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ON THE DECOMPOSITION OF OPERATORS WITH SEVERAL ALMOST-INVARIANT SUBSPACES

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Abstract

We seek a sufficient condition which preserves almost-invariant subspaces under the weak limit of bounded operators. We study the bounded linear operators which have a collection of almost-invariant subspaces and prove that a bounded linear operator on a Banach space, admitting each closed subspace as an almost-invariant subspace, can be decomposed into the sum of a multiple of the identity and a finite-rank operator.

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1. Introduction

The invariant subspace problem is a famous problem in operator theory. An operator without a nontrivial invariant subspace was first found by Enflo [4]. Read constructed such an operator on l_1 [8] and a quasinilpotent operator without a nontrivial invariant subspace [9]. The problem is still open for reflexive Banach spaces.

Androulakis *et al.* [1] introduced almost-invariant subspaces as a modified version of invariant subspaces. For a Banach space X and a bounded linear operator T on X, a closed subspace Y of X is called *almost invariant under* T if there exists a finite-dimensional subspace M of X such that $TY \subseteq Y + M$. If M is chosen with minimum dimension, M and $d_{Y,T} = \dim M$ are respectively called the *error* and *defect* of Y under T. It is easy to see that every finite-dimensional or finite-codimensional subspace of X is always almost invariant under every operator on X. Therefore, the study of almost-invariant subspaces is restricted to *half-spaces*, which are closed subspaces with both infinite dimension and infinite codimension in X. The first question raised was whether every operator on an infinite-dimensional Banach space has an almost-invariant half-space. An affirmative answer was given for reflexive Banach spaces [7] and then for compact and quasinilpotent operators [10]. Finally, Tcaciuc showed that every operator on a separable Banach space has an almost-invariant half-space with defect at most one [12].

In Section 2, we present a sufficient condition for a weak operator convergent sequence to preserve almost-invariant subspaces, improving a theorem of Popov [6]. In Section 3, we show that if a bounded operator on a Banach space admits each closed subspace as an almost-invariant subspace, then it can be decomposed into the sum of a multiple of the identity and a finite-rank operator. This has already been proven for Hilbert spaces [5].

Let Y be a closed subspace of a Banach space X and T a bounded operator on X. By $\operatorname{Alg} Y$, we denote the set of all bounded operators on X which have Y as an invariant subspace. It is known that if Y is almost invariant under T, then T can be expressed in the form $S + F$ for some $S \in \operatorname{Alg} Y$ and a finite-rank operator F [1]. In the following, by putting appropriate conditions on $\mathcal{L}$, a collection of closed subspaces which are almost invariant under T, we achieve a decomposition of T in the form $S + F$ for some $S \in \operatorname{Alg} \mathcal{L}$ and a finite-rank operator F.

Throughout the paper, X is a complex Banach space and $\mathcal{B}(X)$ is the set of all bounded linear operators on X. The terms 'subspace' and 'operator' refer to 'closed subspace' and 'bounded linear operator', respectively.

2. Limit properties of operators with almost-invariant subspaces

Suppose that $(T_\alpha)_{\alpha\in I}$ is a net of bounded operators on X converging to a bounded operator T in the weak operator topology (wot), that is, for each $x^* \in X^*$ and $x \in X$, the net $(x^*(T_\alpha x))_{\alpha\in I}$ converges to $x^*(Tx)$. We denote this limit by wot-lim. If Y is an invariant subspace under each T_α, then Y will also be invariant under T. But this is not valid for almost-invariant subspaces. Indeed, it is enough to consider a sequence $(F_n)_{n=1}^{\infty}$ of finite-rank operators on an infinite-dimensional Hilbert space H, converging to a nonfinite-rank compact operator K. Clearly, each subspace of H is almost invariant under F_n for all n. Nevertheless it is not true for K, since, according to [5, Corollary 4.16], the compact operator K must be in the form $\alpha I + F$ for some nonzero scalar α and a finite-rank operator F, which is a contradiction.

The next proposition provides a sufficient condition. Before that we give two lemmas needed in the proof.

We denote by $\mathcal{F}(X)$ the set of all bounded finite-rank operators on X and by $\mathcal{F}_n(X)$ the set of all bounded finite-rank operators on X with rank $\le n$. Similarly, we use $\mathcal{B}(X, Y)$, $\mathcal{F}(X, Y)$ and $\mathcal{F}_n(X, Y)$ for operators between the Banach spaces X and Y.

