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OPTIMAL STOPPING ZERO-SUM GAMES IN CONTINUOUS HIDDEN MARKOV MODELS

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Abstract

We study a two-dimensional discounted optimal stopping zero-sum (or Dynkin) game related to perpetual redeemable convertible bonds expressed as game (or Israeli) options in a model of financial markets in which the behaviour of the ex-dividend price of a dividend-paying asset follows a generalized geometric Brownian motion. It is assumed that the dynamics of the random dividend rate of the asset paid to shareholders are described by the mean-reverting filtering estimate of an unobservable continuous-time Markov chain with two states. It is shown that the optimal exercise (conversion) and withdrawal (redemption) times forming a Nash equilibrium are the first times at which the asset price hits either lower or upper stochastic boundaries being monotone functions of the running value of the filtering estimate of the state of the chain. We rigorously prove that the optimal stopping boundaries are regular for the stopping region relative to the resulting two-dimensional diffusion process and that the value function is continuously differentiable with respect to the both variables. It is verified by means of a change-of-variable formula with local time on surfaces that the optimal stopping boundaries are determined as a unique solution to the associated coupled system of nonlinear Fredholm integral equations among the couples of continuous functions of bounded variation satisfying certain conditions. We also give a closed-form solution to the appropriate optimal stopping zero-sum game in the corresponding model with an observable continuous-time Markov chain.

Keywords: Optimal stopping zero-sum (or Dynkin) game; generalized geometric Brownian motion; continuous-time Markov chain; filtering estimate; two-dimensional diffusion process; stochastic boundaries; probabilistic regularity; parabolic-type free-boundary problem; coupled system of nonlinear Fredholm integral equations; change-of-variable formula with local time on surfaces; perpetual redeemable convertible bond; random mean-reverting bounded dividend rate

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1. Introduction

Let us consider a probability space $(\Omega, \mathcal{H}, \mathbb{P})$ endowed with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and a continuous-time Markov chain $\mathbf{\Theta} = (\mathbf{\Theta}_t)_{t \geq 0}$ with two states, 0 and 1. It

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is assumed that the processes B and $\mathbf{\Theta}$ are independent under the probability measure $\mathbb{P}$. We define the process $Y = (Y_t)_{t \geq 0}$ by

$$Y_t = y \, \exp\bigg(\int_0^t \Big(r - \frac{\sigma^2}{2} - \eta_0 - (\eta_1 - \eta_0) \, \mathbf{\Theta}_s \Big) \, ds + \sigma \, B_t \bigg) \tag{1.1}$$

which solves the stochastic differential equation

$$dY_t = \big(r - \eta_0 - (\eta_1 - \eta_0) \, \mathbf{\Theta}_t \big) \, Y_t \, dt + \sigma \, Y_t \, dB_t \quad (Y_0 = y) \tag{1.2}$$

where $y \in \mathbb{R}_{++} \equiv (0, \infty)$ is fixed, while $r > 0$, $\eta_j > 0$ for $j = 0, 1$, and $\sigma > 0$ are some given constants, and we may assume that the inequalities $\eta_0 > \eta_1 > 0$ hold, without loss of generality. The main aim of this paper is to study analytic properties of the expected upper and lower values $V^* \geq V_*$ of the discounted optimal stopping zero-sum (or Dynkin) game, arising from the problem of pricing of perpetual *redeemable convertible bonds* described below, given by

$$V^* = \inf_{\zeta} \sup\nolimits_{\tau} \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-r\,s} \, Y_s \, ds + e^{-r\tau} \, K \, I(\tau < \zeta) + e^{-r\zeta} \, L \, I(\zeta \leq \tau) \bigg] \tag{1.3}$$

and

$$V_* = \sup\nolimits_{\tau} \inf_{\zeta} \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-r\,s} \, Y_s \, ds + e^{-r\tau} \, K \, I(\tau < \zeta) + e^{-r\zeta} \, L \, I(\zeta \leq \tau) \bigg] \tag{1.4}$$

for some given constants $L > K > 0$, where $\mathbb{E}$ denotes the expectation taken under the probability measure $\mathbb{P}$, the suprema and infima are taken over all stopping times τ and ζ with respect to the natural filtration $(\mathcal{G}_t)_{t \geq 0}$ of the process Y defined by (1.1) and (1.2), while $I(\,\cdot\,)$ denotes the indicator function.

