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PARTIAL SPREADS AND REPLACEABLE NETS

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1. Introduction. A blocking set S in a projective plane Π is a subset of the points of Π such that every line of Π contains at least one point of S and at least one point not in S. In previous papers [**5**; **6**], we have shown that if Π is finite of order n, then $n + \sqrt{n} + 1 \leqq |S| \leqq n^2 - \sqrt{n}$ (see [**6**, Theorem 3.9]), where $|S|$ stands for the number of points of S. This work is concerned with some applications of the above result to nets and partial spreads, and with some examples of partial spreads which give rise to unimbeddable nets of small deficiency.

In the next section we re-prove a well known result of Bruck which states that if N is a replaceable net of order n and degree k, then $k \geqq \sqrt{n} + 1$, and show how this bound can be improved if $n = 7$, 8, or 11. We also reprove a result of Ostrom, concerning the imbeddability of a net N of order n and deficiency $\sqrt{n} + 1$, which states that N is imbeddable in at most 2 planes Π_1 and Π_2, and that if Π_1 and Π_2 exist they are related to each other by derivation.

In § 3, we show that if W is a maximal strictly partial spread of PG(3, q), then $q + \sqrt{q} + 1 \leqq |W| \leqq q^2 - \sqrt{q}$. This answers a question posed in Mesner [**13**]. Here, $|W|$ denotes the number of lines in W, and q is a prime power.

We improve this by showing that if $q = 3$, $|W|$ must be 7, and we give an example of this case. Maximal strictly partial (msp) spreads W are exhibited such that either (1) $|W| = q^2 - q + 1$ or (2) $|W| = q^2 - q + 2$. We prove that if $q > 2$ the first case always occurs and we construct examples for the second case whenever q is odd and greater than 3. For both cases it can be shown that the partial spread W gives rise to a net which is not imbeddable in a plane, provided that q is a prime greater than two. The proof of the existence of case (1) (see Theorem 3.5) can be extended to show the existence of msp spreads in PG(3, **R**), for example, where **R** denotes the field of real numbers. The existence of msp spreads in PG(3, F), with F countably infinite, has already been shown in [**7**].

§ 4 contains some miscellaneous remarks and sketches of some related results.

2. Blocking Sets and Replaceable Nets. One of the best known examples of a non-Desarguesian plane is the classical Moulton plane. Starting with the Euclidean plane we "bend" the lines with positive slopes, leaving the other

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lines as they are, and in this fashion we obtain an affine plane, namely the Moulton plane, whose projective completion is a non-Desarguesian plane. In modern language, we replace the net consisting of all lines with positive slopes. In fact most of the known planes can be interpreted as having been obtained from a Desarguesian plane by means of net replacement. For basic definitions as well as an excellent account of the above idea we refer the reader to [**17**]. Here we show a connection between replaceable nets N and blocking sets and obtain some well known results on nets as corollaries of Theorem 3.9 in [**6**]. In the sequel 'N is replaceable' means that the net N admits a non trivial replacement (see [**16**] or [**17**]). Let N be a net of order n and degree k which is embedded in an affine plane Π_A of order n, *where we assume that* $n > 2$ *and that* $k < n + 1$. In other words the points of N are the points of Π_A and the lines of N are the lines belonging to some k parallel classes of Π_A. Suppose N is replaceable by the net N^1. We list the slopes associated with $N(N^1)$ as (m_1), (m_2), ..., (m_k), where each m_i is a point on l_∞, the line at infinity of Π_A corresponding to the projective completion Π of Π_A. Since the replacing net N^1 is non-trivial, some line l^1 of N^1 is not a line of N. l^1 is a set of n points of Π and, by definition, any line of Π connecting 2 distinct points of l^1 meets l_∞ in an (m_i), $i = 1, \ldots, k$. Hence any line in Π_A which does not have a slope m_i $(1 \leqq i \leqq k)$ must intersect l^1 (in exactly one point), and a line which does have a slope m_i contains points not in l^1. Thus the points of l^1 together with the (m_i) form a blocking set in Π (in fact a special sort of blocking set S with $|S| = n + k$, and with some line l_∞ of Π containing precisely k points of S). We have therefore established:

