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HARMONIC MAPS BETWEEN ROTATIONALLY SYMMETRIC MANIFOLDS

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Abstract We prove the existence and uniqueness of harmonic maps between rotationally symmetric manifolds that are asymptotically hyperbolic.

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1. Introduction and main results

Let M and N be two Riemannian manifolds of dimension $(m+1)$ and $(n+1)$, whose metrics are given locally by $g = g_{kj}\,\mathrm{d}x^k\,\mathrm{d}x^j$ and $h = h_{\alpha\beta}\,\mathrm{d}y^\alpha\,\mathrm{d}y^\beta$, respectively. Throughout this paper we use the summation convention for repeated indices, i.e. when an index is repeated it means that we have summation in terms of this index. Domains are denoted by Latin letters as indexes, while targets are denoted by Greek letter indexes.

The tension field $\tau(u)$ of a map $u \in C^2(M, N)$ is defined by $\tau(u) = \mathrm{Tr}(\nabla\,\mathrm{d}u)$. In local coordinates $\tau(u)$ is given by

$$\tau^\alpha(u)(x) = \Delta_M u^\alpha(x) + g^{kj}(x)\Gamma^\beta_{\beta\gamma}(u(x))\partial_k u^\alpha(x)\partial_j u^\gamma(x), \quad \alpha = 0, 1, \dots, n, \tag{1.1}$$

where Δ_M is the Laplace–Beltrami operator for M. *Harmonic maps* are defined to be the maps for which the tension field $\tau(u)$ vanishes.

In [**9**, **10**] Li and Tam proved the existence and uniqueness of harmonic maps between real hyperbolic spaces. In the present paper we deal with the same problem in the context of rotationally symmetric manifolds that are asymptotically hyperbolic. This class contains hyperbolic spaces and consequently our results extend those of [**9**, **10**] in this more general context.

More precisely, let us equip $\mathbb{R}^{m+1}$, $m \geqslant 1$, with a Riemannian metric that can be written in polar coordinates in the form

$$g = \mathrm{d}\rho^2 + f(\rho)^2\,\mathrm{d}\eta^2,$$

where $f(0) = 0$, $f'(0) = 1$ and $f(\rho) > 0$ for every $\rho > 0$. Then, $\mathbb{R}^{m+1}$ equipped with the above metric becomes a rotationally symmetric manifold M (see [**3**, pp. 22–29], in which M is called a weak model, and [**1**], in which M is called a Ricci model). The ideal boundary ∂M of M is the 'sphere at infinity' and it is isomorphic to $\mathbb{S}^m$.

We say that M is asymptotically hyperbolic if all sectional curvatures of M at $x \in M$ converge to -1 as $x \to \partial M$ and there exists a constant $d > 0$ such that

$$\lim_{\rho\to\infty} \frac{f(\rho)}{\mathrm{e}^{\rho}} = d. \tag{1.2}$$

Let us recall that the radial curvature $K_M(x)$ of M at $x \in M$ is the restriction of the sectional curvature function to all the planes that contain $\partial_\rho(x)$. It is a function of $\rho(x)$ and it is given by

$$K_M(\rho) = -\frac{f''(\rho)}{f(\rho)}. \tag{1.3}$$

In Remark 2.3, we show that if $\lim_{\rho\to\infty} K_M(\rho) = -1$ and (1.2) hold, then all sectional curvatures of M at $x \in M$ converge to -1 as $x \to \partial M$. Note that the condition $\lim_{\rho\to\infty} K_M(\rho) = -1$ does not in general imply (1.2), as the example $f(\rho) = \mathrm{e}^{\rho} \log \rho$, $\rho > 2$, shows.

The Dirichlet problem for harmonic maps between such manifolds consists of finding a harmonic map $u\colon M \to N$ such that $u(\infty, \eta) = (\infty, \phi(\eta))$, where $\phi\colon \mathbb{S}^m \to \mathbb{S}^n$ is a given map. Then, u is called a harmonic extension of ϕ, and ϕ is referred to as the boundary map at infinity.

Given a map $u \in C^1(M, N)$, its energy density $e(u)$ is defined by

$$e(u)(x) = g^{kj}(x)\partial_k u^\alpha \partial_j u^\beta h_{\alpha\beta}(u(x)). \tag{1.4}$$

In this paper we prove the following theorems.