Let us assume that the process $\mathbf{\Theta}$ has the initial distribution $\{1 - q, q\}$, for $q \in [0, 1]$, the transition probability matrix $\{(\lambda_0 e^{-(\lambda_0 + \lambda_1)t} + \lambda_1)/(\lambda_0 + \lambda_1), \lambda_0(1 - e^{-(\lambda_0 + \lambda_1)t})/(\lambda_0 + \lambda_1); \lambda_1(1 - e^{-(\lambda_0 + \lambda_1)t})/(\lambda_0 + \lambda_1), (\lambda_1 e^{-(\lambda_0 + \lambda_1)t} + \lambda_0)/(\lambda_0 + \lambda_1)\}$, and thus the intensity matrix $\{-\lambda_0, \lambda_0; \lambda_1, -\lambda_1\}$, for all $t \geq 0$ and some $\lambda_j > 0$, for $j = 0, 1$ fixed. In other words, the Markov chain $\mathbf{\Theta}$ changes its state from j to $1 - j$ at exponentially distributed times of intensity λ_j, for every $j = 0, 1$, which are independent of the dynamics of the standard Brownian motion B. Such a process $\mathbf{\Theta}$ is called a *telegraphic signal* in the literature (cf., for example, [61, Chapter IX, Section 4] or [27, Chapter VIII]). Then it is shown by means of standard arguments (cf., for example, [61, Chapter IX] or [27, Chapter VIII]) that the asset price process Y from (1.1) and (1.2) admits the representation

$$dY_t = \big(r - \eta_0 - (\eta_1 - \eta_0) \, Q_t \big) \, Y_t \, dt + \sigma \, Y_t \, d\overline{B}_t \quad (Y_0 = y) \tag{1.5}$$

on its natural filtration $(\mathcal{G}_t)_{t \geq 0}$, and the filtering estimate process $Q = (Q_t)_{t \geq 0}$ given the (observable) filtration $(\mathcal{G}_t)_{t \geq 0}$ defined by $Q_t = \mathbb{E}[\mathbf{\Theta}_t \mid \mathcal{G}_t] \equiv \mathbb{P}(\mathbf{\Theta}_t = 1 \mid \mathcal{G}_t)$ solves the stochastic differential equation

$$dQ_t = \big(\lambda_0 \, (1 - Q_t) - \lambda_1 \, Q_t \big) \, dt + \frac{\eta_0 - \eta_1}{\sigma} \, Q_t (1 - Q_t) \, d\overline{B}_t \quad (Q_0 = q) \tag{1.6}$$

for some $(y, q) \in \mathbb{R}_{++} \times [0, 1]$ fixed. Here, the innovation process $\overline{B} = (\overline{B}_t)_{t \geq 0}$ defined by

$$\overline{B}_t = \int_0^t \frac{dY_s}{\sigma \, Y_s} - \frac{1}{\sigma} \int_0^t \big(r - \eta_0 - (\eta_1 - \eta_0) \, Q_s \big) \, ds \tag{1.7}$$

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is a standard Brownian motion under the probability measure $\mathbb{P}$, according to P. Lévy's characterization theorem (cf., for example, [61, Theorem 4.1]). Thus, (Y, Q) is a continuous (time-homogeneous) strong Markov (as well as strong Feller) process, under the probability measure $\mathbb{P}$ with respect to its natural filtration which clearly coincides with $(\mathcal{G}_t)_{t \geq 0}$, as a unique strong solution to the system of stochastic differential equations in (1.5) and (1.6) (cf., for example, [64, Theorem 7.2.4] and [73, page 170]). The main aim of this paper is to study the game-theoretic problem of (1.3) and (1.4) as the necessarily two-dimensional optimal stopping zero-sum game of (2.1) and (2.2) in the continuous hidden Markov model of (Y, Q) defined above.