THEOREM 2.1. *Let N be a replaceable net of degree k embedded in an affine plane Π_A of order n. Then there exists a blocking set S in Π, the projective extension of Π_A, with $|S| = n + k$.*

Now a blocking set S in Π contains at least $n + \sqrt{n} + 1$ points ([**6**, Theorem 3.9]). Thus we obtain

COROLLARY 2.2. (See [**1**, Theorem 3.1]). *If N is a replaceable net of order n then N contains at least $\sqrt{n} + 1$ parallel classes.*

Using Theorems (4.1), (4.2), and (4.4) of [**6**], we can improve this slightly in certain cases, namely:

COROLLARY 2.3. *If N is a replaceable net which is embedded in a plane of order* 7, 8, 11, *respectively, then N contains at least* 5, 5, 6 *parallel classes, respectively.*

Next, we briefly return to the case of $n = t^2$. In [**5**] we showed that if S is a blocking set in a projective plane Π of order t^2 with $|S| = t^2 + t + 1$, then the points of S must form the points of a subplane of Π of order t. We now make use of this. Suppose N is a replaceable net of degree $t + 1$ in an affine plane Π_A of order t^2. Let N^1 be a replacing net. As before we list the slopes

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corresponding to N as (m_i), $i = 1, 2, \ldots t + 1$. Then an argument similar to that used in 2.1 will show that each line of N^1 is either a line of N or the set of points of an affine subplane of Π_A. This subplane is obtained by deleting the points (m_i) from a projective subplane of Π where, as usual, Π denotes the projective extension of Π_A. In fact one can show that every line of N^1 is an affine subplane in Π_A of the above type. We then obtain

THEOREM 2.4 (see Ostrom [**15**, p. 1382]). *Suppose N is a net of order t^2 and critical deficiency $d = t + 1$. Then N can be embedded in at most two planes Π_1 and Π_2. If Π_1 and Π_2 exist, they are related to each other by derivation.*

3. Blocking Sets and Partial Spreads. In this section we are mainly concerned with partial spreads of $\Sigma = \mathrm{PG}(3, F)$, the 3 dimensional projective space over the field F. For general definitions we refer the reader to [**3**; **4**; **17**]. For our purposes, a *partial spread* of Σ is a collection W of pairwise skew lines in Σ. W is said to be *maximal* if it is not properly contained in any other partial spread; in particular, if every point of Σ is contained in some line of W, then W is called a *spread*. If F happens to be $\mathrm{GF}(q)$, where q is a prime power, and if W is a spread of $\mathrm{PG}(3, q)$, then $|W| = q^2 + 1$. (We write $|W|$ as usual for the number of lines in W). If W is a partial spread of $\mathrm{PG}(3, q)$, then the number $d = q^2 + 1 - |W|$ is called the *deficiency* of W. If W is a maximal partial spread of Σ which is not a spread, we say that W is a *maximal strictly partial spread* (*msp spread*) of Σ. Clearly, in the finite case, $|W|$ will have to be reasonably large in order that W be an msp spread. For example, if $|W| = 1$, then W can always be extended to a larger partial spread. Also, it seems reasonable to suppose that if $|W|$ is large, then W can always be completed to a spread of Σ. This last statement has in fact been proved by Mesner [**13**] who finds an upper bound on the number of lines of an msp spread. In the same paper, he poses the question: Find a lower bound on the number of lines of an msp spread of $\Sigma = \mathrm{PG}(3, q)$. Here we obtain his Theorem 1 and Corollary 1, as well as obtaining such a lower bound, by making use of [**6**, (3.9)] (see section 1), namely

