

ON A CONSTRUCTION IN BORDISM THEORY

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0. Introduction

In [2], R. Arthan and S. Bullett pose the problem of representing generators of the complex bordism ring MU_* by manifolds which are *totally normally split*; i.e. whose stable normal bundles are split into a sum of complex line bundles. This has recently been solved by Ochanine and Schwartz [8] who use a mixture of J -theory and surgery theory to establish several results, including the following.

Theorem 0.1. *Each element of MU_* may be represented by a manifold which is totally normally split.* □

Their proof, however, is not explicitly constructive.

It is our aim here to adapt an old construction of Conner and Floyd [4] to give a simple and direct proof of (0.1), which subsumes an explicit choice of generators for MU_* . Moreover, we attain certain further goals.

Firstly, in answer to a supplementary question of Stong [13], we show that our manifolds have the extra property that they are also totally tangentially split.

Secondly, by considering the symplectic analogue, we exhibit for the first time precise geometrical models for the important elements $\phi_n \in MSp_{8n-3}$ [9] in the symplectic bordism ring.

Additionally, we note that these manifolds, as well as a large family of other Sp bordism representatives, are also totally normally and tangentially split.

The key result is in Section 2, where we offer a straightforward procedure for producing generators of $MU_*(CP^\infty)$ and $MSp_*(HP^\infty)$ over the respective coefficient rings.

Throughout, $E_*(X_+) = E_* \oplus E_*(X)$ stands for the unreduced E homology module of a space X . Also, the trivial real, complex or symplectic line bundle over X will be respectively denoted by \mathbb{R} , \mathbb{C} or \mathbb{H} , whilst an arbitrary line bundle will be systematically confused with its classifying map.

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1. Iterated projective bundles

In this section we describe our basic inductive construction.

To begin, let $B^0 = *$, admitting the complex line bundle $\beta_0 = \mathbb{C}$.

Now suppose given the $2n-2$ dimensional smooth manifold B^{n-1} , carrying line bundles $\beta_0, \dots, \beta_{n-1}$. Then take B^n to be the projectification $CP(\beta_{n-1} \oplus \mathbb{C})$, with projection $\pi: B^n \rightarrow B^{n-1}$ and fibre CP^1 . Each point $\lambda \in B^n$ may be considered as a line in $\beta_{n-1} \oplus \mathbb{C}$. Define $\beta_0, \dots, \beta_{n-1}$ over B^n by pulling back along π^* , and choose β_n to be the canonical line bundle whose fibre over λ consists of all points in λ .

Thus B^n is a projective variety, and also a smooth $2n$ dimensional manifold. Moreover, it is diffeomorphic to the 2-sphere bundle $S(\beta_{n-1} \oplus \mathbb{R})$ via fibrewise application of the usual identification of CP^1 with S^2 . In this guise, and for $n > 0$, it bounds the disc bundle $D(\beta_{n-1} \oplus \mathbb{R})$.

A third description, which we shall not use here, is due to Conner and Floyd [4], and arises from forming the quotient of the cartesian product $(S^3)^n$ by a suitable action of the torus T^n .

We now wish to invest B^n with the structure of a weakly almost complex, or U -manifold, which is not the obvious projective one. Instead, we take the sphere bundle version, and following Szczarba [16] note that its tangent bundle satisfies

$$\tau_n \oplus \mathbb{R} \cong \pi^*(\tau_{n-1} \oplus \beta_{n-1} \oplus \mathbb{R}).$$

Since $\tau_0 \oplus \mathbb{R} \cong \mathbb{R}$, we obtain inductively

$$\tau_n \oplus \mathbb{R} \cong \left(\bigoplus_{i=0}^{n-1} \beta_i \right) \oplus \mathbb{R}. \tag{1.1}$$

This provides an explicit U -structure for the stable tangent bundle τ_n^S , and hence also for the stable normal bundle ν_n^S , such that B^n is totally tangentially split.

Of course, this U -structure may be extended over the corresponding 3-disc bundle.

To study ν_n^S more closely, we recall the well-known splitting principle.