Theorem 1.1. *Let M and N be two rotationally symmetric manifolds whose sectional curvatures are less than $-a^2$ for some $a > 0$. If M and N are asymptotically hyperbolic and $\phi \in C^3(\mathbb{S}^m, \mathbb{S}^n)$ has nowhere-vanishing energy density, then a harmonic map $u\colon M \to N$ exists with ϕ as the boundary map at infinity.*

Let us recall that a harmonic map $u\colon M \to N$ is called proper if, for every sequence (x_j) of elements of M that converges to the ideal boundary of M, the sequence $u(x_j)$ converges to the ideal boundary of N.

Theorem 1.2. *Let M, N and ϕ be as in Theorem 1.1. If $u, v\colon M \to N$ are proper harmonic maps with boundary value ϕ, then $u = v$.*

Let us say a few words about the history of harmonic maps. Harmonic maps were first introduced by Eells and Sampson [**2**]. Hamilton [**4**] investigated the Dirichlet problem for harmonic maps between compact manifolds with a boundary. In [**9**, **10**] Li and Tam proved the existence and uniqueness of harmonic maps between real hyperbolic spaces. Our results generalize the aforementioned results of Li and Tam.

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2. Proof of Theorem 1.1

To prove Theorem 1.1 we shall first construct an appropriate extension $\Phi\colon M \to N$ of ϕ and next we shall show that there exists a harmonic map $u\colon M \to N$ that is a bounded distance from Φ. Finally, we show that the map u is the required harmonic extension of ϕ.

Let $e(\phi)$ (respectively, $e(\Phi)$) be the energy density of ϕ (respectively, Φ) and let $\|\tau(\Phi)\|$ be the norm of the tension field of Φ.

Lemma 2.1. *If $\phi \in C^3(\mathbb{S}^m, \mathbb{S}^n)$ is such that $e(\phi) > 0$, then there exists a map $\Phi \in C^2(M, N)$ with $\Phi(\infty, \eta) = (\infty, \phi(\eta))$, such that*

(i) $e(\Phi)(\infty, \eta) = m + 1$,

(ii) $\sup_{(\rho,\eta)} e(\Phi)(\rho, \eta) < \infty$,

(iii) $\|\tau(\Phi)\|(\infty, \eta) = 0$,

(iv) $\sup_{(\rho,\eta)} \|\tau(\Phi)\|(\rho, \eta) < \infty$.

For the proof of Lemma 2.1 we shall make use of Lemma 2.2 below, the proof of which is postponed till after the proof of Lemma 2.1.

Lemma 2.2. *If $f\colon \mathbb{R} \to \mathbb{R}$ is a C^2 function satisfying (1.2) and $\lim_{\rho\to\infty} f''(\rho)/f(\rho) = 1$, then*

$$\lim_{\rho\to\infty} \frac{f'(\rho)}{f(\rho)} = 1.$$

Proof of Lemma 2.1. Since M and N are rotationally symmetric manifolds, their metrics are written as

$$g = \mathrm{d}\rho^2 + f(\rho)^2\,\mathrm{d}\eta^2 \quad \text{and} \quad h = \mathrm{d}r^2 + F(r)^2\,\mathrm{d}\phi^2, \tag{2.1}$$

respectively.

Furthermore, by our assumption, M and N are asymptotically hyperbolic. Consequently, by Lemma 2.2 we have that

$$\lim_{\rho\to\infty} \frac{f'(\rho)}{f(\rho)} = 1 \quad \text{and} \quad \lim_{r\to\infty} \frac{F'(r)}{F(r)} = 1. \tag{2.2}$$

Also, there are constants $d_1, d_2 > 0$ such that

$$\lim_{\rho\to\infty} \frac{f(\rho)}{\mathrm{e}^\rho} = d_1 \quad \text{and} \quad \lim_{r\to\infty} \frac{F(r)}{\mathrm{e}^r} = d_2. \tag{2.3}$$

Now consider the map $\Psi(\rho, \eta) = (\rho - l(\eta), \phi(\eta))$, where

$$l(\eta) = \log\left(\sqrt{\frac{e(\phi)(\eta) d_2}{m d_1}}\right) \tag{2.4}$$

and $e(\phi)$ is the energy density of ϕ with respect to the metrics on $\mathbb{S}^m$ and $\mathbb{S}^n$.