THEOREM 3.1. *Let W be an msp spread of $\Sigma = \mathrm{PG}(3, q)$. Then $q + \sqrt{q} + 1 \leqq |W| \leqq q^2 - \sqrt{q}$.*

Proof. Let W be an msp spread of Σ. Let Π be a plane of Σ which contains no line of W (see Remark below). Let S be the set of points in which W intersects Π. Here $|S| = |W|$, and S is called the *section* of W by Π. Since W is maximal, every line of Π contains at least one point of S. Suppose l is a line of Π, all of whose points are in S. Then the $q + 1$ lines of W meeting l, along with l, form $q + 1$ distinct planes of Σ containing l. But then Π itself, being one of these $q + 1$ planes, would contain a line of W, which gives a contradiction. Thus S is a blocking set in Π as also is the complement of S in Π. Thus, $|S| = |W|$, and applying [**6**, (3.9)], we obtain the result.

Remark. Such a plane Π always exists since Σ is finite (see [**7**, Theorem 1]). However, if $\Sigma = PG(3, F)$ and F is countably infinite, for example, we may have a case where W is an msp spread of Σ and such that every plane Π of Σ contains exactly one line of W. If S is a spread of Σ which is not a dual spread of Σ then the image W of S under any correlation ρ of Σ will yield such an example of an msp spread (see Bruen and Fisher [**7**]). We can say something also concerning the case of equality in the above Theorem 3.1 (see Mesner [**13**, Theorem 3]).

THEOREM 3.2. *Let W be an msp spread of $\Sigma = PG(3, q)$. Let Π be any plane of Σ which contains no line of W and let S be the section of W by Π. Then*
(i) *if $|W| = q + \sqrt{q} + 1$, then S forms a subplane of Π,*
(ii) *if $|W| = q^2 - \sqrt{q}$, then the complement of S forms a subplane of Π.*

Proof. Use [**6**, Theorem 3.9].

We next examine in detail the case of $q = 3$.

Partial Spreads in $\Sigma = PG(3, 3)$. From (3.1), if W is an msp spread of Σ, then $6 \leqq |W| \leqq 7$. Although blocking sets S with $|S| = 6$ can occur in planes of order 3, (see [**6**]) we show that $|W| = 6$ cannot occur, but that $|W| = 7$ does occur (see the introductory remarks in Mesner [**13**]).

THEOREM 3.3. *Let W be an msp spread of $\Sigma = PG(3, 3)$. Then $|W|$ must be 7; moreover this case occurs.*

Proof. First we establish:

LEMMA. *If l_1, l_2, l_3, l_4, l_5 be any 5 pairwise skew lines of $\Sigma = PG(3, 3)$ and Π a plane of Σ not incident with any of them (see the remark above) then there is at least one line, and at most two lines, incident with Π and skew to each of the 5 lines.*

To see this we note that the l_i $(i = 1, \ldots, 5)$ meet Π in 5 distinct points. As in the proof of (3.1), no line of Π contains more than 3 of these points. Now Π is a plane of order 3 and by [**6**, (3.9)], a blocking set in Π contains at least 6 points. Thus, there is at least one line of Π which does not meet any of the lines l_i. All told, Π has 13 points, and so the rest of the Lemma is clear.

If the space dual of the Lemma were not true we could obtain a contradiction by using a correlation of Σ. Thus the space dual of the Lemma is also true.

Now suppose that W is an msp spread of Σ with $|W| = 6$. Let l_i $(i = 1, \ldots, 5)$ be any 5 distinct lines of W and let l be the 6th line of W. Using the space-dual of the lemma above, and examining the points on l, we obtain two possibilities:

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(a) For some point X on l there is one and only one line of Σ (namely l) incident with X and skew to each line l_i, $1 \leqq i \leqq 5$.

(b) There are precisely two lines of Σ, both of which are skew to the 5 lines l_i, and incident with each point of l.