Lemma 2.1. *For Banach spaces X and Y, $\mathcal{F}_n(X, Y)$ is a closed subset of $\mathcal{B}(X, Y)$ in the weak operator topology.*

Proof. Let $(T_\alpha)_{\alpha\in I}$ be a net in $\mathcal{F}_n(X, Y)$ converging to a bounded operator T in the weak operator topology. Suppose that $\operatorname{rank} T \ge n + 1$. We can choose vectors $x_1, \ldots, x_{n+1}$ such that the collection $\{Tx_i\}_{i=1}^{n+1} \subseteq Y$ is linearly independent. By the Hahn–Banach theorem, there exist linear functionals $y_j^* \in Y^*$, $j = 1, \ldots, n + 1$, with $y_j^*(Tx_j) = 1$ and $y_j^*(Tx_i) = 0$ for $i \ne j$. Now, define the operator $S \in \mathcal{F}_{n+1}(Y)$ by the formula $Sy = \sum_{j=1}^{n+1} y_j^*(y)Tx_j$. Since $T_\alpha x_1, \ldots, T_\alpha x_{n+1}$ converge weakly to $Tx_1, \ldots, Tx_{n+1}$, we

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conclude that $\lim_\alpha S(T_\alpha x_i) = Tx_i$ for $i = 1, \dots, n+1$. Also, since $Tx_1, \dots, Tx_{n+1}$ are linearly independent, the collection $\{S(T_\alpha x_i)\}_{i=1}^{n+1}$ will eventually become linearly independent and so will the preimage $\{T_\alpha x_i\}_{i=1}^{n+1}$, which contradicts the hypothesis that $T_\alpha \in \mathcal{F}_n(X, Y)$. □

The following lemma provides a connection between almost-invariant subspaces and their quotient maps.

Lemma 2.2 [6]. *Let $T \in \mathcal{B}(X)$ and Y be a subspace of X. Let $q : X \longrightarrow X/Y$ be the quotient map. Then Y is an almost-invariant subspace under T if and only if $(qT)|_Y$ is of finite rank. Moreover,* $\dim(qT)(Y) = d_{Y,T}$.

Proposition 2.3. *Suppose that $(T_\alpha)_{\alpha\in I}$ is a net of bounded operators on X converging to a bounded operator T in the weak operator topology. Let Y be an almost-invariant subspace under every T_α with $d_{Y,T_\alpha} \le N$. Then Y is almost invariant under T with $d_{Y,T} \le N$.*

Proof. Let $q : X \longrightarrow X/Y$ be the quotient map. Since $\text{wot-lim}_\alpha T_\alpha = T$ and q is a bounded operator, $\text{wot-lim}_\alpha (qT_\alpha)|_Y = (qT)|_Y$. By Lemma 2.2, each $(qT_\alpha)|_Y$ is a finite-rank operator with rank $\le N$. Now, by Lemma 2.1, $\operatorname{rank}(qT)|_Y \le N$ and again, by Lemma 2.2, Y is almost invariant under T with $d_{Y,T} \le N$. □

Let Y be a closed subspace of X. Similarly to invariant subspaces, the set of all bounded operators which have Y as an almost-invariant subspace is a subalgebra of $\mathcal{B}(X)$, denoted by $\text{Alg}_a Y$. Unfortunately, it is not a closed algebra; by [1, Proposition 1.3], $\text{Alg}_a Y = \text{Alg}\, Y + \mathcal{F}(X)$.

For $T \in \mathcal{B}(X)$, a subspace Y of X is called essentially invariant under T if it is invariant under $T + K$ for some $K \in \mathcal{K}(X)$, where $\mathcal{K}(X)$ denotes the class of compact operators on X. By [11, Corollary 4.3], every bounded operator on a Banach space admits an essentially invariant half-space. The set of all bounded operators which have Y as an essentially invariant subspace is a subalgebra of $\mathcal{B}(X)$, denoted by $\text{Alg}_e Y$. Clearly, $\text{Alg}_e Y = \text{Alg}\, Y + \mathcal{K}(X)$.

Suppose that $(T_n)_{n=1}^\infty$ is a sequence of bounded operators on X converging to T in norm topology and Y is an almost-invariant subspace under each T_n. We can ask, does T admit Y as an essentially invariant subspace? In other words, is $\overline{\text{Alg}_a Y} \subseteq \text{Alg}_e Y$? When Y is a complemented subspace of X, the answer is affirmative. Indeed, let P be a projection on X with range Y. Since Y is an almost-invariant subspace under each T_n, it follows that $(I - P)T_nP$ is a finite-rank operator. Moreover, $(T_n - (I - P)T_nP)Y \subseteq Y$. So, $(T - (I - P)TP)Y \subseteq Y$ and $(I - P)TP$ is a compact operator.

Now, suppose that X has the approximation property, in particular $\overline{\mathcal{F}(X)} = \mathcal{K}(X)$. Then

$$\text{Alg}_e Y = \text{Alg}\, Y + \mathcal{K}(X) = \text{Alg}\, Y + \overline{\mathcal{F}(X)} \subseteq \overline{\text{Alg}\, Y + \mathcal{F}(X)} = \overline{\text{Alg}_a Y}.$$

If $\text{Alg}_e Y$ is also a norm-closed subalgebra of $\mathcal{B}(X)$, then $\overline{\text{Alg}_a Y} = \text{Alg}_e Y$. This motivates and proves the next corollary.