Assume that the process Y describes the risk-neutral dynamics of the ex-dividend price of a dividend-paying asset issued by a certain firm and liquidly traded on a financial market under a risk-neutral (or martingale) probability measure $\mathbb{P}$, under which the associated (discounted) cum-dividend process forms a martingale. Here, the discounting rate r coincides with the interest rate of a riskless bank account, and σ is the volatility coefficient. Suppose that the process $\boldsymbol{\Theta}$ describes the running economic state of the firm issuing the asset, which can be either in the so-called *good* state when $\boldsymbol{\Theta} = 0$, or in the so-called *bad* state when $\boldsymbol{\Theta} = 1$, which is unobservable by the usual investors trading in the market, but has an influence on the dividend rate of the asset paid to the shareholders of the issuing firm. More precisely, we assume that the random dividend rate dynamics is modelled by the continuous mean-reverting bounded process $\eta_0(1 - Q) + \eta_1 Q$, so that the process $\eta_0(1 - \boldsymbol{\Theta}) + \eta_1 \boldsymbol{\Theta}$ can in its turn be used as an estimate for the dividend rate process $\eta_0(1 - Q) + \eta_1 Q$. In this respect, the solution to the associated optimal stopping zero-sum game of (5.1) and (5.2) below, in which the suprema and infima are taken with respect to the filtration $(\mathcal{H}_t)_{t \geq 0}$ generated by the processes Y and $\boldsymbol{\Theta}$, provides natural estimates for the solution to the original problem of (1.3) and (1.4), which is embedded into that of (2.1) and (2.2) below. We further assume that the process $\boldsymbol{\Theta}$ has the same probability characteristics including the initial distribution $\{1 - q, q\}$ and the intensity matrix $\{-\lambda_0, \lambda_0; \lambda_1, -\lambda_1\}$ with respect to the martingale measure $\mathbb{P}$ and the initial or physical probability measure existing in the considered financial market model. This property enables us to specify the pricing measure $\mathbb{P}$ from the set of all martingale measures in the incomplete market model defined by (1.1) and (1.2).

According to the conditions of the contract considered, the writer of the bond continuously delivers the *cash flow* generated by the assets of the issuing dividend-paying firm at the rate Y to the bond holder up to the maturity time of the contract $\tau \wedge \zeta$, as well as paying to the holder either the *redemption price* L for the withdrawal at the time ζ that the writer may choose, or the *conversion price* K for the exercise at the time τ that the holder may choose. In this respect, the writer of the bond looks for the *redemption time* ζ_* minimizing the expected cumulative discounted amount delivered to the holder in (1.3) and (1.4), while the holder of the bond looks for the *conversion time* τ^* maximizing the expected cumulative discounted amount delivered by the writer in (1.3). We may thus conclude that this contract has a structure of a (zero-sum) game-type (or Israeli) contingent claim introduced by Kifer [54] and further developed by Kyprianou [57], Kühn and Kyprianou [56], Kallsen and Kühn [51], Baurdoux and Kyprianou [3–5], and Ekström and Villeneuve [26] among others. The two-person zero-sum games related to *(redeemable) convertible bonds* (with *floating conversion prices*) were considered by Sîrbu, Pikovsky, and Shreve [75] and Sîrbu and Shreve [76] within the structural firm-value models based on geometric Brownian motions on infinite and finite time intervals, as well as in Gapeev and Kühn [38] and Baurdoux, Kyprianou, and Pardo [6] in the reduced-form models with infinite horizon based on geometric jump-diffusion processes and spectrally positive

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Lévy processes. In the view of the considered model of financial markets for the dividend-paying asset described above, the (actually coinciding) values of (1.3) and (1.4) may therefore be interpreted as the rational (or no-arbitrage) net present values of a perpetual *redeemable convertible bond* (with a *fixed conversion price*) in an extension of the Black–Merton–Scholes model with random continuous mean-reverting bounded dividend rates. The rational valuation of other contingent claims (with sole active participants) on the infinite time horizon was studied in Leland [58] in the classical model based on a one-dimensional geometric Brownian motion with constant coefficients, and in [36] in the hidden Markov model formulated above.

The study of optimal stopping zero-sum games was initiated by Dynkin [22]. The purely probabilistic approach to the analysis of such game-theoretic problems involving applications of the martingale theory was developed by Neveu [63], Krylov [55], Bismut [10], Stettner [77, 78], and Lepeltier and Mainguenau [59] among others. Bensoussan and Friedman [8, 9] developed the analytical theory of stochastic differential games with stopping times in continuous Markovian diffusion models. Relations between Dynkin games and bounded-variation control problems were observed by Taksar [79] and further developed by Karatzas and Wang [53] and, more recently, by Ferrari and Rodosthenous [31] among others. Connections between the values of optimal stopping games and the solutions of (doubly) reflected backward stochastic differential equations with general (random) coefficients were established by Cvitanić and Karatzas [15], where a pathwise approach for studying these games was also provided. Further developments of the general theory of optimal stopping games were recently provided by Ekström and Peskir [24], Peskir [68, 69], and Bayraktar and Sîrbu [7] among others.