We suppose case (a). The 5 lines l_i form, with X, 5 distinct planes through X. There are exactly 4 planes incident with l. Thus, there are precisely 4 planes which are incident with X and which do not contain any line of W. Let Π be any one of these 4 planes. By the Lemma above, there is at least one line m of Π which is skew to the l_i, $i = 1, \ldots, 5$. This line m cannot be l. Thus, by the assumed property of X, m cannot contain X. It follows that m does not meet any line of W. But then, W can be extended to a larger partial spread containing m, and this contradicts the assumed maximality of W.

Next, we suppose case (b). Now, by hypothesis, l is skew to each of the l_i. We draw the other line through each point of l which is skew to each of the l_i. Call the 4 lines so obtained: a, b, c and d. Now the 5 lines l_i account for exactly 20 points. The 5 lines l, a, b, c, d account for precisely 16 points. The total number of points in Σ is 40. Thus there is a point X' of Σ which is not on any of the above 10 lines. By the dual of the Lemma, there is at least one line m through X' which is skew to each of the l_i. Since m contains X', m is different from l, a, b, c, d. Thus, by the second half of the dual of the Lemma, m cannot meet l. But this yields that W can be extended to a larger partial spread (containing m), again contradicting the assumed maximality of W. Thus in all cases, the assumption that W is an msp spread of Σ with $|W| = 6$ is contradictory. Thus $|W|$ can only be 7, and that $|W| = 7$ actually occurs will follow from Theorem 3.5.

In Theorem 3.1, we have obtained bounds on $|W|$, with W an msp spread of $\Sigma = \mathrm{PG}(3, q)$. We proceed to show that the upper bound, at least, is reasonably good for any q. First, however, we mention briefly a few results on spreads. If Π_A is $\mathrm{AG}(2, q^2)$, the affine plane over $\mathrm{GF}(q^2)$, then the lines through $O = (0, 0)$ of Π will give rise to a spread S of $\Sigma = \mathrm{PG}(3, q)$ (see [**17**]). Moreover, it can be shown that S is a regular spread: that is, if a, b, *and* c are 3 distinct lines of S, then a, b *and* c determine a unique regulus $R(a, b, c)$, and S contains all lines of $R(a, b, c)$. Equivalently, if l is any line of Σ which is not in S, then the lines of S meeting l form a regulus. Not all spreads of Σ are regular: for example, if S is regular and if R is any regulus of S, then the spread S' obtained from S on replacing the regulus R by its opposite regulus R' is not regular. In fact, examples of non-regular spreads of Σ are furnished by any non-Desarguesian translation plane of order q^2 whose kernel is isomorphic to $\mathrm{GF}(q)$. Some of these ideas are discussed in [**1**; **2**]. We now prove:

THEOREM 3.4. Suppose $q > 2$. Then there exist msp spreads W in $\Sigma = \mathrm{PG}(3, q)$ with $q^2 - q + 1 \leqq |W| \leqq q^2 - q + 2$.

Proof. Let S be any spread of Σ such that S is not regular. This means that there is some line l of Σ, with l not in S, such that the lines of S meeting l do not form a regulus. Let A denote the set of these lines. Let W be the partial spread $W = (S - A) \cup l$. If u is any line of Σ which extends W (i.e. is skew to all of the lines in W) then the points of u are all contained on the lines of A, and u meets any line of A in at most one point. In other words, u must be a transversal to A. Now $|A| = q + 1 \geqq 4$ and A is not a regulus. Thus there are at most 2 transversals to A, and l is one of them. Thus, there is at most one line u which extends W. Hence W is an msp spread of Σ and either $|W| = q^2 - q + 1$ or $|W| = q^2 - q + 2$. Theorem 3.3 is also now completed.

