we obtain, as in relation (3.10) $g(\frac{3}{4}cA\varphi U + \mu\varphi U, X) = 0$, for any $X \in T_0$. Thus we get $A\varphi U = -\frac{4\mu}{3c}\varphi U$, which, together with $g(AV, V) = 0$, implies that $\mu = 0$ and so

$$A\varphi U = 0. \tag{3.16}$$

Now, from (3.7) we get $h(A\varphi - \varphi A)U = (A^2\varphi - \varphi A^2)U$ or $\alpha h - \alpha^2 - \beta^2 = 0$. Thus $\kappa = \frac{c}{2}(n-1)$. Now, from (1.3), since $\mu = 0$, we get

$$\begin{aligned} 0 &= g\big((\nabla_U S)\varphi U, \xi\big) \\ &= g\big(\varphi U, (\nabla_U S)\xi\big) \\ &= g\big(\varphi U, \nabla_U(S\xi) - S\nabla_U\xi\big) \\ &= \kappa g(\varphi U, \varphi AU) - (h-\alpha)g(\varphi U, S\varphi U) \\ &= -\frac{3c}{4}(h-\alpha). \end{aligned}$$

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Thus $c = 0$, which is impossible.

Now, let us continue with our discussion on the open set $\mathcal{U}$ with $h-\alpha = 0$, $g(AV, U) = 0$ and $g(AU, U) = 0$. Putting in (3.1) $Z = V$ or $Z = \varphi V$ we obtain the relations (3.5), which give $\big(\theta(Y)^2+\theta(\varphi Y)^2\big)AV = 0$ and $\big(\theta(Y)^2+\theta(\varphi Y)^2\big)A\varphi V = -\beta^2\big(\theta(Y)^2+\theta(\varphi Y)^2\big)\xi$. We claim that $\theta(X) = 0$ for any $X \in T_0$. Indeed, if there exists $Y \in T_0$ such that $\theta(Y) \neq 0$, then $AV = 0$ and $A\varphi V = -\beta^2\xi$. The last one gives $AU = \beta\xi$. Now, from (2.4) we obtain $S\xi = \big(\frac{c}{2}(n-1) - \beta^2\big)\xi$, which combined with (1.4) implies $\theta(X) = 0$ for any $X \in T_0$, a contradiction.

Consequently we have always $\theta(X) = 0$ for any $X \in T_0$. Now, from (1.4) and (2.4) we obtain $AU = \beta\xi$. Also, from (2.4) $S\xi = \kappa\xi$ with $\kappa = \frac{c}{2}(n-1) - \beta^2$ and $SU = \rho U$, with $\rho = \frac{c}{4}(2n+1) - \beta^2$. Then,

$$\begin{aligned}\mu &= g\big((\nabla_U S)\varphi U, \xi\big)\\ &= g(\varphi U, \nabla_U(S\xi) - S\nabla_U\xi)\\ &= \kappa g(\varphi U, \varphi AU) - g(\varphi U, S\varphi AU)\\ &= 0.\end{aligned}$$

Now, from (1.3) and (2.5) we get

$$\begin{aligned}0 &= g\big((\nabla_X S)U, \xi\big)\\ &= -\frac{3c}{4}g(\varphi AX, U) + (Xh)g(AU, \xi) + g\big((hI - A)(\nabla_X A)U, \xi\big) - g\big((\nabla_X A)AU, \xi\big)\\ &= \frac{3c}{4}g(A\varphi U, X).\end{aligned}$$

Finally we have $A\varphi U = 0$.

Now, from $S\varphi = \varphi S$ we obtain $h(A\varphi - \varphi A)U = (A^2\varphi - \varphi A^2)U$ and so $\beta = 0$. This results in a contradiction.

The set $\mathcal{U}$ should be empty. Thus there does not exist such an open neighborhood $\mathcal{U}$ in M, which means that the structure vector field ξ is principal.

4. **Proof of the Theorem.** Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, $n \geq 3$ under the assumptions (1.3) and (1.4). According to Lemma 3.2 the structure vector field ξ is principal. Namely $A\xi = \alpha\xi$. Thus from (2.4) we have $S\xi = \kappa\xi$, with $\kappa = \frac{c}{2}(n-1) + \alpha h - \alpha^2$. Now, from (1.4) we obtain $S\varphi = \varphi S$. Then, by using Propositions A and B of M. Kimura [9], [10] for $c > 0$ and of U.-H. Ki and Y. J. Suh [6] for $c < 0$ we get our result.

ACKNOWLEDGMENT. The author heartily thanks Professors Sadahiro Maeda and Young Jin Suh for their kind advice during the preparation of this paper. He also greatly appreciates the referee's valuable suggestions.

https://doi.org/10.4153/CMB-1997-031-5 Published online by Cambridge University Press

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