One class of two-dimensional optimal stopping problems is formed by the optimal stopping problems for one-dimensional continuous (time-homogeneous strong) Markov processes on finite time intervals, which were initiated and further studied by van Moerbeke [80], Jacka [46], Broadie and Detemple [12], and Carr, Jarrow, and Myneni [14] among others (see also Myneni [62] for the review of contemporary results in the area). These problems are normally reduced to the equivalent free-boundary problems for the infinitesimal (parabolic) operators of the original diffusion processes. It was shown by Peskir [65, 66] (see also Peskir and Shiryaev [71, Chapter VI, Sections 21 and 22, Chapter VII, Sections 25–27] and other related subsequent articles on optimal stopping problems for one-dimensional diffusion processes on finite time intervals) that the value functions and optimal stopping boundaries are uniquely characterized by parabolic free-boundary problems, which are equivalent to (systems of) nonlinear Volterra integral equations for the boundaries.

Another class of such problems is formed by the optimal stopping problems for two-dimensional diffusion processes with time-independent rewards on infinite-time intervals. Such problems particularly appear in relation to the Bayesian sequential hypothesis testing and quickest change-point (or disorder) detection problems for observable diffusion processes. It was shown in Liptser and Shiryaev [61, Chapter IX, Section 4] (see also Gapeev and Shiryaev [42, 43]) that the sufficient statistics processes containing the observable processes and the appropriated posterior probabilities as their state space components are driven by the same (one-dimensional innovation) standard Brownian motion, so that the original problems are equivalent to free-boundary problems for partial differential operators of parabolic type. These problems of statistical sequential analysis were taken further and solved by Johnson and Peskir [49, 50] for the cases of models with observable Bessel processes. It was shown, by using the change-of-variable formula with local time on surfaces derived by Peskir [67, Theorem 3.1], that the value functions and (single) optimal stopping boundaries are uniquely characterized by the parabolic-type free-boundary problems which are equivalent to the (single) nonlinear

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Fredholm integral equations for the boundaries. More recently, Ernst, Peskir, and Zhou [29] solved the optimal stopping problem for a two-dimensional diffusion process related to the optimal real-time detection of a drifting Brownian coordinate which is equivalent to a free-boundary problem, the associated partial differential operator of elliptic type. The important recent results in this area comprise the continuity of the optimal stopping boundaries in optimal stopping problems for two-dimensional diffusions proved by Peskir [70] and the global C^1-regularity of the value function in two-dimensional optimal stopping problems studied by De Angelis and Peskir [18].

In the present paper, we study the necessarily two-dimensional optimal stopping zero-sum game of (2.1) and (2.2) which is associated with that of (1.3) and (1.4) for the firm value expressed by a generalized geometric Brownian motion Y having the drift rate involving the continuous mean-reverting filtering estimate Q of an unobservable continuous-time two-state Markov chain $\boldsymbol{\Theta}$. Such a hidden Markov model was proposed by Shiryaev [74, Chapter III, Section 4a] for the description of interest rate dynamics, and then applied by Elliott and Wilson [28] for the computation of zero-coupon bond prices and other quantities in the interest rate framework. Some preliminary results including the monotonicity of the optimal stopping boundaries and the verification assertion related to a different optimal stopping problem in a similar model with the resulting two-dimensional continuous (strong Markov) diffusion process were presented in [35] (see also [41] for a study of another optimal stopping problem in the same model with one-sided continuation regions). A rigorous proof of the probabilistic regularity of the optimal stopping boundary as well as an assertion about the continuous differentiability of the value function of another different optimal stopping problem with a one-sided continuation region were recently given in [36]. In contrast to previous results, we give rigorous proofs of the probabilistic regularity of the optimal stopping boundaries and establish the continuous differentiability of the value function in the context of the optimal stopping zero-sum games of two players in the model with a two-dimensional non-trivial degenerate diffusion process and a two-sided continuation region. Moreover, we prove the existence and uniqueness of solutions to the appropriate coupled system of two nonlinear Fredholm integral equations for the boundaries (among couples of continuous functions of bounded variation), which appear to be novel for the existing literature, to the best of our knowledge. Note that another proof of the probabilistic regularity of the optimal stopping boundaries in optimal stopping games with incomplete information but static two-state Markov chain with non-switching parameters was recently presented in De Angelis, Gensbittel, and Villeneuve [17, Section 6]. We also derive a closed-form solution to the associated optimal stopping zero-sum game of two players of (5.1) and (5.2) in the model with a diffusion process Y and an observable continuous-time two-state Markov chain $\boldsymbol{\Theta}$ with a two-sided continuation region, which has not been done before, to the best of our knowledge.

