to them. The number of stones to be put in each of the three spaces between the pairs of pearls is given in the following table, with the total number of corresponding necklaces in each case :—

0	0	6	gives	10	necklaces,
0	1	5	„	20	„
0	2	4	„	20	„
0	3	3	„	10	„
1	1	4	„	10	„
1	2	3	„	20	„
2	2	2	„	4	„
				94	

Sixth Meeting, 21*th April* 1890.

R. E. Allardice, Esq., M.A., Vice-President, in the Chair.

On a hydromechanical theorem.

By Dr A. C. Elliott.

Giffard's injector appeared more than thirty years ago. The first serious attempt to explain its action on dynamical principles was made by the late William Froude at the Oxford Meeting of the British Association in 1860. The history of mechanical science is almost everywhere deeply marked by Rankine; and it seems, just as it ought to be, that he should be found to have contributed not a little to the literature of this particular subject in a paper presented to the Royal Society of London in 1870. As serving to show how far the problem is still interesting, even from a high standpoint, attention may be directed to the exceedingly curious procedure of Professor Greenhill, where he deals cursorily with the matter at the page numbered 448 of his article on Hydromechanics in the *Encyclopædia Britannica.*

When first announced, the statement that the particles of a mere steam jet could, by the agency of this somewhat simple apparatus, force for themselves, in addition to a considerable quantity of more or less cold feed water, re-entrance into the identical boiler from whence they had escaped, seemed to involve an impossibility. But the mystery of that aforetime paradox would have been as nothing had it then been farther known what is now familiar—namely, that

https://doi.org/10.1017/S0013091500030558 Published online by Cambridge University Press

the steam for the steam jet might be in some cases taken from the exhaust-pipe, instead of the boiler, with equally effective results.

In dealing with the theory of Giffard's injector, the author found that it was convenient to employ the principle of impulse and momentum; by which is meant simply a statement of Newton's second law in the form that the time-integral of the forces acting on the masses concerned, and reckoned in a specified direction, is equal to the resulting gain or destruction of momentum reckoned in the same direction.

In the ordinary symbols but merely altered in respect that f and v are regarded as component vectors, Newton's second law is represented by

$$f=m\frac{dv}{dt};$$

and hence

$$\int fdt=\int mdv.$$

The idea is applied in hydromechanics by considering a fluid-filled closed surface; and then the sum of the time-integrals of the external forces acting on or through the bounding surface is equal to the generation of momentum, all reckoned in one and the same direction. It is an instance of Mr Froude's remarkable insight that, in the paper referred to above, his explanation of Giffard's injector was based, so far as it went, on these principles. And it is worthy of remark, too, that Professor Greenhill, in the article already mentioned, by the help of Green's theorem, deduces Euler's equations directly from the fundamental principle.

But first we must obtain the equation of energy. Suppose a fluid to move steadily, irrotationally, and frictionlessly along a tube of stream lines. Let A_1, p_1, v_1 and ρ_1 be respectively the area, the pressure, the velocity, and the density at one section; and let similar symbols with the subscript $_2$ represent the same quantities for another and second section. Farther, let q be the volume corresponding to the mass-flow Q and density ρ. Then neglecting for our purpose differences of level, and equating the work done by the pressures on the end sections per unit time to the increase of energy, we have

$$p_1A_1v_1-p_2A_2v_2=Q\left(\frac{v_2^2}{2g}-\frac{v_1^2}{2g}\right)-\int_1^2 pdq \quad \ldots \quad \ldots \quad (1)$$

where the pressures are measured in terms of the weight of unit-mass. The equation of continuity gives

https://doi.org/10.1017/S0013091500030558 Published online by Cambridge University Press

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2 = Q; \quad \ldots \quad \ldots \quad (2)$$

and

$$q\rho = Q. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Also

$$\int p dq = pq - \int q dp. \quad \ldots \quad \ldots \quad (4)$$

$\therefore$

$$\int_1^2 p dq = A_2 v_2 p_2 - A_1 v_1 p_1 - Q \int_1^2 \frac{dp}{\rho}. \quad \ldots \quad (5)$$

$\therefore$ (1) becomes

$$O = \frac{v_2^2}{2g} - \frac{v_1^2}{2g} + \int_1^2 \frac{dp}{\rho}; \quad \ldots \quad \ldots \quad (6)$$

or

$$\frac{v^2}{2g} + \int \frac{dp}{\rho} = \text{const} = H. \quad \ldots \quad \ldots \quad (7)$$

For the case of a liquid ρ is practically constant; and the equation of energy becomes

$$\frac{v^2}{2g} + \frac{p}{\rho} = H, \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where H may be called the total head.