From (1.4) and (2.1) it follows that the energy density of Ψ is given by

$$e(\Psi)(\rho,\eta) = 1 + \frac{1}{f(\rho)^2}\langle \mathrm{d}l(\eta), \mathrm{d}l(\eta)\rangle + e(\phi)(\eta)\frac{F(\rho - l(\eta))^2}{f(\rho)^2}, \tag{2.5}$$

where $\langle \mathrm{d}l, \mathrm{d}l\rangle$ is the inner product with respect to the metric on $\mathbb{S}^m$.

From (2.3) we deduce that

$$\lim_{\rho\to\infty} \frac{1}{f(\rho)^2}\langle \mathrm{d}l(\eta), \mathrm{d}l(\eta)\rangle = 0. \tag{2.6}$$

Also, from (2.3) and (2.4), it follows that

$$\lim_{\rho\to\infty} \frac{F(\rho - l(\eta))^2}{f(\rho)^2} = \frac{m}{e(\phi)(\eta)}. \tag{2.7}$$

As a result of (2.5)–(2.7), we have that

$$\lim_{\rho\to\infty} e(\Psi)(\rho,\eta) = m + 1. \tag{2.8}$$

Thus, $e(\Psi)(\rho,\eta)$ is bounded for ρ big enough, say $\rho > 2$.

Define $\Phi\colon M \to N$ as follows. $\Phi(\rho,\eta) = \Psi(\rho,\eta)$ for $\rho \geqslant 2$ and Φ is defined in a smooth way in the rest of M (e.g. take $\psi = \Psi|_{\rho=2}$, find a harmonic extension from $\rho \leqslant 2$ to N and then deform the two maps so that they match smoothly when $\rho = 2$). We shall then show that Φ is the required map.

From (2.8) it follows that Φ satisfies claims (i) and (ii).

From (1.1) and (2.1) it follows that the components of the tension field of Ψ are given by

$$\tau^0(\Psi)(\rho,\eta) = -\frac{1}{f(\rho)^2}\Delta l(\eta) + \left(m\frac{f'(\rho)}{f(\rho)} - e(\phi)(\eta)\frac{F'(\rho - l(\eta))F(\rho - l(\eta))}{f(\rho)^2}\right)$$

and

$$\tau^\alpha(\Psi)(\rho,\eta) = \frac{1}{f(\rho)^2}\left(\tau^\alpha(\phi)(\eta) + \frac{F'(\rho - l(\eta))}{F(\rho - l(\eta))}\langle \mathrm{d}\phi^\alpha(\eta), \mathrm{d}L(\eta)\rangle\right), \quad \alpha = 1,2,\dots,n,$$

where the $\tau^\alpha(\phi)$ are the components of the tension field of ϕ with respect to the metrics on $\mathbb{S}^m$ and $\mathbb{S}^n$, and Δl and $\langle \mathrm{d}\phi^\alpha, \mathrm{d}l\rangle$ are computed with respect to the metric on $\mathbb{S}^m$.

Observe that by (2.2) and (2.7) it follows that

$$\lim_{\rho\to\infty} |\tau^0(\Psi)| = 0 \quad \text{and} \quad |\tau^\alpha(\Psi)(\rho,\eta)| \leqslant \frac{C}{f(\rho)^2} \tag{2.9}$$

for some constant $C > 0$.

Let $H_{\alpha\beta}$ be the components of the metric on $\mathbb{S}^n$. Taking into account (2.9) and (2.7), we find that

$$\begin{aligned}\|\tau(\Psi)\|^2 &= |\tau^0(\Psi)|^2 + F(\rho - l(\eta))^2\tau^\alpha(\Psi)(\rho,\eta)\tau^\beta(\Psi)(\rho,\eta)H_{\alpha\beta}(\phi(\eta)) \\ &\leqslant |\tau^0(\Psi)|^2 + C\frac{F(\rho - l(\eta))^2}{f(\rho)^4}H_{\alpha\beta}(\phi(\eta)) \to 0.\end{aligned}$$

Thus, $\|\tau(\Psi)\|^2$ is bounded for ρ big enough, say $\rho > 2$. This proves claims (iii) and (iv). □

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Proof of Lemma 2.2. Since

$$\lim_{\rho\to\infty}\frac{f''(\rho)}{f(\rho)}=1,$$

it follows that for every $\epsilon>0$ there exists $R_1>0$ such that $f''(\rho)/f(\rho)<1+\epsilon$, for every $\rho>R_1$.