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Corollary 2.4. *Suppose that* X *has the approximation property and* Y *is a subspace of* X. *Then* $\overline{\mathrm{Alg}_a Y} = \mathrm{Alg}_e Y$ *if and only if* $\mathrm{Alg}_e Y = \mathrm{Alg}\, Y + \mathcal{K}(X)$ *is norm-closed in* $\mathcal{B}(X)$. *In particular, if* Y *is a complemented subspace of* X, *then the subalgebra* $\mathrm{Alg}\, Y + \mathcal{K}(X)$ *is a norm-closed subspace of* $\mathcal{B}(X)$.

We denote by $\mathrm{Lat}_a T$ the set of all almost-invariant subspaces under T. According to [1, Proposition 1.3], $\mathrm{Lat}_a T = \bigcup_{F \in \mathcal{F}(X)} \mathrm{Lat}(T + F)$. Similarly to invariant subspaces, $\mathrm{Lat}_a T$ is a complete lattice. Indeed, if Y_1 and $Y_2 \in \mathrm{Lat}_a T$, then there exist finite-dimensional subspaces M_1 and M_2 such that $TY_1 \subseteq Y_1 + M_1$ and $TY_2 \subseteq Y_2 + M_2$. So, $T(Y_1 + Y_2) \subseteq Y_1 + Y_2 + M_1 + M_2$ and, since $M_1 + M_2$ is of finite dimension,

$$T(\mathrm{cl}(Y_1 + Y_2)) \subseteq \mathrm{cl}(Y_1 + Y_2) + M_1 + M_2.$$

Therefore, $\mathrm{cl}(Y_1 + Y_2) \in \mathrm{Lat}_a T$. Also, by [2, Proposition 2.2], there exist finite-codimensional subspaces N_1 and N_2 such that $T(Y_1 \cap N_1) \subseteq Y_1$ and $T(Y_2 \cap N_2) \subseteq Y_2$. Hence,

$$T(Y_1 \cap Y_2 \cap N_1 \cap N_2) \subseteq T(Y_1 \cap N_1) \cap T(Y_2 \cap N_2) \subseteq Y_1 \cap Y_2.$$

Since $N_1 \cap N_2$ is still of finite codimension, this shows that $Y_1 \cap Y_2 \in \mathrm{Lat}_a T$.

For a subspace Y of X, we denote by $\Lambda_a^n Y$ the set of all bounded operators which have Y as an almost-invariant subspace with defect $\leq n$. Clearly, $\Lambda_a^n Y = \mathrm{Alg}\, Y + \mathcal{F}_n(X)$. By Proposition 2.3, $\Lambda_a^n Y$ is a closed subset of $\mathcal{B}(X)$ in the weak operator topology. If $\mathcal{L}$ is a collection of subspaces of X, we can similarly define $\mathrm{Alg}_a \mathcal{L}$ and $\Lambda_a^n \mathcal{L}$. Clearly, $\mathrm{Alg}_a \mathcal{L} = \bigcap_{Y \in \mathcal{L}} \mathrm{Alg}_a Y$ and $\Lambda_a^n \mathcal{L} = \bigcap_{Y \in \mathcal{L}} \Lambda_a^n Y$.

Popov stated the following theorem and gave a rather lengthy and technical proof.

Theorem 2.5 [6]. *Let* $\mathcal{A}$ *be a norm-closed subspace of* $\mathcal{B}(X)$. *Suppose that* Y *is a subspace of* X *that is almost invariant under* $\mathcal{A}$. *Then* $\sup\{d_{Y,S} : S \in \mathcal{A}\} < \infty$.

We extend this theorem and give a much shorter proof.

Theorem 2.6. *Let* $\mathcal{L}$ *be a finite collection of subspaces of* X. *Let* C *be a norm-closed convex subset of* $\mathcal{B}(X)$ *such that* $C \subseteq \mathrm{Alg}_a \mathcal{L}$. *Then there exists an integer* $n \geq 0$ *such that* $C \subseteq \Lambda_a^n \mathcal{L}$.

Proof. Set $C_k = C \cap \Lambda_a^k \mathcal{L}$. By Proposition 2.3, C_k is a closed subset of C for all k. Also, since $\mathcal{L}$ is a finite collection, $C = \bigcup_{k=1}^{\infty} C_k$. Considering C as a complete metric space, by the Baire category theorem, there exists an integer $k > 0$ such that the interior of C_k in C is nonempty. Choose an operator T_0 in the interior of C_k in C. Since $C - T_0 = \{T - T_0 : T \in C\}$ is still convex and $0 \in C - T_0$, we have $t(T - T_0) \in C - T_0$ for $0 \leq t \leq 1$ and $T \in C$. Now, fix an operator $T \in C$ and consider the continuous map $f : [0, 1] \longrightarrow C - T_0$ given by $f(t) = t(T - T_0)$. Since $C_k - T_0$ contains an open ball in the metric space $C - T_0$ of positive radius at 0, there is a real number $s > 0$ such that

$$s(T - T_0) = f(s) \in C_k - T_0 \subseteq \Lambda_a^k \mathcal{L} + \Lambda_a^k \mathcal{L} \subseteq \Lambda_a^{2k} \mathcal{L}.$$

Therefore,

$$T \in \Lambda_a^{2k} \mathcal{L} + T_0 \subseteq \Lambda_a^{3k} \mathcal{L}$$

and setting $n = 3k$ completes the proof. □

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The finiteness of $\mathcal{L}$ in the previous theorem is necessary. Indeed, if $\mathcal{L}$ includes a chain $Y_1 \subsetneq Y_2 \subsetneq Y_3 \subsetneq \cdots$ of finite-dimensional subspaces of an infinite-dimensional Banach space X, then $\mathcal{B}(X) = \mathrm{Alg}_a\mathcal{L}$. However, there is no integer $n \geq 1$ such that $\mathcal{B}(X) \subseteq \Lambda_a^n\mathcal{L}$.

