ON TWO PROBLEMS CONCERNING THE GENERALIZED LOTOTSKY TRANSFORMS

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1. The generalized Lototsky transform (or the $[F, d_n]$ transform) of a sequence $\{s_n\}$ into a sequence $\{t_n\}$ was defined in **(2)** in the following way: Let $\{d_n\}$ $(d_n \neq -1)$ be a fixed complex sequence and let

$$\prod_{k=1}^{n} \frac{d_k + x}{d_k + 1} = \sum_{m=0}^{n} c_{nm}\, x^m, \tag{1.1}$$

then the sequence $\{t_n\}$ is defined by

$$t_n = \sum_{m=0}^{n} c_{nm}\, s_m, \qquad n = 1, 2, \ldots .$$

In a recent paper by V. F. Cowling and C. L. Miracle **(1)** on these transformations, the following two problems were left open.

Denote

$$d_n = \rho_n\, e^{i\theta_n}, \qquad n = 1, 2, \ldots, \tag{1.2}$$

where $-\pi < \theta_n \leqslant \pi$, and $\rho_n > 0$.

In **(**1, Theorem 3.1**)** it is proved that if

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \theta_n^{2}\, \rho_n^{-1} < +\infty,$$

then the $[F, d_n]$ transformation is regular. The authors left open the question: (i) Do

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty \quad \text{and} \quad \lim_{n\to\infty} \theta_n = 0 \tag{1.3}$$

imply the regularity of the $[F, d_n]$ transform? We shall prove here that this question is to be answered in the negative. We shall show even more, namely that there exists a sequence $\{d_n\}$, such that

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty, \qquad \lim_{n\to\infty} \theta_n = 0, \tag{1.4}$$

and

$$\sum_{n=1}^{\infty} \theta_n^{2+\epsilon}\, \rho_n^{-1} < +\infty \tag{1.5}$$

for every $\epsilon > 0$, and the $[F, d_n]$ transformation is *not* regular.

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In (**1**, Theorems 2.2 and 2.3) it is proved that if

$$\sum_{n=1}^{\infty} \rho_n = \infty, \qquad \pi \geqslant \theta_n \geqslant \alpha > 0 \text{ or } -\pi < \theta_n \leqslant -\alpha < 0 \text{ for } n \geqslant N,$$

then the $[F, d_n]$ transformation is not regular. The authors left open the question: (ii) Does there exist a sequence $\{d_n\}$ such that θ_n does not tend to zero and the $[F, d_n]$ transformation is regular? We shall show that the answer is affirmative.

2. Proofs.

(i) Let

$$d_n = e^{i\theta_n}, \tag{2.1}$$

where

$$\theta_n = \left(\frac{\log(n+1)}{n}\right)^{\frac{1}{2}}, \qquad n = 1, 2, \dots . \tag{2.2}$$

Clearly (1.4) and (1.5) hold.

Suppose that the $[F, d_n]$ transformation defined by the sqeuence (2.1) is regular. Then there exists a constant $H < +\infty$ independent of n such that

$$\sum_{m=0}^{n} |c_{nm}| \leqslant H, \qquad n = 1, 2, \dots . \tag{2.3}$$

By (2.1)

$$|d_n| = 1, \qquad n = 1, 2, \dots . \tag{2.4}$$

Thus from (2.3) for $n \geqslant 1$

$$H \geqslant \sum_{m=0}^{n} |c_{nm}| = \sum_{m=0}^{n} |c_{nm}||d_n|^m \geqslant \left| \sum_{m=0}^{n} c_{nm}\, d_n{}^m \right| . \tag{2.5}$$

By (1.1) and (2.1), this is equal to

$$\prod_{m=1}^{n} \left| \frac{e^{i\theta_m} + e^{i\theta_n}}{e^{i\theta_m} + 1} \right| \geqslant \prod_{m=1}^{n} \{1 + \sin \tfrac{1}{2}\theta_n \sin(\theta_m - \tfrac{1}{2}\theta_n)\}^{\frac{1}{2}}.$$

The sequence $\{\theta_n\}$ being monotonic, this is

$$\geqslant n^{\frac{1}{2}} \sin \tfrac{1}{2}\theta_n \sim \tfrac{1}{2}(\log n)^{\frac{1}{2}}.$$

Therefore (2.5) yields a contradiction when $n \to \infty$ and the $[F, d_n]$ transform under consideration is not regular.

(ii) Let

$$d_n = \begin{cases} i, & \text{if } n = 2k, \\ -i, & \text{if } n = 2k-1, \end{cases} \qquad k = 1, 2, \dots \tag{2.6}$$

where $i = \sqrt{(-1)}$.