If $g(\rho)=\sinh(\rho\sqrt{1+\epsilon})$, then

$$\frac{f''(\rho)}{f(\rho)}<1+\epsilon=\frac{g''(\rho)}{g(\rho)}$$

holds for every $\rho>R_1$.

Observe next that

$$\left(g^2\left(\frac{f}{g}\right)'\right)'=fg\left(\frac{f''}{f}-\frac{g''}{g}\right)\leqslant 0$$

for every $\rho>R_1$.

Thus,

$$f'g-fg'=\left(g^2\left(\frac{f}{g}\right)'\right)$$

is a decreasing function. Consequently, we get that

$$\frac{f'(\rho)}{f(\rho)}-\frac{g'(\rho)}{g(\rho)}\leqslant\frac{f'(R_1)g(R_1)-f(R_1)g'(R_1)}{f(\rho)g(\rho)} \tag{2.10}$$

holds for every $\rho>R_1$.

But

$$\lim_{\rho\to\infty}\frac{g'(\rho)}{g(\rho)}=\sqrt{1+\epsilon}$$

and by (1.2) it follows that $\lim_{\rho\to\infty}f(\rho)=\infty$. Thus, we get from (2.10) that for all $\epsilon>0$ there exists R_1' such that

$$\frac{f'(\rho)}{f(\rho)}\leqslant\sqrt{1+\epsilon}+2\epsilon\leqslant 1+3\epsilon\quad\text{for all }\rho>R_1'.$$

Similarly, there exists $R_2>0$ such that $f''(\rho)/f(\rho)>1-\epsilon$ for every $\rho>R_2$. Choosing $g(\rho)=\sinh(\rho\sqrt{1-\epsilon})$, one can prove that for all $\epsilon>0$ there exists R_2' such that

$$\frac{f'(\rho)}{f(\rho)}\geqslant\sqrt{1-\epsilon}-2\epsilon\geqslant 1-3\epsilon\quad\text{for all }\rho>R_2'.$$

Thus, we conclude that

$$\lim_{\rho\to\infty}\frac{f'(\rho)}{f(\rho)}=1,$$

and the proof is complete. □

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Remark 2.3. Let $k_M(x)$ be the restriction of the sectional curvatures of M at $x \in M$ to all the planes that do not contain $\partial_\rho(x)$. It is a function of $\rho(x)$ and it is given by

$$k_M(\rho) = \frac{1-(f'(\rho))^2}{(f(\rho))^2}.$$

If $\lim_{\rho\to\infty} K_M(\rho) = -1$, then by (1.3) and Lemma 2.2 it follows that $\lim_{\rho\to\infty} k_M(\rho) = -1$ as well. Thus, if $\lim_{\rho\to\infty} K_M(\rho) = -1$ and (1.2) hold, then all sectional curvatures of M at $x \in M$ converge to -1 as $x \to \partial M$.

2.1. Proof of Theorem 1.1

We shall proceed as in [**10**, pp. 631–634] and prove that there is a harmonic map u at a bounded distance from Φ. Next, we shall complete the proof of Theorem 1.1 by showing that u is in fact an extension of ϕ. The proof will be given in three steps.

Recall first that $\rho = \rho(x)$ is the distance of the point $x \in M$ from the point that corresponds to the radial coordinate $\rho = 0$.

Step 1. *For any $R > 0$, there is a harmonic map u_R from the ball $\{\rho \leqslant R\}$ into N such that*

$$d_R(x) := d(u_R(x), \Phi(x)) < c$$

for some constant $c > 0$ independent of R.

Let $R > R_0$, where R_0 is a large positive number to be determined later. Consider u_R to be the harmonic map from the ball $\{\rho \leqslant R\}$ into N, such that $u_R = \Phi$ when $\rho = R$. Such a map u exists, since N has negative sectional curvatures [**4**].

Arguing as in [**10**, p. 632], one can show that there is a $C_1 > 0$ such that, for $\rho \leqslant R$,

$$\Delta_M d_R(x) \geqslant -2C_1 \tag{2.11}$$

holds in the sense of distributions.