For two different subspaces Y and Z of X, there exists a rank-one operator T on X such that Y is invariant under T, but Z is not. In particular, $\mathrm{Alg}\, Y \neq \mathrm{Alg}\, Z$. Now, we obtain a similar result for almost-invariant subspaces.

For the subspaces Y_1 and Y_2, we say that Y_1 is almost equivalent to Y_2 if there exist finite-dimensional subspaces M_1 and M_2 such that $Y_1 + M_1 = Y_2 + M_2$.

Proposition 2.7. *For a subspace Y and a half-space Z of X, which are not almost equivalent, there exists an operator $T \in \overline{\mathcal{F}(X)}$ such that Y is almost invariant under T, but Z is not. In particular, if both Y and Z are half-spaces, then $\mathrm{Alg}_a Z \nsubseteq \mathrm{Alg}_a Y$ and $\mathrm{Alg}_a Y \nsubseteq \mathrm{Alg}_a Z$.*

Proof. First, we suppose that Y is not a half-space. Then $\mathrm{Alg}_a Y = \mathcal{B}(X)$ and we show that $\overline{\mathcal{F}(X)} \nsubseteq \mathrm{Alg}_a Z$.

Let Z be an almost-invariant half-space under every operator in $\overline{\mathcal{F}(X)}$. Since $\overline{\mathcal{F}(X)}$ is a norm-closed algebra, by [11, Theorem 1.1], there exists a half-space Z' which is invariant under every operator in $\overline{\mathcal{F}(X)}$. This contradicts the transitivity of $\overline{\mathcal{F}(X)}$.

Now, suppose that both Y and Z are half-spaces. Since Y and Z are not almost equivalent, we can assume, without loss of generality, that $Z \nsubseteq Y + \mathrm{span}\{z_i\}_{i=1}^n$ for all integers $n > 0$ and each set of linearly independent vectors $\{z_i\}_{i=1}^n \subseteq Z$. We show that $\mathrm{Alg}_a Z \nsubseteq \mathrm{Alg}_a Y$ and $\mathrm{Alg}_a Y \nsubseteq \mathrm{Alg}_a Z$.

If $\mathrm{Alg}_a Y \subseteq \mathrm{Alg}_a Z$, then $\mathrm{Alg}\, Y \subseteq \mathrm{Alg}_a Z$ and, by Theorem 2.6, there is an integer $k > 0$ such that $\mathrm{Alg}\, Y \subseteq \Lambda_a^k Z$. We can choose linearly independent vectors $\{y_i\}_{i=1}^{k+1} \subseteq Y$ and linearly independent vectors $\{z_i\}_{i=1}^{k+1} \subseteq Z$ such that $\mathrm{span}\{z_i\}_{i=1}^{k+1} \cap Y = \{0\}$. Since $y_1, \ldots, y_{n+1}$ are linearly independent, there are linear functionals $\{x_i^*\}_{i=1}^{k+1}$ with $x_i^*(y_i) = 1$ and $x_i^*(y_j) = 0$ for $j \neq i$. Now, define the operator $T \in \mathcal{F}(X)$ by $Tx = \sum_{i=1}^{k+1} x_i^*(x) z_i$. It is easily seen that $TZ \subseteq Z$ and $d_{Y,T} \geq k + 1$, which is a contradiction.

If $\mathrm{Alg}_a Z \subseteq \mathrm{Alg}_a Y$, then $\mathrm{Alg}\, Z \subseteq \mathrm{Alg}_a Y$ and, by Theorem 2.6, there is a $k > 0$ such that $\mathrm{Alg}\, Z \subseteq \Lambda_a^k Y$. Since Z is a half-space, we can choose linearly independent vectors $\{z_i\}_{i=1}^{k+1} \subseteq Z$ and linearly independent vectors $\{w_i\}_{i=1}^{k+1} \subseteq X$ with $\mathrm{span}\{w_i\}_{i=1}^{k+1} \cap Z = \{0\}$ and $\mathrm{span}\{z_i\}_{i=1}^{k+1} \cap Y = \{0\}$. By the Hahn–Banach theorem, there are linear functionals $\{x_i^*\}_{i=1}^{k+1}$ with $x_i^*|Y = 0$, $x_i^*(z_i) = 1$ and $x_i^*(z_j) = 0$ for $j \neq i$. If we define the operator $S \in \mathcal{F}(X)$ by $Sx = \sum_{i=1}^{k+1} x_i^*(x) w_i$, then $SY \subseteq Y$ and $d_{Z,S} \geq k + 1$, which is a contradiction. □

3. Properties of operators having a collection of almost-invariant subspaces

If $T \in \mathcal{B}(X)$ and each subspace of X is invariant under T, then T must be a multiple of the identity. What happens if each subspace of X is almost invariant under T? In [1], it is shown that T has a nontrivial invariant subspace of finite codimension. If X is a Hilbert space, then T has the form $\alpha I + F$ for some scalar α and a finite-rank operator F [3, Corollary 4.16]. We extend this result to a Banach space X.