We proceed to show that the case $|W| = q^2 - q + 1$ occurs. For this we need another result on spreads. In [**2**], Bruck exhibits an isomorphism between a regular spread S of $PG(3, q)$ and the inversive plane $I = IP(q)$. In this isomorphism, lines of S correspond to points of I, and reguli of S correspond to circles of I. This makes it easy to see that in a regular spread S, there are many pairs of reguli which have exactly one line or exactly two lines in common (see [**18**] for the case of infinite fields). We now show:

THEOREM 3.5. If $q > 2$, there exist msp spreads W of $\Sigma = PG(3, q)$ with $|W| = q^2 - q + 1$.

Proof. The idea is to produce some spread S and some line l not in S such that the lines A of S meeting l have exactly one transversal (see 3.4). For this purpose, let S_0 be a regular spread of Σ and let R_1 and R_2 be two reguli of S_0 which have exactly one line c in common. Let R_1' (R_2') denote the opposite regulus of R_1 (R_2). Let m_1' (m_2') be any line of R_1' (R_2') and suppose m_1' meets m_2'; let $m_1' \cap m_2' = X$. Through X there passes a line u of R_1 and a line v of R_2. Now $S_0 \supset R_1$ and $S_0 \supset R_2$. Also there is a unique line x of S_0 through X. Hence $u = v = x$ and $x \in R_1 \cap R_2$, that is, $x = c$. Thus a line of R_1' can meet a line of R_2' only in points of c.

Now let Y be any point of c, and let l_1' (l_2') be the unique line of R_1' (R_2') through Y (see Figure 3.5).

Denote the set of all lines in $S_0 - (R_1 \cup R_2)$ by B. Now the set

$$B \cup (R_1 - c) \cup R_2'$$

is a spread S_1 of Σ. l_1' is a line of Σ which is not a line of S_1. Let A denote the set of lines of S_1 meeting l_1'. As in the proof of Theorem 3.4, let W be the partial spread given by $W = (S_1 - A) \cup l_1'$. Any line u extending W will have to be a transversal of A. Since $q \geqq 3$, $R_1 - c$ contains at least 3 lines. Thus u meets at least 3 lines of R_1; thus $u \in R_1'$. By our opening remarks, u can meet l_2' only in a point of c. But then, u passes through Y and meets l_1', and so there is no line extending W. It follows that W is maximal and W has deficiency q, that is, $|W| = q^2 - q + 1$.

Remarks. Substituting $q = 5$ in the above, we obtain an example of an msp spread in $\Sigma = PG(3, 5)$ with deficiency 5. David Foulser [**11**] has kindly