Other analytic properties of the value functions of optimal stopping problems in continuous diffusion models with unobservable regime-switching as well as non-switching parameters were studied by Buffington and Elliott [13], Décamps, Mariotti, and Villeneuve [19], and De Angelis, Gensbittel, and Villeneuve [17] among others. Further properties of the value functions (including numerical approximations for the optimal stopping boundaries) of the problems of optimal liquidation of risky assets described by generalized geometric Brownian motions with unobservable random drift rates were studied by Ekström and Lu [23] and Ekström and Vaicenavicius [25] among others. Note that this paper represents a complement to the articles mentioned above in the sense that it deals with the model containing unobservable stochastic drift rates with the dynamics essentially changing over the infinite time horizon.

Recall that closed-form solutions to other optimal stopping problems in the appropriate model with an observable two-state Markov chain were obtained by Guo [44], Guo and Zhang [45], and Gapeev, Kort, and Lavrutich [37] for the perpetual American standard and lookback (or Russian) option problems, respectively. Such problems in models with observable regime-switching parameters were also studied by Jobert and Rogers [48], Dalang and Hongler [16], and Jiang and Pistorius [47] among others.

The rest of the paper is organized as follows. In Section 2, we embed the original game of (1.3) and (1.4) into the optimal stopping game of (2.1) and (2.2) for the two-dimensional continuous Markov diffusion process (Y, Q) defined in (1.1) and (1.2), and (1.5) and (1.6) with (1.7), where the process Q is the filtering estimate of the state of the issuing firm $\boldsymbol{\Theta}$. It is shown that the stopping times τ^* and ζ_* forming a Nash equilibrium are expressed as the first times at which the firm value process Y hits lower or upper stochastic boundaries $g^*(Q)$ and $h_*(Q)$ which represent decreasing functions of the running value of the process Q (Lemma 2.1). We also introduce the auxiliary optimal stopping problems of (2.6) and (2.7) in the associated model with the observable Markov chain $\boldsymbol{\Theta}$ and apply the comparison arguments to specify the location of the boundaries $g^*(Q)$ and $h_*(Q)$ relative to the optimal stopping boundaries $\underline{g}(\boldsymbol{\Theta})$ and $\overline{h}(\boldsymbol{\Theta})$ for the process Y in the problems introduced.

In Section 3, we formulate the equivalent free-boundary problem for a parabolic-type partial differential equation and present rigorous proofs of the facts that the optimal stopping boundaries are regular for the stopping region relative to the process (Y, Q) and the value function of the optimal stopping zero-sum game is continuously differentiable in the both variables (Lemmas 3.1–3.4).

In Section 4, in order to be able to apply the change-of-variable formula with local times on surfaces derived by Peskir [67, Theorem 3.1], we introduce an appropriate change of variables that allows us to reduce the resulting parabolic-type partial differential equation to the normal (or canonical) form. It is verified that the optimal stopping boundaries are determined as a unique solution to the associated coupled system of nonlinear Fredholm integral equations among couples of continuous functions of bounded variation satisfying certain appropriate conditions (Lemma 4.1). We state the main result concerning the rational valuation of the perpetual redeemable convertible bonds in the hidden Markov model considered (Theorem 4.1). In particular, we give closed-form solutions to the optimal stopping problem of (2.1) and (2.2) in terms of the Gauss hypergeometric functions under certain relations on the parameters of the model, for the case of unobservable random (Bernoulli) drift rates (Corollary 4.1).

Finally, in Section 5, we derive a closed-form solution to the auxiliary optimal stopping game of (5.1) and (5.2) in the model with an observable continuous-time Markov chain $\boldsymbol{\Theta}$ (Theorem 5.1) for completeness. We also give closed-form solutions to the appropriate optimal stopping problems in that model (Corollaries 5.1 and 5.2), which give natural bounds $\overline{g}(\boldsymbol{\Theta})$ and $\underline{h}(\boldsymbol{\Theta})$ for the optimal stopping boundaries $g^*(Q)$ and $h_*(Q)$ in the original zero-sum game of (2.1) and (2.2).

2. Formulation of the problem

In this section, we introduce the setting and notation of the optimal stopping zero-sum game which is related to the problem of pricing perpetual redeemable convertible bonds in the underlying diffusion-type model with random dividends and an *unobservable* economic state of the firm described by a continuous-time Markov chain with two states.