Proposition 1.2. *Over B^n , there is a $U(1)$ bundle β_n^* satisfying*

$$\beta_{n-1} \oplus \mathbb{C} \cong \beta_n^* \oplus \beta_n.$$

Proof. Using the standard inner product in β_{n-1} , choose as fibre for β_n^* over $\lambda \in B^n$ the orthogonal complement λ^\perp in $\beta_{n-1} \oplus \mathbb{C}$. □

Note that, by taking pull-backs, we have also defined β_k^* over B^n for $k=1, \dots, n-1$ satisfying

$$\beta_{k-1} \oplus \mathbb{C} \cong \beta_k^* \oplus \beta_k.$$

Iterating, we deduce

$$\mathbb{C}^{n+1} \cong \left(\bigoplus_{i=1}^n \beta_i^* \right) \oplus \beta_n. \tag{1.3}$$

and substituting successively in (1.1) finally leads to

$$\tau_n \oplus \left(\bigoplus_{i=1}^{n-1} (n-i)\beta_i^* \right) \oplus \mathbb{R} \cong \mathbb{C}^{\frac{1}{2}n(n+1)} \oplus \mathbb{R}. \tag{1.4}$$

This formula allows us to make our first crucial observation.

Proposition 1.5. *With the U-structure of (1.1), B^n is totally normally split.*

Proof. From (1.4), we have an isomorphism

$$v_n^S \cong \bigoplus_{i=1}^{n-1} (n-i)\beta_i^*. \quad \square$$

Using the results of Borel and Hirzebruch [3], we can now compute the integral cohomology ring $H^*(B^n)$. Writing $b_0, b_1, \dots, b_n \in H^2(B^n)$ for the respective first Chern classes of $\beta_0, \beta_1, \dots, \beta_n$ (so that $b_0=0$), we deduce that $H^*(B^n)$ is free over $H^*(B^{n-1})$ on generators $1, b_n$, and that

$$b_n(b_n - b_{n-1}) = 0.$$

By induction, this yields the following answer.

Proposition 1.6. (see [4, Section 42]). *The ring $H^*(B^n)$ is given by*

$$H^*(B^n) \cong \mathbb{Z}[b_1, \dots, b_n] / (b_k^2 - b_k b_{k-1}, 1 \leq k \leq n)$$

for all $n > 0$. □

We observe that we can mimic all the preceding constructions and computations in the symplectic case, beginning with $Q^0 = *$ and the trivial $Sp(1)$ bundle $\chi_0 = \mathbb{H}$. We produce a smooth $4n$ dimensional manifold Q^n , carrying $Sp(1)$ bundles χ_0, \dots, χ_n . The tangent bundle satisfies

$$\tau_n \oplus \mathbb{R} \cong \left(\bigoplus_{i=0}^{n-1} \chi_i \right) \oplus \mathbb{R}, \tag{1.7}$$

so τ_n^S and v_n^S admit Sp -structures such that Q^n is totally tangentially split. This Sp -structure may be extended over the corresponding 5-disc bundle.

We then define $Sp(1)$ bundles χ_k^* over Q^n for $k=1, \dots, n-1$ satisfying

$$\chi_{k-1} \oplus \mathbb{C} \cong \chi_k^* \oplus \chi_k,$$

and so acquire an isomorphism

$$v_n^S \cong \bigoplus_{i=1}^{n-1} (n-i)\chi_i^*. \tag{1.8}$$

So Q^n is also totally normally split.

Finally, if we write $q_0, q_1, \dots, q_n \in H^4(Q^n)$ for the respective first symplectic Pontrjagin classes of $\chi_0, \chi_1, \dots, \chi_n$ (so that $q_0 = 0$), we infer the integral cohomology ring structure

$$H^*(Q^n) \cong \mathbb{Z}[q_1, \dots, q_n]/(q_k^2 - q_{k-1}q_k, 1 \leq k \leq n). \tag{1.9}$$

2. Generators for $MU_*(CP^\infty)$ and $MSP_*(HP^\infty)$

In this section we utilise Section 1 to introduce a direct approach to creating generators for the reduced bordism modules $MU_*(CP^\infty)$ and $MSP_*(HP^\infty)$ over the respective coefficient rings. Such a construction does not seem to appear in the literature, although indirect methods have been attempted [5], [7].

Using the notation of [10], we can state the following well-known definitive property.

Proposition 2.1. *$MU_*(CP_+^\infty)$ is a free MU_* module on generators bf_0, bf_1, bf_2, \dots , where $bf_0 = 1$, and $bf_n \in MU_{2n}(CP^\infty)$. Furthermore, the bf_n 's are uniquely specified by*

$$cf_1 \cap bf_n = bf_{n-1}, \quad n = 1, 2, \dots,$$

where $cf_1 \in MU^2(CP_+^\infty)$ is the universal first Conner–Floyd Chern class.

Similarly, $MSP_*(HP_+^\infty)$ is a free MSP_* module on generators qf_0, qf_1, qf_2, \dots , where $qf_0 = 1$, and $qf_n \in MSP_{4n}(HP^\infty)$. Furthermore, the qf_n 's are uniquely specified by

$$pf_1 \cap qf_n = qf_{n-1}, \quad n = 1, 2, \dots,$$

where $pf_1 \in MSP^4(HP_+^\infty)$ is the universal first Conner–Floyd Pontrjagin class.