From (8)

$$p = \rho\left\{H - \frac{v^2}{2g}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The equation of continuity gives

$$Av\rho = A_1 v_1 \rho = A_2 v_2 \rho = Q.$$

$\therefore$

$$p = \rho\left\{H - \frac{1}{2g}\frac{Q^2}{\rho^2 A^2}\right\}. \quad \ldots \quad \ldots \quad (10)$$

Multiplying by dA and integrating the impulse of the sides of the tube per second

$$= \int_1^2 p dA = \rho\left[H(A_2 - A_1) + \frac{Q^2}{\rho^2 2g}\left(\frac{1}{A_2} - \frac{1}{A_1}\right)\right]$$

$$= 2\rho H(A_2 - A_1) + p_1 A_1 - p_2 A_2. \ldots \quad \ldots \quad \ldots \quad (11)$$

Now, considering the impulse on the areas A_1, A_2 it appears that, to obtain the total impulse, there falls to be added to the last expression

$$p_1 A_1 - p_2 A_2.$$

Hence the total impulse per second, say

$$J = 2\{p_1 A_1 - p_2 A_2 - \rho H(A_1 - A_2)\} \quad \ldots \quad \ldots \quad (12)$$

$$= \frac{\rho}{g}\left\{A_2 V_2^2 - A V_1^2\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

https://doi.org/10.1017/S0013091500030558 Published online by Cambridge University Press

$$= 2\,(p_1 - p_2)\,\frac{A_1 A_2}{A_1 + A_2}\,. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (14)$$

(13) may be obtained almost directly; for since impulse is equal to momentum generated

$$J = \frac{Q}{g}(v_2 - v_1)$$

$$= \frac{\rho}{g}\left\{ A_2 v_2^2 - A_1 v_1^2 \right\},$$

as before.

Take for an illustration of (14) the case of a liquid escaping from a large reservoir by a bell-mouthed tube into a space where the pressure is zero. Then $A_1 = \infty$, $p_2 = 0$. Therefore

$$J = 2p_1 A_2, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

and *not* simply $p_1 A_2$, as might be hastily and erroneously concluded.

Secondly, suppose the fluid to be a gas and that the relation of pressure to density can be represented by

$$p = k\rho^{\gamma} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (16)$$

where k and γ are constants.

Reverting to (7) we have

$$\int \frac{dp}{\rho} = \frac{\gamma}{\gamma - 1}\,\frac{p}{\rho}$$

$$= \lambda\frac{p}{\rho}, \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17)$$

where $\lambda = \frac{\gamma}{\gamma - 1}$.

The gain of momentum is as before

$$J = \frac{Q}{g}(v_2 - v_1);$$

and the equation of continuity gives

$$Q = Av\rho = \text{const.}$$

$$\therefore\; J = A_2\rho_2\frac{v_2^2}{g} - A_1\rho_1\frac{v_1^2}{g}. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (18)$$

Substituting from (17) in (6), and by the help of that result and (2) eliminating v_1 and v_2 from (18),

$$J = 2\lambda\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)\frac{A_1\rho_1 \cdot A_2\rho_2}{A_1\rho_1 + A_2\rho_2}. \qquad \ldots \qquad \ldots \qquad (19)$$

(14) and (19) are of course analogous; and the first may be obtained from the second by putting $\rho = \text{const.}$ The author has

found these expressions to possess some interest; and it was principally with the object of drawing attention to them that the present paper was entitled as above.

Let steam escape from a reservoir at pressure p_1 into a space at pressure p_2; and suppose the law

$$p = k\rho^{\gamma}$$

to hold. Then from (6)

$$\frac{v_2^2}{2g} = \int_2^1 \frac{dp}{\rho} = \lambda\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right); \quad \ldots \quad \ldots \quad (20)$$

$$\text{or,} \quad v = \sqrt{2g\lambda}\sqrt{\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}}.$$

Hence the mass-flow

$$Q = A_2\rho_2\sqrt{2g\lambda}\sqrt{\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}}. \quad \ldots \quad \ldots \quad (21)$$

Q is maximum when for a fixed value of p_1, p_2 has such a value that

$$\frac{p_1}{\rho_1}\rho_2^2 - p_2\rho_2 \text{ a maximum.}$$

That is when

$$\frac{p_2}{\rho_2} = \frac{2}{\gamma+1}\frac{p_1}{\rho_1}. \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Therefore

$$Q_{max.} = \rho_2 A_2 \sqrt{2g\lambda}\sqrt{\frac{\gamma-1}{\gamma+1}}\sqrt{\frac{p_1}{\rho_1}}$$

$$= \rho_2 A_2 \sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\sqrt{\frac{p_1}{\rho_1}}$$

$$= \sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} A_2\sqrt{p_1\rho_1}. \quad (23)$$