Set

$$h(x) = C_2 \exp(-\epsilon\sqrt{\rho^2(x)+1}), \quad \epsilon > 0.$$

Note that h satisfies (see [**7**, pp. 200–201] and [**10**, p. 632])

$$\Delta_M h \leqslant 0 \quad \forall x \in M. \tag{2.12}$$

Now choose C_2 such that

$$\Delta_M h < -2C_1 \quad \forall x \in \{\rho < R_0\}. \tag{2.13}$$

Let

$$H_R = d_R - h.$$

We claim that H_R is bounded above by a bound d independent of R. This implies that $d_R(x) < d + C_2$ for all $x \in \{\rho \leqslant R\}$, i.e. the required bound of d_R.

Since H_R is continuous on the compact set $\{\rho \leqslant R\}$, it suffices to show that $H_R(x_0) \leqslant d$, where x_0 is the point where the maximum of H_R in $\{\rho \leqslant R\}$ is attained.

First we show that x_0 is not contained in the open ball $\{\rho < R_0\}$. Indeed, by (2.11) and (2.13), it follows that

$$\Delta_M H(x) \geqslant 0 \quad \forall x \in \{\rho < R_0\}.$$

Thus, H_R is superharmonic in the ball $\{\rho < R_0\}$, which implies that the maximum of H_R is attained at some point outside the open ball $\{\rho < R_0\}$. Therefore, x_0 is contained in the annulus $R_0 \leqslant \rho \leqslant R$.

Next we assume that $x_0 \in \{\rho = R\}$. Then $d_R(x_0) = d_R(u_R(x_0), \Phi(x_0)) = 0$, since $u_R = \Phi$ on $\{\rho = R\}$. It follows that $H_R(x_0) = -h(x_0) \leqslant 0$ and hence

$$H_R(x) \leqslant 0 \quad \text{in } \{\rho \leqslant R\}. \tag{2.14}$$

It remains to treat the case $R_0 \leqslant \rho(x_0) < R$. Using that $\|\tau(\Phi)\| = 0$ at infinity, we shall prove that $H_R(x_0)$ is bounded when $R_0 \leqslant \rho(x_0) < R$.

By the construction of Φ in Lemma 2.1 and proceeding as in [**10**, p. 632] (see also [**6**] and [**11**, pp. 361–368]) one can prove that there exists a $C_3 > 0$ independent of R such that

$$\Delta_M d_R \geqslant -2\|\tau(\Phi)\| + 4C_3 \tanh(\tfrac{1}{2} a d_R), \tag{2.15}$$

where $-a^2$ is the upper bound of the sectional curvatures of N.

Since $\|\tau(\Phi)\| = 0$ at infinity, we can choose $R_0 > 0$ such that $\|\tau(\Phi)\| < C_3$ holds outside the open ball $\{\rho < R_0\}$.

But, the maximum of H_R is attained at some point x_0 with $R > \rho(x_0) \geqslant R_0$, thus by (2.12), (2.15) and the maximum principle, it follows that

$$0 \geqslant \Delta_M H_R(x_0) = \Delta_M d_R(x_0) - \Delta_M h(x_0) \geqslant 4C_3(-\tfrac{1}{2} + \tanh(\tfrac{1}{2} a d_R(x_0))).$$

This implies that

$$\tanh(\tfrac{1}{2} a d_R(x_0)) \leqslant \tfrac{1}{2},$$

hence

$$d_R(x_0) \leqslant C \quad \text{for some constant } C > 0 \text{ independent of } R. \tag{2.16}$$

Consequently, $H_R(x_0)$ is bounded by a bound independent of R and the proof of Step 1 is complete.