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2.1. The optimal stopping game

It is seen that the problem of (1.3) and (1.4) can be embedded into the optimal stopping zero-sum game for the two-dimensional continuous (time-homogeneous) strong Markov process $(Y, Q) = (Y_t, Q_t)_{t \geq 0}$ with the upper and lower value functions $V^*(y, q) \geq V_*(y, q)$ given by

$$V^*(y, q) = \inf_{\zeta} \sup_{\tau} \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs}\, Y_s^{(y,q)}\, ds + e^{-r\tau}\, K\, I(\tau < \zeta) + e^{-r\zeta}\, L\, I(\zeta \leq \tau) \bigg] \tag{2.1}$$

and

$$V_*(y, q) = \sup_{\tau} \inf_{\zeta} \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs}\, Y_s^{(y,q)}\, ds + e^{-r\tau}\, K\, I(\tau < \zeta) + e^{-r\zeta}\, L\, I(\zeta \leq \tau) \bigg] \tag{2.2}$$

for some $r > 0$ and $L > K > 0$ fixed, where the suprema and infima are taken over all stopping times τ and ζ of the process (Y, Q). Hereafter, we indicate by $(Y^{(y,q)}, Q^{(q)})$ the dependence of the solution (Y, Q) of the system of stochastic differential equations in (1.5) and (1.6) on the starting point $(y, q) \in \mathbb{R}_{++} \times [0, 1]$. We may conclude from the results of [24, Theorem 2.1] and [68, Theorem 2.1] that the optimal stopping game of (2.1) and (2.2) has a value providing a Stackelberg equilibrium, so that the equality $V^*(y, q) = V_*(y, q)$ holds, for each $(y, q) \in \mathbb{R}_{++} \times [0, 1]$ fixed. It follows from the explicit form of the process Y in (1.1) and (1.2) and the representations for the processes Y and Q in (1.5) and (1.6) that the discounted process $(e^{-rt} Y_t)_{t \geq 0}$ is a strict supermartingale closed at zero, under the assumptions that $r > 0$ and $\eta_0 > \eta_1 > 0$. Thus, the value functions in (2.1) and (2.2) are restricted between K and L, for each $(y, q) \in \mathbb{R}_{++} \times [0, 1]$ fixed.

Observe that the arguments of the proofs from [24, Theorem 2.1] and [68, Theorem 2.1], which are also applicable to strong Markov time-space processes with constant killing rates, can be naturally extended to the case of the discounted optimal stopping zero-sum game of (2.1) and (2.2) for the process (Y, Q) solving the system of stochastic differential equations in (1.5) and (1.6) (see also [69, Theorem 3.1] for the related result in a model based on a standard Brownian motion in the interval [0,1] absorbed at either 0 or 1). Note that the natural analogues of the conditions of [24, Formula (2.1)] and [68, Formulae (2.9) and (2.12)] are clearly satisfied for the discounted payoffs $e^{-rt}K$ and $e^{-rt}L$ from (2.1) and (2.2). Hence, by applying the resulting extensions of the results on optimal stopping zero-sum games for continuous (time-homogeneous) strong Markov processes of [24, Theorem 2.1] and [68, Theorem 2.1], we obtain from the structure of the reward functional in the game of (2.1) and (2.2) with infinite time horizon that the stopping times forming a Nash equilibrium exist and have the form

$$\tau^* = \inf\big\{ t \geq 0 \,\big|\, V^*(Y_t, Q_t) = K \big\} \quad \text{and} \quad \zeta_* = \inf\big\{ t \geq 0 \,\big|\, V_*(Y_t, Q_t) = L \big\} \tag{2.3}$$

so that the appropriate continuation and stopping regions C_* and D_* are given by

$$C_* = \big\{ (y, q) \in \mathbb{R}_{++} \times [0, 1] \,\big|\, K < V^*(y, q) = V_*(y, q) < L \big\} \tag{2.4}$$

and

$$D_* = \big\{ (y, q) \in \mathbb{R}_{++} \times [0, 1] \,\big|\, \text{either } V^*(y, q) = V_*(y, q) = K \text{ or } V^*(y, q) = V_*(y, q) = L \big\}, \tag{2.5}$$

respectively. We show in parts (i) and (ii) of Section 2.3 below that the sets C_* and D_* in (2.4) and (2.5) are non-empty. We will prove in Lemma 3.1 below that $V^*(y, q) = V_*(y, q)$ forms a continuous function, so that the set C_* is open and the set D_* is closed.