It is found by experiment that this maximum value of Q occurs; but that on continuously diminishing p_2 below the point corresponding to the maximum hardly any appreciable change takes place. In fact where $p_2 < \frac{3}{5}p_1$, an expression of the form (23) holds; and consequently

$$Q = mA_2\sqrt{p_1\rho_1} \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

where m is a constant not much differing from

$$\sqrt{2g}\sqrt{\frac{\gamma}{\gamma+1}}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}$$

Accepting the guidance of these facts, and reverting to (19), we find when $A_1 = \infty$ that

$$J = 2\lambda\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)A_2\rho_2. \qquad \ldots \qquad \ldots \qquad (25)$$

This expression will be maximum for a given value of $\frac{p_1}{\rho_1}$ when

$$\frac{p_1}{\rho_1}\rho_2 - p_2$$

is maximum. That is when

$$\frac{p_2}{\rho_2} = \frac{1}{\gamma}\frac{p_1}{\rho_1}.$$

Hence the maximum value of J is

$$J_{max.} = 2\lambda\frac{\gamma - 1}{\gamma}\frac{p_1}{\rho_1}A_2\rho_2$$

$$= 2\frac{p_1}{\rho_1}A_2\rho_2$$

$$= 2\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}p_1A_2 \qquad \ldots \qquad \ldots \qquad (26)$$

$$= 2np_1A_2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (27)$$

where n is put for $\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$. We shall take this as a probable value of J for the steady state.

Fig. 55 represents diagramatically the essential parts of an injector. S is the steam pipe; W, the water inlet; O, the overflow; and B the pipe leading through the check valve to the boiler. Let p_1 be the boiler pressure; A_2 the area of the steam nozzle; and A_3 the area of the water-steam throat. The author intends on some future occasion to make an experimental determination of the pressure in the chamber where the steam and water mix. In the meantime, if the feed water be not too hot, we may assume that the pressure is but little greater than zero.

Let us suppose also that the water and steam pipes are large, and that the passages and nozzles are well and gently tapered. Then in applying (14), A_1 may be taken as $= \infty$. Farther, let v_3 represent the velocity of the water-steam jet, p_3 its pressure, and ρ the density; and put p_a for the atmospheric pressure (diminished by a quantity corresponding to the lift of the injector, if any). Then taking the sum of the impulses and equating to the gain of momentum, there results

$$2np_1A_2 + 2p_aA\cos a - p_3A_3 - f_1\rho\frac{v_3^2}{2g}A_3 = A_3v_3\rho \,.\, v_3 \,|\, g, \quad \ldots \quad (28)$$

where f_1 is a friction and eddy resistance factor, A is the sectional area of the water passage projected on a plane at right angles to the stream lines, and a is the angle which a tangent to the stream lines at the same place makes with the axis of the injector. Now, from the equation of energy,

$$\frac{p_3}{\rho} = \frac{p_1}{\rho} - \frac{v_3^2}{2g} + f_0\frac{v_3^2}{2g};$$

or

$$\frac{p_3}{\rho} = \frac{p_1}{\rho} - f_2\frac{v_3^2}{2g} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where f_0 is a friction factor, and f_2 is written for $1 - f_0$.

Hence putting $A\cos a = sA_2$

$$v_3 = \sqrt{\frac{2g}{f_2}}\sqrt{\frac{2(np_1 + sp_a)A_2 - p_1A_3}{\rho(1+f_1)A_3}}. \quad \ldots \quad (30)$$

The mass of steam used per second is say

$$S = mA_2\sqrt{p_1\rho_1};$$

and the mass of water passed on to the boiler per second is

$$A_3v_3\rho.$$

Therefore the mass of water injected per second, say

$$W = A_3v_3\rho - mA_2\sqrt{p_1\rho_1}. \quad \ldots \quad \ldots \quad (31)$$

Hence the ratio of the mass of water injected to the mass of steam used is

$$\frac{W}{S} = \frac{A_3v_3\rho}{mA_2\sqrt{p_1\rho_1}} - 1. \quad \ldots \quad \ldots \quad (32)$$