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First, we give some lemmas needed in the proof.

Lemma 3.1. *Let $T \in \mathcal{B}(X)$ and M be a finite-dimensional subspace of X such that M and $M + \operatorname{span}\{x\}$ are invariant under T for every $x \in X$. Then $T = \alpha I + F$ for some scalar α and a finite-rank operator F.*

Proof. Consider the operator $\tilde{T} : X/M \to X/M$ given by $\tilde{T}(x + M) = Tx + M$. Since the subspace $M + \operatorname{span}\{x\}$ is invariant under T for all $x \in X$, every one-dimensional subspace of X/M is invariant under $\tilde{T}$. This implies that $\tilde{T} = \alpha I$ for some scalar α. Now, we define the operator F on X by $Fx = Tx - \alpha x$. It is clear that $FX \subseteq M$ and $T = \alpha I + F$. □

Lemma 3.2. *Suppose that $T \in \mathcal{B}(X)$ and every subspace of X is almost invariant under T. Then, for every $x \in X$, the subspace $\operatorname{cl}(\operatorname{span}\{T^n x\}_{n=0}^{\infty})$ is of finite dimension.*

Proof. Suppose that for some $x_1 \in X$ the subspace $\operatorname{cl}(\operatorname{span}\{T^n x_1\}_{n=0}^{\infty})$ is of infinite dimension. Since $\operatorname{span}\{T^n x_1\}_{n=0}^{\infty}$ is also of infinite dimension, $T^k x_1 \notin \operatorname{span}\{T^n x_1\}_{n=0}^{k-1}$ for all $k \geq 1$. We will construct a subspace of X that is not almost invariant under T.

Consider $x_1^* \in X^*$ such that $x_1^*(x_1) \neq 0$. Let $P_1(x) = x - (x_1^*(x)/x_1^*(x_1))x_1$ be the projection on X with kernel $\operatorname{span}\{x_1\}$ and image $\ker x_1^*$. Define $x_2 = P_1 T x_1$. It is easily seen that $\operatorname{span}\{x_1, Tx_1\} = \operatorname{span}\{x_1, x_2\}$ and $x_2 \notin \operatorname{span}\{x_1\}$, since $Tx_1 \notin \operatorname{span}\{x_1\}$.

We claim that for each $n \geq 1$, there exist sequences $\{x_n\}$ of vectors, $\{x_n^*\}$ of functionals and $\{P_n\}$ of projections on X such that:

(i) $x_i^*(x_j) = 0$ if and only if $i \neq j$;
(ii) $P_n(x) = x - \sum_{k=1}^{n}(x_k^*(x)/x_k^*(x_k))x_k$ is the projection with kernel $\operatorname{span}\{x_1, \ldots, x_n\}$ and image $\bigcap_{i=1}^{n} \ker x_i^*$;
(iii) $x_n = P_{n-1} T x_{n-1}$;
(iv) $\operatorname{span}\{x_1, \ldots, T^{n-1}x_1\} = \operatorname{span}\{x_1, \ldots, x_n\}$;
(v) $x_n \notin \operatorname{span}\{x_1, \ldots, x_{n-1}\}$.

Indeed, suppose that we have defined x_i, x_{i-1}^* and P_{i-1}, for $1 \leq i \leq n$, satisfying (i)–(v). Since $x_n \notin \operatorname{span}\{x_1, \ldots, x_{n-1}\}$, we can choose $x_n^* \in X^*$ such that $x_n^*(x_i) = 0$ for $1 \leq i \leq n-1$ and $x_n^*(x_n) \neq 0$. Let $P_n(x) = x - \sum_{k=1}^{n}(x_k^*(x)/x_k^*(x_k))x_k$ be the projection with kernel $\operatorname{span}\{x_1, \ldots, x_n\}$ and image $\bigcap_{i=1}^{n} \ker x_i^*$. Define $x_{n+1} = P_n T x_n$. There exists $y_n \in \operatorname{span}\{x_1, \ldots, x_n\}$ such that $x_{n+1} = Tx_n + y_n$. By (iv), $x_n, y_n \in \operatorname{span}\{x_1, \ldots, T^{n-1}x_1\}$ and so $x_{n+1} \in \operatorname{span}\{x_1, \ldots, T^n x_1\}$. On the other hand, $T^{n-1}x_1 \in \operatorname{span}\{x_1, \ldots, x_n\}$ and $Tx_i \in \operatorname{span}\{x_1, \ldots, x_{i+1}\}$ for $1 \leq i \leq n$, so

$$T^n x_1 \in \operatorname{span}\{Tx_1, \ldots, Tx_n\} \subseteq \operatorname{span}\{x_1, \ldots, x_{n+1}\}.$$