We have by (**1**, (2.2)) that

$$c_{2\nu, 2\mu+1} = 0, \qquad \nu, \mu = 0, 1, 2, \dots \tag{2.7}$$

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and thus by **(2**, (1.2)**)**

$$|c_{2\nu+1,2\mu+1}| + |c_{2\nu+1,2\mu}| = \sqrt{2}.|c_{2\nu,2\mu}|, \qquad \nu, \mu = 0, 1, \ldots . \tag{2.8}$$

By **(1**, (2.2)**)**, it follows that

$$|c_{2\nu,2\mu}| = \binom{\nu}{\mu} 2^{-\nu}, \qquad \nu, \mu = 0, 1, \ldots . \tag{2.9}$$

By (2.7), (2.8), and (2.9)

$$\lim_{n\to\infty} c_{nm} = 0, \qquad m = 0, 1, \ldots ,$$

$$\sum_{m=0}^{n} |c_{nm}| \leqslant \sqrt{2}, \qquad n = 0, 1, \ldots ,$$

and since by definition

$$\sum_{m=0}^{n} c_{nm} = 1, \qquad n = 0, 1, \ldots ,$$

the regularity of the $[F, d_n]$ transformation defined by (2.6) follows. Clearly, θ_n does not tend to zero.

3. Remarks. We use this occasion to point out that the paper of V. F. Cowling and C. L. Miracle **(1)** contains a few inexact statements.

(i) Theorem 2.2 is not true without the further condition that

$$\sum_{n=1}^{\infty} \rho_n = +\infty .$$

Take, for example, $d_n = in^{-2}$ $(n \geqslant 1)$; it is easy to see that the sufficient conditions of Jakimovski **(2)** for regularity of the $[F, d_n]$ transformation are satisfied.

(ii) The above-mentioned example shows also that Theorem 2.4 is true only if

$$\sum_{n=1}^{\infty} \rho_n = +\infty .$$

(iii) Theorem 2.3 is not true without the same condition concerning the sequence $\{d_n\}$. Take, for example, $d_n = -in^{-2}$ $(n \geqslant 1)$.

(iv) In the proof of Theorem 3.1 (p. 425) the authors state that

$$\sum_{n=1}^{\infty} \theta_n^{\,2} \rho_n^{\,-1} < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \rho_n^{\,-1} = +\infty$$

imply that

$$\lim_{n\to\infty} \theta_n = 0.$$

But this is not true in general. Take, for example, $\theta_n = 1$ for $n = 2^k$ $(k \geqslant 1)$, $\theta_n = 0$ otherwise, and $\rho_n = n (n \geqslant 1)$.

At the end of the same proof the authors state that the assumption

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$$

implies that

$$\sum_{j=N}^{\infty} \rho_j(1 + \rho_j)^{-2} = +\infty$$

but this is true only if

$$\sum_{j=1}^{\infty} \rho_j = +\infty.$$

(v) In the proof of Theorem 4.1 after formula (4.5), the authors state that

$$\lim_{n\to\infty} \theta_n = 0,$$

but this does not follow from the conditions, as the example mentioned in (iv) shows.

(vi) In Corollary 4.1 the authors state that the conditions imply that

$$\lim_{n\to\infty} \theta_n = 0,$$

which is not true in general. Take, for example, $\rho_n = n (n \geqslant 1)$; $\theta_n = 1$ if $n = 2^k (k \geqslant 0)$ and $\theta_n = 1/(\log n)$ otherwise.

(vii) In the Introduction the authors state that the sufficient conditions proved in Theorem 3.1 *include* as special case Theorem 3.1 of Jakimovski's paper **(2)**. The real relation is the opposite. Namely, if

$$\sum_{n=1}^{\infty} \theta_n^{\,2} \rho_n^{-1} < +\infty$$

is satisfied, then by easy consideration we obtain that also

$$\prod_{n=1}^{\infty} \frac{1 + |d_n|}{|1 + d_n|} \leqslant H < +\infty$$

is satisfied; thus, together with

$$\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty,$$

the regularity of the $[F, d_n]$ transformation follows by Jakimovski's theorem. Conversely, from Theorem 3.1 of **(1)** one cannot prove Jakimovski's conditions as the example $d_n = in^{-2}$ $(n \geqslant 1)$ shows.

References

1. V. F. Cowling and C. L. Miracle, *Some results for the generalized Lototsky transform*, Can. J. Math., *14* (1962), 418–435.
2. A. Jakimovski, *A generalization of the Lototsky method of summability*, Mich. Math. J. (1959), 277–290.

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