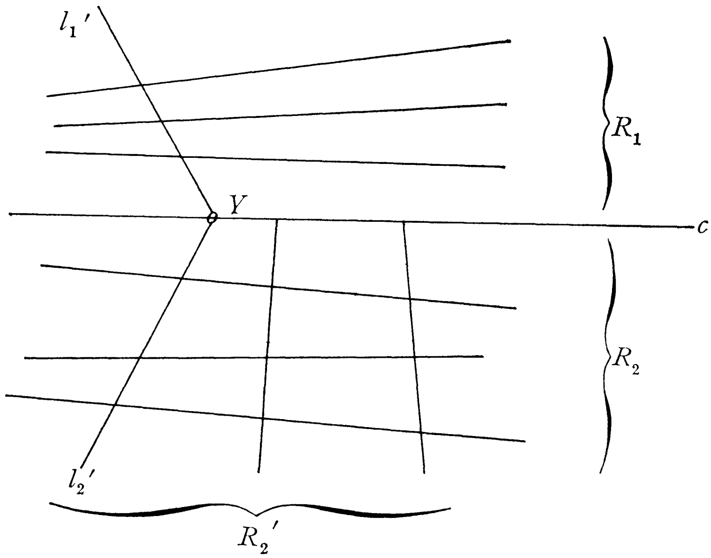

FIGURE 3.5

shown me an example of an msp spread in PG(3, 5) with deficiency 4, using subnets of the exceptional nearfield plane of order 25. Stimulated by this result we proceed to show the existence of msp spreads with deficiency $q - 1$ in PG$(3, q)$ for any odd q with $q > 3$. However, some more results from spread theory are necessary. In accordance with Bruck and Bose [**3**; **4**], we have defined a regular spread S of Σ = PG$(3, q)$ as a spread of Σ such that if the distinct lines a, b and c are in S then the lines of $R(a, b, c)$, the unique regulus determined by a, b and c, are also contained in S. We say 'unique regulus' because GF(q) is commutative. (The notion of reguli in a projective space over an arbitrary skew field is more difficult, as in this case, 3 pairwise skew lines may not determine a unique regulus.) However, the notion of a regular spread is quite a classical one. In fact Veblen and Young [**18**] define a *linear congruence* in a 3-dimensional projective space Σ to be the set of all lines which are linearly dependent on four linearly independent lines. It is assumed only that Σ is defined over some commutative field which is not of characteristic 2 (see [**18**, bottom of p. 298]), and it is shown that any 4 pairwise skew lines a, b, c and d, such that d is skew to all lines of the regulus $R(a, b, c)$, determine one and only one *elliptic linear congruence*, which is in fact what we have called a regular spread (see [**18**, Theorem 18 and the Corollary, p. 318]). The assumption concerning the characteristic of the field is not needed for the above result. Thus any regulus R and any line skew to R determine one and only one regular spread. (This fact may be used to construct msp spreads of PG(3, **R**), for example, using the method of Theorem 3.5.) The above theorem is also proved for the finite case in Bruck [**2**, Theorem 4.3].

The notion of a linear congruence is also discussed in Coxeter [**8**]. We are now in a position to prove:

THEOREM 3.6. If $q > 3$ is odd, then there exist msp spreads W in $\Sigma = PG(3, q)$, of deficiency $q - 1$, that is, such that $|W| = q^2 - q + 2$.

Proof. Here the idea is to produce some spread S and some line l not in S, such that the lines of S meeting l have exactly 2 transversals.

Thus, let R be a regulus of Σ determining the quadric $H(R)$. Let the line l be a secant to $H(R)$, that is, l meets $H(R)$ in precisely two distinct points A and B. Let u and v be the lines of R through A and B, and list the remaining lines of R as $r_3, r_4, \ldots, r_{q+1}$. Denote the lines of R' through A and B by u' and v', where R' is the opposite regulus of R. Let $v \cap u' = P, v' \cap u = Q$, and let PQ be the line m (see Figure 3.6).

Clearly, l and m must be skew. We wish to show the existence of a line which meets both l and m and is skew to all lines of R.

For this purpose, let X be any point of l different from A and B. Thus X is not on $H(R)$, nor is X on m. m is skew to each of the lines r_i, $3 \leqq i \leqq q + 1$. For otherwise, m would meet at least 3 lines of R and thus would be a line of R'. From X we draw the unique transversal t_i to m and r_i, for each r_i, $3 \leqq i \leqq q + 1$. Now XP(resp. XQ) is a line in the tangent plane to $H(R)$ at P(resp. Q) which passes through P(resp. Q). Thus XP(resp. XQ) meets $H(R)$ only in P(resp. Q). Hence each t_i is different from XP and from XQ. It follows that no line t_i meets u or v for this would force l and m to intersect. We claim that some two of the lines t_i coincide. For suppose this is not the case. Then each line t_i is a tangent to $H(R)$, that is, each line t_i intersects $H(R)$ in exactly one point which is on r_i.

Let Π denote the plane formed by X and m. Then $\Pi \cap H(R)$ is either

(a) 2 lines, one line being a line of R, the other a line of R', or

(b) a conic (see Veblen and Young [**18**, p. 300] or Coxeter [**9**, p. 260]).

Suppose case (a). Let g be the line of R in $\Pi \cap H(R)$. Then every line of $\Pi = \Pi(X, m)$ must meet g. XP is a line of Π. XP meets v and no other line of R. Thus $g = v$. Similarly, using XQ, we get $g = u$. But v is different from u, and we obtain a contradiction.