It follows that $\operatorname{span}\{x_1, \ldots, T^n x_1\} = \operatorname{span}\{x_1, \ldots, x_{n+1}\}$. Also, since

$$T^n x_1 \notin \operatorname{span}\{x_1, \ldots, T^{n-1}x_1\} = \operatorname{span}\{x_1, \ldots, x_n\}$$

and

$$T^n x_1 \in \operatorname{span}\{x_1, \ldots, T^n x_1\} = \operatorname{span}\{x_1, \ldots, x_{n+1}\},$$

we have $x_{n+1} \notin \operatorname{span}\{x_1, \ldots, x_n\}$.

Now, set $Z = \mathrm{cl}(\mathrm{span}\{x_{2n-1}\}_{n=1}^{\infty})$. By assumption, there exists a finite-dimensional subspace M such that $TZ \subseteq Z + M$. So, $Tx_{2n-1} = z_n + m_n$ for some $z_n \in Z$ and $m_n \in M$. Also, since $P_{2n-1}Tx_{2n-1} = x_{2n}$, we have $Tx_{2n-1} = x_{2n} + u_n$ for some $u_n \in \mathrm{span}\{x_1, \ldots, x_{2n-1}\}$.

Let j and n be natural numbers and $j > n$. Since $x_{2j}^*(x_{2n}) = x_{2j}^*(u_n) = x_{2j}^*(z_n) = 0$, we have $x_{2j}^*(m_n) = 0$. On the other hand, $x_{2n}^*(x_{2n}) \neq 0$, $x_{2n}^*(u_n) = 0$ and $x_{2n}^*(z_n) = 0$. Therefore, $x_{2n}^*(m_n) \neq 0$. We conclude that $x_{2n}^*(m_n) \neq 0$ and $x_{2j}^*(m_n) = 0$ for all n and $j > n$, contradicting $\dim M < \infty$. □

Proposition 3.3. *Suppose that $T \in \mathcal{B}(X)$ and every subspace of X is almost invariant under T. Then $T = \alpha I + F$ for some scalar α and $F \in \mathcal{F}(X)$.*

Proof. Suppose that T cannot be expressed in the form $\alpha I + F$ for any scalar α and $F \in \mathcal{F}(X)$. Start with the subspace $\{0\}$ of X. By Lemma 3.1, there is $x_1 \in X$ such that $Tx_1 \notin \mathrm{span}\{x_1\}$. Set $M_1 = \mathrm{span}\{x_1\}$ and choose $x_1^* \in X^*$ such that $x_1^*|M_1 = 0$ and $x_1^*(Tx_1) \neq 0$. Also, set $M_1' = \mathrm{cl}(\mathrm{span}\{T^k x_1\}_{k=0}^{\infty})$, which is invariant under T. By Lemma 3.2, M_1' is of finite dimension and again, by Lemma 3.1, there is $x_2 \in X$ such that $M_1' + \mathrm{span}\{x_2\}$ is not invariant under T. Since $X = \ker x_1^* \oplus \mathrm{span}\{Tx_1\}$ and $Tx_1 \in M_1'$, we can choose x_2 in $\ker x_1^*$.

Continuing inductively in this way, we can construct sequences $\{x_n\}$ of vectors, $\{x_n^*\}$ of functionals and $\{M_n\}$ and $\{M_n'\}$ of finite-dimensional subspaces of X such that, for $n = 1, 2, \ldots$:

(i) $x_i^*(x_j) = 0$ for all i and j;
(ii) $x_i^*(Tx_j) \neq 0$ if $i = j$, and $x_i^*(Tx_j) = 0$ if $i > j$;
(iii) $M_n = M_{n-1}' + \mathrm{span}\{x_n\}$;
(iv) $M_n' = M_n + \mathrm{cl}(\mathrm{span}\{T^k x_n\}_{k=0}^{\infty})$ and M_n' is invariant under T.

Indeed, suppose that we have defined x_i, x_i^*, M_i and M_i', for $1 \leq i \leq n$, satisfying (i)–(iv). Since M_n' is of finite dimension, by Lemma 3.1, there exists $z_{n+1} \in X$ such that $M_n' + \mathrm{span}\{z_{n+1}\}$ is not invariant under T. By (ii),