We can now describe our main observation.

Proposition 2.2. *The singular manifolds*

$$\beta_n: B^n \rightarrow CP^\infty, \quad \chi_n: Q^n \rightarrow HP^\infty$$

represent $bf_n \in MU_{2n}(CP^\infty)$ and $qf_n \in MSP_{4n}(HP^\infty)$ respectively.

Proof. We treat the complex case in detail, noting that $\beta_0: * \rightarrow CP^\infty$ clearly represents 1.

Since B^n bounds for $n > 0$, β_n represents an element in the reduced bordism module, so by (2.1) it suffices to determine its image under $cf_1 \cap$. This we can do by making β_n transverse to $CP^{\infty-1}$.

As required for induction, this yields $\beta_{n-1}: B^{n-1} \rightarrow CP^\infty$, where B^{n-1} is identified with the subspace of B^n consisting of lines in β_{n-1} , and as such has U -structure equivalent to that given in (1.1).

The symplectic case is entirely analogous. □

3. Generators of MU_*

In this section we concentrate on the complex case, and manufacture our set of

generators for MU_* . Our detection criterion is the well-documented Milnor genus, e.g. see [14].

First we consider the cartesian product $B^q \times B^r$, with $q, r = 1, 2, \dots$, and the product U -structure. We write $\beta_{1,j}, \beta_{2,k}$ respectively for the bundles $\beta_j \otimes \mathbb{C}, \mathbb{C} \otimes \beta_k$, and hence $b_{1,j}, b_{2,k}$ respectively for their first Chern classes $b_j \otimes 1$ and $1 \otimes b_k$.

Now dualise the cohomology class $b_{1,q} + b_{2,r}$ to obtain the codimension 2 embedding

$$B(q, r) \xrightarrow{e} B^q \times B^r,$$

where the normal bundles satisfy

$$v_{q,r} \cong e^* \{ v_{B^q \times B^r} \oplus (\beta_{1,q} \otimes \beta_{2,r}) \}. \tag{3.1}$$

This isomorphism invests $B(q, r)$ with a U -structure. It may also, and distinctly, be considered as a projective algebraic variety.

Proposition 3.2. *For each q and r , $B(q, r)$ is totally normally and tangentially split.*

Proof. Combining (1.5) with (3.1) describes the chosen U -structure by

$$v_{q,r}^S \cong e^* \left\{ \bigoplus_{j=1}^{q-1} (q-j)\beta_{1,j}^* \oplus \bigoplus_{k=1}^{r-1} (r-k)\beta_{2,k}^* \oplus (\beta_{1,q} \otimes \beta_{2,r}) \right\}.$$

On the other hand, using (1.3) twice, we obtain

$$\begin{aligned} \mathbb{C}^{(q+1)(r+1)} &\cong \left(\bigoplus_{j=1}^q \beta_{1,j}^* \oplus \beta_{1,q} \right) \otimes \left(\bigoplus_{k=1}^r \beta_{2,k}^* \oplus \beta_{2,r} \right) \\ &\cong (\beta_{1,q} \otimes \beta_{2,r}) \oplus \gamma \end{aligned}$$

over $B(q, r)$, where

$$\gamma = \bigoplus_{j,k=1}^{q,r} \{ (\beta_{1,j}^* \otimes \beta_{2,r}) \oplus (\beta_{1,q} \otimes \beta_{2,k}^*) \oplus (\beta_{1,j}^* \otimes \beta_{2,k}^*) \}.$$

Thus we can amalgamate (1.1) and (3.1) to describe the same U -structure by

$$\tau_{q,r}^S \cong e^* \left\{ \bigoplus_{j=0}^{q-1} \beta_{1,j} \oplus \bigoplus_{k=0}^{r-1} \beta_{2,k} \oplus \gamma \right\},$$

as sought. □

Corollary 3.3. *Any product of the form*

$$B(\mathbf{q}, \mathbf{r})^m = B(q_1, r_1)^{m_1} \cdots B(q_k, r_k)^{m_k}$$

is totally normally and tangentially split.

We may now use the standard methods of [14] to compute the normal Milnor genus by

$$s_{q+r-1}(B(q, r)) = \langle (b_{1,q} + b_{2,r})^{q+r}, \mu \rangle = \binom{q+r}{r} \in \mathbb{Z},$$

where $\mu \in H_{2(q+r)}(B^q \times B^r)$ is the fundamental class.