The efficiency of the apparatus will be maximum when

$$\frac{A_3v_3\rho}{mA_2\sqrt{p_1\rho_1}}$$

is maximum; that is when

$$\frac{A_3}{A_2}v_3$$

is maximum. Writing r for the ratio A_3/A_2, it appears from (30) that there is to be made maximum

$$2(np_1 + sp_a)r - pr^2.$$

Hence for best working

$$r = \frac{np_1 + sp_a}{p_1} \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

This value of r makes

$$v=\sqrt{\frac{2gp_1}{(1+f_1)\rho}}; \quad \ldots \quad \ldots \quad \ldots \qquad (34)$$

and

$$p_3=\frac{f_1}{1+f_1}p_1. \quad \ldots \quad \ldots \quad \ldots \qquad (35)$$

If f_1 were small enough, then this condition could not be fulfilled without

$$p_3<p_a$$

which is inadmissible with an open overflow.

On the other hand, if f_1 were large enough, this value for v could only be consistent with pressure above atmospheric in the overflow chamber, which, under the same conditions as formerly, is equally inadmissible.

It appears, therefore, that the best adjustment will tend towards the production of pressure in the overflow chamber above or below the atmosphere according as

$$\frac{f_1}{1+f_1}p_1 \text{ is } > \text{ or } <p_a.$$

With an open overflow we must have

$$f_2\frac{v_3^2}{2g}=\frac{p_1-p_a}{\rho}. \quad \ldots \quad \ldots \quad \ldots \qquad (36)$$

Therefore from (28) putting $p_3=p_a$,

$$\frac{p_1-p_a}{f_2\rho}=\frac{1}{\rho(2+f_1)}\left\{\frac{2}{r}(np_1+sp_a)-p_a\right\};$$

or

$$r=\frac{2(np_1+sp_a)}{\frac{2+f_1}{f_2}(p_1-p_a)+p_a}. \quad \ldots \quad \ldots \qquad (37)$$

It appears also from (28) that in any case

$$r \text{ must be } < \frac{2(np_1+sp_a)}{p_1}. \quad \ldots \quad \ldots \qquad (38)$$

If T_1 and T_f be respectively the temperatures of the steam and of the feed water in degrees Fahr., then in order that there might possibly just be condensation at the throat,

$$s=\frac{m\sqrt{p_1\rho_1}\{912+\cdot3(T_1-32)\}}{\sqrt{2g\rho p_a}\{212-T_f\}}\cos a;$$

but the mutual action is not actually so perfect, and it is well to allow

https://doi.org/10.1017/S0013091500030558 Published online by Cambridge University Press

$$s = \frac{m\sqrt{p_1\rho_1}\{974 + 3(T_1 - 32)\}}{\sqrt{2g\rho p_a}\{150 - T_f\}}\cos a. \qquad \ldots \qquad (39)$$

Taking $\gamma = 10/9$

$$n = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma - 1}} = \cdot 4 \text{ nearly.}$$

Suppose, further, for example, that the boiler pressure is 160 lbs. per sq. in. absolute; that the diameter of the water-steam throat is ·25 inch; and assume the following values:—

$$T_f = 60°F.; \ a = 13°; \ f = \cdot 5;$$
$$f_2 = \cdot 9; \ m = 3\ 6, \text{ allowing for contraction.}$$

Then $r = 42$; and diameter of steam nozzle

$$= \frac{\cdot 25}{\sqrt{\cdot 42}} = \cdot 38 \text{ inch.}$$

$$W = A_3\left\{\sqrt{2g\rho(p_1 - p_a)/f_2} - \frac{m}{r}\sqrt{p_1\rho_1}\right\} \quad \ldots \qquad (40)$$

$$= 3 \cdot 0 \text{ lbs. per second.}$$

For the exhaust steam injector we have merely to substitute p_a for p_1 where it is associated with n, and in (31) where it appears under the radical. Thus

$$r = \frac{2p_a(n + s)}{\frac{2 + f_1}{f_2}(p_1 - p_a) + p_a} \qquad \ldots \qquad \ldots \qquad (41)$$

$$s = \frac{m\sqrt{\rho_a} \times 1028}{\sqrt{2g\rho}\{150° - T_f\}}; \qquad \ldots \qquad \ldots \qquad (42)$$

and

$$W = A_3\{\sqrt{2g\rho(p_1 - p_a)/f_2} - \frac{m}{r}\sqrt{p_a\rho_a}\}. \ldots \qquad (43)$$

On Rankine's Formula for Earth Pressure.

By Dr A. C. Elliott.

In a short course of lectures on "Railway Practice," the author was recently called upon to deal with the mechanical principles involved in the design of retaining walls. What has come to be known as Rankine's method had to be explained, at all events, in its