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2.2. The structure of optimal stopping times

Let us now show the form of the optimal stopping times τ^* and ζ_* in (2.3) and clarify the structure of the associated continuation and stopping regions C_* and D_* in (2.4) and (2.5), respectively.

(i) In order to provide the upper and lower bounds for the value functions and the existence of the optimal stopping boundaries in the optimal stopping zero-sum game of (2.1) and (2.2), let us consider the optimal stopping problems with the value functions

$$\underline{W}(y, j) = \sup_{\tau'} \mathbb{E}\bigg[\int_0^{\tau'} e^{-rs}\, Y_s^{(y,j)}\, ds + e^{-r\tau'} K \bigg] \tag{2.6}$$

and

$$\overline{W}(y, j) = \inf_{\zeta'} \mathbb{E}\bigg[\int_0^{\zeta'} e^{-rs}\, Y_s^{(y,j)}\, ds + e^{-r\zeta'} L \bigg] \tag{2.7}$$

for some $r > 0$ and $L > K > 0$ fixed, where the supremum and infimum are taken over all stopping times τ' and ζ' with respect to the natural filtration $\mathcal{H} = (\mathcal{H}_t)_{t \ge 0}$ of the process $(Y, \mathbf{\Theta})$ with the first component explicitly given by (1.1) and (1.2). Hereafter, we indicate by $(Y^{(y,j)}, \mathbf{\Theta}^{(j)})$ the dependence of the process $(Y, \mathbf{\Theta})$ on its starting point $(y, j) \in \mathbb{R}_{++} \times \{0, 1\}$.

It is shown by means of arguments similar to those used in [36, Appendix] as well as in [44, Section 2] and [45, Section 2] applied to solving other optimal stopping problems in models with observable continuous-time Markov chains (or using the easily shown convexity and concavity of the value functions) that the optimal stopping times in the problems of (2.6) and (2.7) have the form

$$\underline{\tau} = \inf\big\{ t \ge 0 \,\big|\, Y_t \le \underline{g}(\mathbf{\Theta}_t) \big\} \quad \text{and} \quad \overline{\zeta} = \inf\big\{ t \ge 0 \,\big|\, Y_t \ge \overline{h}(\mathbf{\Theta}_t) \big\} \tag{2.8}$$

for some numbers $0 < \underline{g}(j) \le rK < rL \le \overline{h}(j)$, for $j = 0, 1$, to be determined. It follows from the arguments in [36, Appendix] and Section 5.5 below resulting at the statements of Corollaries 5.2 and 5.3 below (see also [36; Corollary A.2] as well as [44, Theorem 2.1] and [45, Theorem 2]) that the value functions $\underline{W}(y, j)$ and $\overline{W}(y, j)$ of the auxiliary optimal stopping problems from (2.6) and (2.7) admit the explicit expressions in (5.56) and (5.57), and the numbers $\underline{g}(j)$ and $\overline{h}(j)$, for $j = 0, 1$, in the associated optimal stopping times $\underline{\tau}$ and $\overline{\zeta}$ from (2.8) are determined from the expressions in (5.49)–(5.50) and (5.54)–(5.55) with (5.15) as well as in (5.68)–(5.69), respectively.

Note that, since the suprema and infima in (2.1) and (2.2) are taken over all stopping times τ and ζ with respect to the filtration $(\mathcal{G}_t)_{t \ge 0}$, which is smaller than the appropriate filtration $(\mathcal{H}_t)_{t \ge 0}$ for the stopping times τ' and ζ' over which the supremum and infimum are taken in (2.6) and (2.7), respectively, the inequalities $V^*(y, j) \le \underline{W}(y, j)$ and $V_*(y, j) \ge \overline{W}(y, j)$ should hold, for all $y \in \mathbb{R}_{++} \times \{0, 1\}$. The latter inequalities also yield the fact that the points $((0, \underline{g}(1)] \cup [\overline{h}(0), \infty)) \times [0, 1]$ surely belong to the stopping region D_* in (2.5).