Step 2. *Existence of a harmonic map at a bounded distance from Φ.*

Let us recall that from Lemma 2.1 we have that there is a $C_1 > 0$ such that

$$e(\Phi)(x) < C_1 \quad \forall x \in M. \tag{2.17}$$

Combining (2.16) with (2.17) we find as in [**10**, p. 633] that there exists a constant $\lambda > 0$ such that, if $\rho(x) < R - 1$, then

$$u_R(\{d(x, y) < 1\}) \subset \{d(u_R(x), u_R(y)) < \lambda\}. \tag{2.18}$$

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Now, by (2.18) and [**5**, Lemma 2.1], it follows that there is a $b = b(\lambda) > 0$ independent of R such that

$$e(u_R)(x) < b \quad \forall x \in \{\rho < R - 1\}. \tag{2.19}$$

From (2.16) and (2.19), by using the Arzelà–Ascoli Theorem, we find a subsequence R_j such that, on each compact subset of M, the sequence u_{R_j} converges to a harmonic map u such that

$$d(u(x), \Phi(x)) < c \quad \forall x \in M. \tag{2.20}$$

Step 3. *u has boundary values ϕ at infinity.*

From (2.20) and the fact that N has negative sectional curvatures, it follows, by contradiction, that u has boundary values ϕ at infinity.

Indeed, let us assume that for some $\eta \in \mathbb{S}^m$ we have $\Phi(\infty, \eta) = (\infty, \phi(\eta)) = p$ and $u(\infty, \eta) = (\infty, \phi_1(\eta)) = q$, with $p \neq q$. Let $\gamma(t)$ be geodesic in N with $\gamma(0) = p$ and $\gamma(0) = q$.

Recall that h is the metric of N and let h_{-a^2} be the metric of $\mathbb{R}^{n+1}$ with constant sectional curvature equal to the upper bound $-a^2$ of the sectional curvatures of N. Then, by the Metric Comparison Theorem [**8**, Corollary 11.4], we have that

$$h(\gamma'(t), \gamma'(t)) \geqslant h_{-a^2}(\gamma'(t), \gamma'(t)). \tag{2.21}$$

If $\tilde{d}(x, y)$ is the distance on $(\mathbb{R}^{n+1}, h_{-a^2})$, then (2.21) implies that

$$d(p, q) \geqslant \int_0^\infty \sqrt{h_{-a^2}(\gamma'(t), \gamma'(t))}\,\mathrm{d}t \geqslant \tilde{d}(p, q).$$

The last inequality is due to the fact that γ is not necessarily a geodesic in $(\mathbb{R}^{n+1}, h_{-a^2})$. But $(\mathbb{R}^{n+1}, h_{-a^2})$ is in fact a hyperbolic space, so $\tilde{d}(p, q) = \infty$ for $p, q \in \mathbb{S}^n$. Hence,

$$d(u(\infty, \eta), \Phi(\infty, \eta)) = d(p, q) \geqslant \tilde{d}(p, q) = \infty,$$

and this contradicts (2.20).

This completes the proof of Step 3 and the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let u, v be two proper harmonic maps that are C^1 up to the boundary, and let $u|_{\mathbb{S}^m} = v|_{\mathbb{S}^m} = \phi$. Arguing as in [**9**] we shall show that $u \equiv v$.

Since the function $d^2(u, v)(x) = d^2(u(x), v(x))$ is subharmonic [**11**], by the maximum principle it follows that it is enough to show that

$$\lim_{x \to \partial M} d^2(u, v)(x) = 0.$$

Setting

$$h(\rho) = \frac{f}{2}\left(\log\left(\frac{1+\rho}{1-\rho}\right)\right)(1 - \rho^2), \quad \rho \in [0, 1),$$

and

$$H(r) = \frac{F}{2}\left(\log\left(\frac{1+r}{1-r}\right)\right)(1-r^2), \quad r \in [0,1),$$

we can represent M and N by a disc model, where the metrics on M and N are given by

$$g = \frac{4}{(1-\rho^2)^2}(\mathrm{d}\rho^2 + h^2(\rho)\,\mathrm{d}\eta^2) \quad \text{and} \quad h = \frac{4}{(1-r^2)^2}(\mathrm{d}r^2 + H^2(r)\,\mathrm{d}\phi^2), \tag{3.1}$$

respectively.