Suppose case (b), that is, $\Pi \cap H(R)$ is a non-degenerate conic C. In particular C is an oval. Since each line t_i is assumed to be tangent to $H(R)$ then each line t_i is a tangent to C passing through X. Thus there are $q + 1$ tangents to C in the plane Π all passing through X (XP and XQ are also tangents to C). But since q is odd, there are at most 2 tangents from any point of Π to C [**12**, p. 381]. Now $q + 1 > 2$, since $q > 3$, and again we have a contradiction.

Thus, some 2 of the lines t_i coincide. Now besides XP and XQ, there are exactly $q - 1$ distinct lines meeting X and intersecting m. Each of these $q - 1$ lines is either a line t_i or a line skew to all lines of R. There are exactly $q - 1$ lines r_i since $3 \leqq i \leqq q + 1$. Since some two of the lines t_i coincide, we deduce that there is at least one line x through X intersecting m and such that x is skew to all lines of the regulus R. Thus R and x will determine a regular spread S of Σ. S contains u, v and x and thus also $G(u, v, x)$, the regulus determined by u, v and x. l and m meet u, v and x and thus all lines of G meet l and m (see Figure 3.6).

Denote the set of lines in $S - (R \cup G)$ by B. Let G^* denote those lines of G different from u and v. Next, the set $S_1 = B \cup G^* \cup R'$ is a spread of Σ.

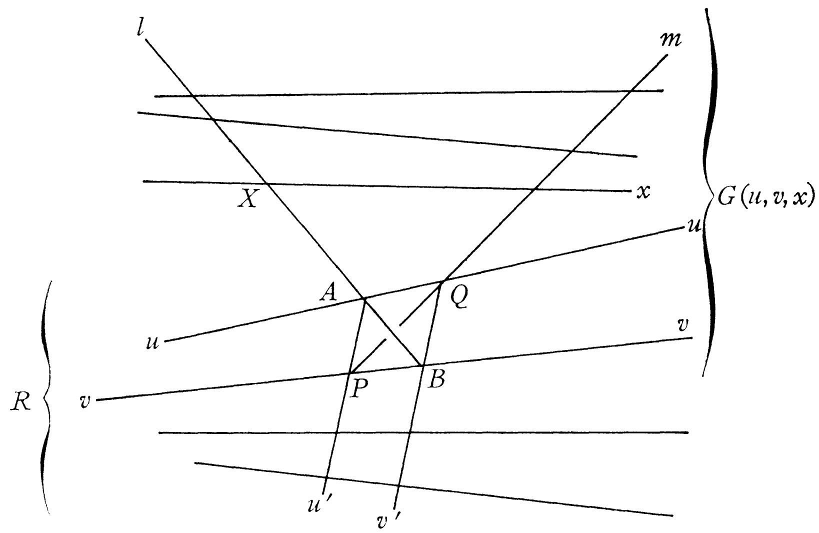

FIGURE 3.6

l is not in S_1. As in (3.4), let A denote those lines of S_1 which meet l. Now $A \supset G^*$. If the lines of A formed a regulus, such a regulus would have to contain G^*. Now $|G^*| > 3$, since $q \geqq 5$. Thus there is only one regulus containing the lines of G^*, namely G. But u and v are in G and neither u nor v is in A. It follows that the lines of A do not form a regulus. Thus, there are at most 2 transversals to A; in this case A has exactly 2 transversals, l and m.

Finally, let W denote the partial spread $W = (S_1 - A) \cup \{l, m\}$. Then W is an msp spread of $\Sigma = \mathrm{PG}(3, q)$ with $|W| = q^2 - q + 2$.