This suffices to show that any element $x \in MU_*$ may be expressed as an integral linear combination of the form

$$x = \sum x_{q,r}^m B(q, r)^m. \tag{3.4}$$

In particular, for any prime p , $B(p-1, 1)$ represents an algebra generator in MU_{2p-2} , since $s_{p-1}(B(p-1, 1)) = p$.

To prove our theorem, it remains only to establish three lemmas.

Lemma 3.5. *Given two U -manifolds which are totally normally and tangentially split, so is their connected sum.*

Proof. Use the fact that $U(n)$ is connected for all n . □

The symplectic analogue is also true.

Lemma 3.6. *The involution $t: B^{n-1} \rightarrow B^{n-1}$ defined by $t(\lambda) = \lambda^\perp$ (see (1.2)) satisfies*

$$t^* \beta_{n-1} = \beta_{n-1}^* \quad \text{and} \quad t^* \beta_k = \beta_k \quad \text{for any } k = 0, \dots, n-2.$$

Proof. Essentially, $t^* \beta_{n-1} = \beta_{n-1}^*$ by definition. The other equations follow since the diagram

$$\begin{array}{ccc} B^{n-1} & \xrightarrow{t} & B^{n-1} \\ \pi \searrow & & \pi \\ & & B^{n-2} \end{array}$$

commutes. □

Corollary 3.7. *The singular manifold*

$$\beta_n^*: B^n \rightarrow CP^\infty$$

represents $-bf_n \in MU_{2n}(CP^\infty)$, for $n > 0$.

Proof. We appeal to (2.1) again, noting first that β_1^* clearly represents $-bf_1$.

Now suppose that $n > 1$. As in (2.2) we apply $cf_1 \cap$ to the element represented by β_n^* . Once more, a simple transversality argument shows that we obtain $\beta_{n-1}^*: B^{n-1} \rightarrow CP^\infty$,

where B^{n-1} is identified by the involution t with the subspace of B^n consisting of lines in β_{n-1} .

Hence

$$\tau_{n-1} \oplus \beta_{n-1}^* \cong t^* \tau_n$$

so

$$\tau_{n-1} \oplus \beta_{n-1}^* \oplus \mathbb{R} \cong t^* \left(\bigoplus_{i=0}^{n-2} \beta_i \right) \oplus \mathbb{R} \tag{by (1.1)}$$

$$= \beta_{n-1}^* \oplus \bigoplus_{i=0}^{n-3} \beta_i \oplus \mathbb{R} \tag{by (3.5)}$$

Thus

$$\tau_{n-1}^S \cong \bigoplus_{i=0}^{n-3} \beta_i$$

as sought, and our induction is done. □

As before, the symplectic analogue also holds.

Lemma 3.8. *For each q and r , we can represent the bordism class $-B(q, r)$ by a manifold $A(q, r)$ which is totally normally and tangentially split.*

Proof. By (3.7), we can construct $A(q, r)$ by dualising $c_1(\beta_{1,q}^* \otimes \beta_{2,r})$ in $H^2(B^q \times B^r)$, where

$$c_1(\beta_{1,q}^* \otimes \beta_{2,r}) = b_{1,q-1} - b_{1,q} + b_{2,r} \tag{by (1.2)}$$

Following (3.2), this yields

$$A(q, r) \xrightarrow{e} B^q \times B^r$$

whose U -structure is given by

$$v_{q,r}^S \cong e^* \left\{ \bigoplus_{j=1}^{q-1} (q-j) \beta_{1,j}^* \oplus \bigoplus_{k=1}^{r-1} (r-k) \beta_{2,k}^* \oplus (\beta_{1,q}^* \otimes \beta_{2,r}) \right\}$$

and

$$\tau_{q,r}^S \cong e^* \left\{ \bigoplus_{j=0}^{q-1} \beta_{1,j} \oplus \bigoplus_{k=0}^{r-1} \beta_{2,k} \oplus \gamma \right\}$$

where

$$\gamma = (\beta_{1,q} \otimes \beta_{2,r}) \oplus \bigoplus_{j,k=1}^{q-1,r} \{ (\beta_{1,j}^* \otimes \beta_{2,r}) \oplus (\beta_{1,q-1} \otimes \beta_{2,k}^*) \oplus (\beta_{1,j}^* \otimes \beta_{2,k}^*) \}. \tag{□}$$

These results may now be combined.

Theorem 3.9. *Any element of MU_* may be represented by a connected U -manifold which is totally normally and tangentially split.*

Proof. Let the element of MU_* be x , and expand it as in (3.4).