Note that the Laplacian of M with respect to the metric g above is written as

$$\Delta_M = \frac{(1-\rho^2)^2}{4}\left(\partial^2_{\rho\rho} + m\frac{h'(\rho)}{h(\rho)}\partial_\rho + \frac{1}{h^2(\rho)}\Delta_{\mathbb{S}^m}\right) + \frac{2(m-1)\rho(1-\rho)}{4}\partial_\rho.$$

Note also that by Lemma 2.2 we can assume without loss of generality that

$$\lim_{\rho\to\infty}\frac{f(\rho)}{\mathrm{e}^\rho} = \frac{1}{2} \quad \text{and} \quad \lim_{r\to\infty}\frac{F(r)}{\mathrm{e}^r} = \frac{1}{2},$$

which imply that

$$\lim_{\rho\to 1} h(\rho) = 1 \quad \text{and} \quad \lim_{r\to 1} H(r) = 1. \tag{3.2}$$

Let us set $\bar{g} = \mathrm{d}\rho^2 + h^2(\rho)\,\mathrm{d}\eta^2$ and let us denote by $\bar{\nabla}$ the connection of the metric $\bar{g}$.

From (3.1) it follows that the components of the tension field of a harmonic map $u = u(\rho,\eta) = (r,\theta)$ are given by

$$\tau^0(u) = \Delta_M r + \frac{(1-\rho^2)^2}{4(1-r^2)}\{2r|\bar{\nabla}r|^2 - H(r)(H'(r)(1-r^2) + 2rH(r))H_{\beta\gamma}(\theta)\langle\bar{\nabla}\theta^\beta, \bar{\nabla}\theta^\gamma\rangle\} \tag{3.3}$$

and

$$\begin{aligned}\tau^\alpha(u) = \Delta_M\theta^\alpha &+ \frac{(1-\rho^2)^2}{4}\Gamma^\alpha_{\beta\gamma}\langle\bar{\nabla}\theta^\beta, \bar{\nabla}\theta^\gamma\rangle \\ &+ \frac{(1-\rho^2)^2}{2H(r)(1-r^2)}(H'(r)(1-r^2) + 2rH(r))\langle\bar{\nabla}r, \bar{\nabla}\theta^\alpha\rangle.\end{aligned} \tag{3.4}$$

Then, multiply (3.3) and (3.4) by $(1-r^2)/(1-\rho^2)^2$ and use (3.2) to deduce, as in [**9**], that

$$\partial_\rho r|_{\rho=1} = \sqrt{\frac{e(\phi)(\eta)}{m}} \quad \text{and} \quad \partial_\rho\theta^\alpha|_{\rho=1} = 0. \tag{3.5}$$

Set

$$\bar{r}(\rho,\eta) = 1 - (1-\rho)\sqrt{\frac{e(\phi)}{m}},$$

and consider the map $\Phi\colon M \to N$ defined by

$$\Phi(\rho,\eta) = (\bar{r},\phi).$$

We shall show that

$$\lim_{\rho\to 1} d(u,\Phi)(\rho,\eta)=0 \quad \text{and} \quad \lim_{\rho\to 1} d(v,\Phi)(\rho,\eta)=0. \tag{3.6}$$

Then, by the triangular inequality, we obtain that

$$\lim_{\rho\to 1} d(u,v)(\rho,\eta)=0, \tag{3.7}$$

i.e. the required result.

For the proof of (3.6) we use (3.5) and the triangular inequality as follows. We have

$$d(u,\Phi)\leqslant d((r,\theta),(\bar r,\theta))+d((\bar r,\theta),(\bar r,\phi)).$$

But

$$d((r,\theta),(\bar r,\theta))=\left|\log\frac{1+r}{1-r}-\log\left(\frac{1+\bar r}{1-\bar r}\right)\right|\to 0 \quad \text{as } \rho\to 1$$

and

$$d((\bar r,\theta),(\bar r,\phi))\leqslant\frac{2H(\bar r)}{1-\bar r^2}d_{\mathbb{S}^m}(\theta,\phi)\leqslant\frac{C}{1-\bar r^2}\int_\rho^1\sqrt{H_{\alpha\beta}(\theta)\partial_s u^\alpha\partial_s u^\beta}\,\mathrm{d}s\to 0 \quad \text{as } \rho\to 1.$$

Thus, we have proved the first equation in (3.6). One can prove the second equation in (3.6) in a similar way; hence, (3.7) is valid and the proof of Theorem 1.2 is complete.

Remark 3.1. Instead of requiring that M and N are asymptotically hyperbolic, one can require that $\lim_{\rho\to\infty}K_M(\rho)=-\alpha^2$ and $\lim_{\rho\to\infty}K_N(\rho)=-\beta^2$ hold for some $\alpha,\beta>0$.

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