$$X = \bigcap_{i=1}^{n} \ker x_i^* \oplus \mathrm{span}\{Tx_1, \ldots, Tx_n\}.$$

Since $\mathrm{span}\{Tx_1, \ldots, Tx_n\} \subseteq M_n'$, there exists $x_{n+1} \in \bigcap_{i=1}^{n} \ker x_i^*$ with $M_n' + \mathrm{span}\{x_{n+1}\} = M_n' + \mathrm{span}\{z_{n+1}\}$. This means that $M_n' + \mathrm{span}\{x_{n+1}\}$ is not invariant under T and, so, $Tx_{n+1} \notin M_n' + \mathrm{span}\{x_{n+1}\}$. Define $M_{n+1} = M_n' + \mathrm{span}\{x_{n+1}\}$ and choose $x_{n+1}^* \in X^*$ such that $x_{n+1}^*|M_{n+1} = 0$ and $x_{n+1}^*(Tx_{n+1}) \neq 0$. Then $x_{n+1}^*(x_j) = 0$, for $j = 1, \ldots, n+1$, and $x_{n+1}^*(Tx_j) = 0$, for $j = 1, \ldots, n$. Set $M_{n+1}' = M_{n+1} + \mathrm{cl}(\mathrm{span}\{T^k x_{n+1}\}_{k=0}^{\infty})$, which is invariant under T by Lemma 3.2. Also, M_{n+1}' is of finite dimension.

Now, define $Z = \mathrm{cl}(\mathrm{span}\{x_n\}_{n=1}^{\infty})$. By assumption, there exists a finite-dimensional subspace M such that $TZ \subseteq Z + M$. So, for each $x_n \in Z$, there exist $z_n \in Z$ and $m_n \in M$ such that $Tx_n = z_n + m_n$. Since $x_n^*(Tx_n) \neq 0$ and $x_n^*(z_n) = 0$, we have $x_n^*(m_n) \neq 0$. Also, for $k > n$, we have $x_k^*(Tx_n) = x_k^*(z_n) = 0$. Therefore, $x_k^*(m_n) = 0$. It follows that $x_n^*(m_n) \neq 0$ and $x_k^*(m_n) = 0$ for all n and $k > n$, contradicting $\dim M < \infty$. □

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Let T be an operator on a Banach space X. It is known that if T commutes with every operator on X, then T must be a multiple of the identity. Using Proposition 3.3, we show that if X is a separable Banach space and $TS - ST$ is a finite-rank operator, for all $S \in \mathcal{B}(X)$, then T will be of the form $\alpha I + F$, where rank $F < \infty$.

Corollary 3.4. *Let T be an operator on a separable Banach space X and suppose that $TS - ST \in \mathcal{F}(X)$ for every $S \in \mathcal{B}(X)$. Then $T = \alpha I + F$ for some scalar α and $F \in \mathcal{F}(X)$.*

Proof. According to Proposition 2.3, it is sufficient to show that every subspace of X is almost invariant under T.

Let Y be an arbitrary closed subspace of X. Since both X and X/Y are separable, by [3, Proposition 3.1], there exists a bounded linear operator Φ from X/Y to X that is one-to-one. Also, if $q : X \longrightarrow X/Y$ is the quotient map, then $S = \Phi q$ will be a bounded operator on X such that $Y = \ker S$. By assumption, there exists $F \in \mathcal{F}(X)$ such that $ST - TS = F$. So, $ST(\ker S) \subseteq FX$ and then $T(\ker S) \subseteq S^{-1}(FX)$. Since $FX \cap SX$ is of finite dimension, there exists a finite-dimensional subspace M such that $FX \cap SX = SM$. Now,

$$S^{-1}(FX) = S^{-1}(FX \cap SX) = S^{-1}(SM) = M + \ker S.$$

Therefore, $T(\ker S) \subseteq M + \ker S$ and $Y = \ker S$ is almost invariant under T. □

Let $\mathcal{L}$ be a collection of closed subspaces of a Banach space X. It is clear that $\mathrm{Alg}\,\mathcal{L} + \mathcal{F}(X) \subseteq \mathrm{Alg}_a Y$. Now, we can ask, under which conditions on $\mathcal{L}$ will we have $\mathrm{Alg}_a\mathcal{L} = \mathrm{Alg}\,\mathcal{L} + \mathcal{F}(X)$?

For a single subspace $\mathcal{L} = \{Y\}$, we have $\mathrm{Alg}_a Y = \mathrm{Alg}\,Y + \mathcal{F}(X)$. In view of Proposition 3.3, if $\mathcal{L}$ is the set of all subspaces of X, then $\mathrm{Alg}_a\mathcal{L} = \mathrm{Alg}\,\mathcal{L} + \mathcal{F}(X)$. However, this is not true in general. It is enough to consider $\mathcal{L}$ as the collection of all finite-dimensional subspaces of X. In the next two propositions, we examine some conditions under which the conclusion does hold.

Proposition 3.5. *If $\mathcal{L} = \{Y_1, \dots, Y_n\}$ is a finite collection of subspaces of X such that $X = Y_1 \oplus \cdots \oplus Y_n$, then $\mathrm{Alg}_a\mathcal{L} = \mathrm{Alg}\,\mathcal{L} + \mathcal{F}(X)$.*

Proof. Since X is a direct sum of subspaces $Y_1, \dots, Y_n$, there exist bounded projections $P_1, \dots, P_n$ such that $P_iX = Y_i$ and $\ker P_i = \sum_{k=1,k\neq i}^{n} Y_k$ for $1 \le i \le n$. Also, $P_iP_j = 0$ whenever $i \neq j$ and $\sum_{i=1}^{n} P_i = I$.