Remarks. We have assumed $q > 3$; this is necessary because of (3.3). (In the proof above we used $|G^*| > 3$.) We have also assumed that q is odd. This assumption may not be necessary. However, in the discussion of case (b) above, we used the fact that in a plane Π of odd order there are at most 2 tangents to an oval C which pass through a point X of Π. This is not the case for planes of even order since the tangents to an oval are concurrent [**12**, p. 381], and so our proof does not generalize to the case of even q. On the other hand, Mesner [**13**] does show the existence of an msp spread W in $\mathrm{PG}(3, 4)$ which has deficiency 3, i.e. $|W| = 4^2 - 4 + 2 = 14$.

4. Concluding Remarks. In what follows, W denotes an msp spread. In Theorem 3.1 we showed that $q + \sqrt{q} + 1 \leqq |W| \leqq q^2 - \sqrt{q}$. Theorem 3.6 showed that there exists W with $|W| = q^2 - q + 2$. Thus our upper bound is best possible for small q and is reasonably good for large q. However we do not have much information on the lower bound. Since both inequalities of Theorem 3.1 were a consequence of [**6**, Theorem 3.9], we feel that it should be possible to construct W with $|W|$ reasonably small, but we have not been successful to date. It seems possible, for reasonably large q at any rate, to duplicate the construction of the proof of Theorem 3.5 to get a W with deficiency $2q$. The idea is that we examine 2 pairs of reguli (R_1, R_2) and (R_3, R_4) which are all contained in a regular spread S and such that

(1) R_1 and R_2 have a line in common,

(2) R_3 and R_4 have a line in common,

(3) Every line of $R_1 \cup R_2$ is skew to every line of $R_3 \cup R_4$.

Then, proceeding as in Theorem 3.5, assuming q is sufficiently large, we can obtain a W in PG(3, q) of deficiency $2q$. A more detailed knowledge of lines and reguli in a regular spread S seems required for a generalization of this sort of technique and, in this connection, the isomorphism (pointed out in Bruck [**2**]) between the lines and reguli of S and between the points and circles of the inverse plane IP(q) over GF(q), may be helpful. It is possible to obtain a net (in fact a translation net) from any partial spread (see Ostrom [**17**]). In the cases of Theorems 3.5 and 3.6, each W gives rise to a translation net N of order q^2 whose deficiency as a net is the same as that of W, regarded as a partial spread of Σ. Thus part of Theorem 3.1 might be phrased as follows: *If N is a translation net of order q^2 whose deficiency is less than $\sqrt{q} + 1$, then N can be completed to (extended to, embedded in) a translation plane of order q^2 that is representable as a spread of* PG(3, q). Using special cases of Theorems 4.3 and 3.1 of Bruck [**1**] we obtain the result that if N is any net of order n^2 whose deficiency is less than $\sqrt{n} + 1$, then N is uniquely embeddable in some affine plane of order n^2. However, it is clear that this does not imply the above result of (3.1).

In connection with (3.1), David Foulser [**11**] has mentioned to me the possibility of obtaining an analagous result for msp spreads in PG($2t - 1$, q), perhaps by suitably generalizing the idea of blocking sets.

In (3.5) and (3.6) we constructed W with $|W| = q^2 - q + 1$ and $q^2 - q + 2$ respectively. By definition, the net N obtained from W has no transversals [**16**] which are 2-dimensional subspaces of the associated 4-dimensional vector space over GF(q). Thus N can certainly not be embedded in a translation plane of order q^2 which is representable as a spread of PG(3, q) containing W. In fact, it is possible to show that if q is a prime, then the net N cannot be embedded in any plane Π whatsoever. This result will be discussed elsewhere. However it might be of interest to see if the net N can be embedded in any larger net. Here we are also thinking of the connection between nets and orthogonal latin squares [**1**]. It may be added that the example of a W in

PG(3, 5) with deficiency 4, when interpreted as a net, shows that the embeddability equation (B) of [**1**, p. 422] is best possible for nets of order 25. Finally we wish to pose the following:

Problem. If q is sufficiently large, do there exist msp spreads W of $\Sigma = \mathrm{PG}(3, q)$ with $q^2 - q + 2 < |W| \leqq q^2 - \sqrt{q}$?

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