For each positive integer $x_{q,r}^m$, the corresponding term $x_{q,r}^m B(q,r)^m$ is suitably represented by utilising (3.3) and (3.5). On the other hand, if $x_{q,r}^m$ is negative we use (3.8) and (3.5). Final appeal to (3.5) then allows us to bring all terms together. \square

4. Representing ϕ 's in MSP_*

In this section we turn our attention to some implications of the Sp version of (2.2).

First we recall some results of [9].

Let η denote the real Hopf line bundle over S^1 , and ξ its symplectic counterpart over HP^∞ (or any HP^n). Thus $\eta \otimes_{\mathbb{R}} \xi$ is also an $Sp(1)$ bundle over $S^1 \times HP^\infty$, and as such it has a first Conner–Floyd Pontrjagin class

$$pf_1(\eta \otimes_{\mathbb{R}} \xi) \in MSP^4(S^1 \times HP^\infty).$$

But

$$MSP^*(S^1 \times HP^\infty) \cong MSP^*[[pf_1, h]]/(h^2),$$

so there is an expansion of the form

$$pf_1(\eta \otimes_{\mathbb{R}} \xi) = pf_1 + \sum_{n \geq 0} h \theta_n pf_1^n, \tag{4.1}$$

where $\theta_n \in MSP_{4n-3}$, $n = 1, 2, \dots$. In fact θ_1 is the generator of $MSP_1 \cong \mathbb{Z}/2$, represented by S^1 with the twisted framing, and $\theta_{2m+1} = 0$ for $m > 0$.

In contrast, the elements θ_{2m} , suitably relabelled ϕ_m , are non-zero and indecomposable elements of order 2 in MSP_* . By all accounts, (see [6], [17]), they play a fundamental role in MSP_* , whose overall structure remains, of course, a mystery.

We now show how to give precise representatives for the elements ϕ_n using Section 2. This has been attempted before in [1] and [11], but with success only modulo certain decomposables.

Note first from (4.1) that

$$\phi_n = \langle pf_1(\eta \otimes_{\mathbb{R}} \xi), g \otimes qf_{2n} \rangle$$

where $g \in MSP_1(S^1)$ is the dual generator to h , and represented by the identity on S^1 . But employing (2.2) with $n > 0$, $g \otimes qf_{2n}$ is represented by $t: R^{8n+1} \rightarrow HP^{2n}$, where $R^{8n+1} = S^1 \times Q^{2n}$ and t classifies $\eta \otimes_{\mathbb{R}} \chi_{2n}$. So ϕ_n arises from making t transverse to HP^{2n-1} .

Proposition 4.2. *The element ϕ_n may be represented on R^{8n-3} by an Sp structure which is totally normally and tangentially split.*

Proof. The transversality described above yields

$$R^{8n-3} = S^1 \times Q^{2n-1} \xrightarrow{j} S^1 \times Q^{2n}$$

where j embeds Q^{2n-1} in Q^{2n} as the subspace of quaternionic lines in χ_{2n-1} . The corresponding normal Sp structure on $S^1 \times Q^{2n-1}$ is provided by the isomorphism

$$v_{R^{8n-3}} \cong v(j) \oplus j^* v_{R^{8n+1}}. \tag{4.3}$$

Moreover, $v(j) \cong \eta \otimes_{\mathbb{R}} \chi_{2n-1}$ by construction, whilst from (1.8)

$$v_{R^{8n+1}} \cong \mathbb{R} \otimes_{\mathbb{R}} \left(\bigoplus_{k=1}^{2n-1} (2n-k) \chi_k^* \right).$$

Thus

$$v_{R^{8n-3}} \cong (\eta \otimes_{\mathbb{R}} \chi_{2n-1}) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} \left(\bigoplus_{k=1}^{2n-1} (2n-k) \chi_k^* \right) \right),$$

and R^{8n-3} is totally normally split.

Similarly, recalling (1.7) and the fact that $2\eta \cong \mathbb{R}^2$ gives

$$\tau_{R^{8n-3}} \cong (\eta \otimes_{\mathbb{R}} \chi_{2n-1}) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} \left(\bigoplus_{k=1}^{2n-1} \chi_i \right) \right),$$

completing the proof. □

In fact a simply-connected representative for ϕ_n may also be given by these methods, by choosing a suitable Sp structure on the S^5 bundle $S(\chi_{2n-2} \oplus \mathbb{R}^2)$ over Q^{2n-2} .

In conclusion, we remark that the procedures introduced in this paper appear to have a wider applicability to MSp_* . Most of the constructions of Sp manifolds given in [12] and [15] can be adapted to give results which are totally normally and tangentially split. This suggests that the infinite product $\bigwedge_{\infty} HP^{\infty}$ may have MSp as a wedge summand, thus mimicking the complex case.

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