Let $T \in \mathrm{Alg}_a\mathcal{L}$. Since each Y_i is almost invariant under T, there exists a finite-dimensional subspace M_i such that $TY_i \subseteq Y_i + M_i = P_iX + M_i$. For $i \neq j$,

$$P_jTP_iX = P_jTY_i \subseteq P_j(Y_i + M_i) \subseteq P_jM_i.$$

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Therefore, the operator P_jTP_i is of finite rank whenever $i \neq j$. On the other hand,

$$P_k\Big(T - \sum_{i,j=1,j\neq i}^{n} P_jTP_i\Big) = P_kT - P_kT \sum_{i=1,i\neq k}^{n} P_i = P_kT - P_kT(I - P_k) = P_kTP_k$$
$$= TP_k - (I - P_k)TP_k = TP_k - \Big(\sum_{i=1,i\neq k}^{n} P_i\Big)TP_k$$
$$= \Big(T - \sum_{i,j=1,j\neq i}^{n} P_jTP_i\Big)P_k$$

for $k = 1, \ldots, n$.

This shows that $T - \sum_{j=1,j\neq i}^{n} P_jTP_i \in \operatorname{Alg}\mathcal{L}$ and, since

$$T = \Big(T - \sum_{i,j=1,j\neq i}^{n} P_jTP_i\Big) + \sum_{i,j=1,j\neq i}^{n} P_jTP_i,$$

the proof is complete. □

Remark 3.6. For an operator T and an almost-invariant subspace Y, there exists a finite-dimensional subspace M with $TY \subseteq Y + M$ and $Y \cap M = \{0\}$. We can find a projection P on X with range M and kernel containing Y such that $(T - PT)Y \subseteq Y$.

Indeed, if $q : X \longrightarrow X/Y$ for the quotient map, then $q(M)$ is a finite-dimensional subspace of X/Y. There is a subspace $L' \subseteq X/Y$ such that $L' \oplus q(M) = X/Y$. Since $Y \cap M = \{0\}$, by setting $L = q^{-1}(L')$, we have $M \oplus L = X$ and $L \supseteq Y$. Now, if we consider the projection on X with kernel L and range M, then $(T - PT)Y \subseteq Y$.

Proposition 3.7. *Let $\mathcal{L} = \{Y_1, \ldots, Y_n\}$ be a finite collection of subspaces of X with $Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_n$. Then $\operatorname{Alg}_a\mathcal{L} = \operatorname{Alg}\mathcal{L} + \mathcal{F}(X)$.*

Proof. Given $T \in \operatorname{Alg}_a\mathcal{L}$, let M_1 be a finite-dimensional subspace of X such that $Y_1 \cap M_1 = \{0\}$ and $TY_1 \subseteq Y_1 + M_1$. By Remark 3.6, there exists a projection P_1 on X with range M_1 and kernel containing Y_1 such that Y_1 is invariant under $T - P_1T$. Set $S_1 = T - P_1T$. Since P_1T is of finite rank, Y_2 is almost invariant under S_1 and, by [6, Lemma 2.1], we can choose a finite-dimensional subspace M_2 such that $M_2 \subseteq S_1Y_2 \subseteq S_1Y_1 \subseteq Y_1$, $Y_2 \cap M_2 = \{0\}$ and $S_1Y_2 \subseteq Y_2 + M_2$. Consider a projection P_2 on Y_1 with range M_2 and kernel containing Y_2. Since P_2 is of finite rank, it can be extended to a bounded linear operator $\tilde{P}_2$ on all of X with the same range as P_2. It is easily seen that Y_1 and Y_2 are invariant under the operator $S_1 - \tilde{P}_2S_1$.

Continuing this process, we obtain operators $\{S_i, P_i, \tilde{P}_i\}_{i=1}^n$ and finite-dimensional subspaces $\{M_i\}_{i=1}^n$ of X such that, for $i = 1, \ldots, n$:

(i) $S_{i-1}Y_i \subseteq Y_i + M_i$, $Y_i \cap M_i = \{0\}$ and $M_i \subseteq S_{i-1}Y_i \subseteq S_{i-1}Y_{i-1} \subseteq Y_{i-1}$ for $i = 2, \ldots, n$;

(ii) P_i is a projection on Y_{i-1} with range M_i and kernel including Y_i;

(iii) $\tilde{P}_i$ is an extension of P_i on X with the same range as P_i;

(iv) $S_i = S_{i-1} - \tilde{P}_iS_{i-1}$, $S_0 = T$ and $\tilde{P}_1 = P_1$;

(v) the subspaces $Y_1, \ldots, Y_i$ are invariant under S_i.

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So,

$$T = S_n + \tilde{P}_n S_{n-1} + \tilde{P}_{n-1} S_{n-2} + \cdots + \tilde{P}_2 S_1 + P_1 T$$

and finally $S_n \in \operatorname{Alg} \mathcal{L}$ and $\tilde{P}_n S_{n-1} + \tilde{P}_{n-1} S_{n-2} + \cdots + \tilde{P}_2 S_1 + P_1 T \in \mathcal{F}(X)$. □

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