(ii) We now observe that, by means of straightforward computations, it is seen that the expressions

$$\begin{aligned} &\mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs}\, Y_s^{(y,q)}\, ds + e^{-r\tau} K\, I(\tau < \zeta) + e^{-r\zeta} L\, I(\zeta \le \tau) \bigg] \\ &= K + \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs} \left(Y_s^{(y,q)} - rK \right) ds + e^{-r\zeta} (L - K)\, I(\zeta \le \tau) \bigg] \end{aligned} \tag{2.9}$$

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and

$$\mathbb{E}\Big[\int_0^{\tau\wedge\zeta} e^{-rs}\, Y_s^{(y,q)}\, ds + e^{-r\tau}\, K\, I(\tau < \zeta) + e^{-r\zeta}\, L\, I(\zeta \le \tau) \Big]$$
$$= L + \mathbb{E}\Big[\int_0^{\tau\wedge\zeta} e^{-rs}\, \big(Y_s^{(y,q)} - rL\big)\, ds - e^{-r\tau}\, (L-K)\, I(\tau < \zeta) \Big] \tag{2.10}$$

hold, for any stopping times τ and ζ. Then it follows from (2.9) and (2.10) and the structure of the stopping times in (2.3) that the (actually coinciding) value functions of the optimal stopping game in (2.1) and (2.2) admit the representations

$$V^*(y,q) = K + \mathbb{E}\Big[\int_0^{\tau^*\wedge\zeta_*} e^{-rs}\, \big(Y_s^{(y,q)} - rK\big)\, ds + e^{-r\zeta_*}\, (L-K)\, I(\zeta_* \le \tau^*) \Big] \tag{2.11}$$

and

$$V_*(y,q) = L + \mathbb{E}\Big[\int_0^{\tau^*\wedge\zeta_*} e^{-rs}\, \big(Y_s^{(y,q)} - rL\big)\, ds - e^{-r\tau^*}\, (L-K)\, I(\tau^* < \zeta_*) \Big] \tag{2.12}$$

for all $(y, q) \in \mathbb{R}_{++} \times [0, 1]$. Here, we denote by $\tau^* = \tau^*(y, q)$ and $\zeta_* = \zeta_*(y, q)$ the stopping times forming a Nash equilibrium in (2.1) and (2.2) associated with the starting point $(y, q) \in \mathbb{R}_{++} \times [0, 1]$ of the process (Y, Q). On the one hand, it is seen from the structure of the integrands in (2.11) and (2.12) that it is not optimal for the holder of the bond (maximizer of the expected reward) to exercise the contract (convert the bond) earlier than the writer may withdraw it when $Y_t > rK$, while it is not optimal for the writer of the bond (minimizer of the expected reward) to withdraw the contract (redeem the bond) earlier than the holder may exercise it when $Y_t < rL$, for $t \ge 0$. These facts mean that the points $(y, q) \in (rK, rL) \times [0, 1]$, for which both inequalities $y < rL$ and $y > rK$ hold simultaneously, belong to the continuation region C_* in (2.4). On the other hand, the structure of the integrands and payoffs in (2.11) and (2.12) also implies that the holder of the bond should exercise the contract at some time when $Y_t \le rK$, while the writer of the bond should withdraw the contract at some time when $Y_t \ge rL$, for $t \ge 0$. These facts mean that the points $(y, q) \in ((0, rK] \cup [rL, \infty)) \times [0, 1]$ cover the stopping region D_* in (2.5).

(iii) Let us now fix some $(y, q) \in C_*$ such that either $y < rK$ or $y > rL$ holds, and consider the optimal stopping times $\tau^* = \tau^*(y, q)$ and $\zeta_* = \zeta_*(y, q)$ for the writer and the holder of the game-type contingent claim considered. Then, by means of the results of general optimal stopping theory for Markov processes (cf., for example, [71, Chapter I, Section 2.2]), we conclude from the structure of the continuation region C_* in (2.4) and the form of the stopping times in (2.3) as well as from the equalities in (2.11) and (2.12) that either

$$V^*(y,q) - K = \mathbb{E}\Big[\int_0^{\tau^*\wedge\zeta_*} e^{-rs}\, \big(Y_s^{(y,q)} - rK\big)\, ds + e^{-r\zeta_*}\, (L-K)\, I(\zeta_* \le \tau^*) \Big] > 0 \tag{2.13}$$

or

$$V_*(y,q) - L = \mathbb{E}\Big[\int_0^{\tau^*\wedge\zeta_*} e^{-rs}\, \big(Y_s^{(y,q)} - rL\big)\, ds - e^{-r\tau^*}\, (L-K)\, I(\tau^* < \zeta_*) \Big] < 0 \tag{2.14}$$

holds. Moreover, it follows from the comparison results for solutions of (one-dimensional time-homogeneous) stochastic differential equations of [32, Theorem 1] applied to the processes Y

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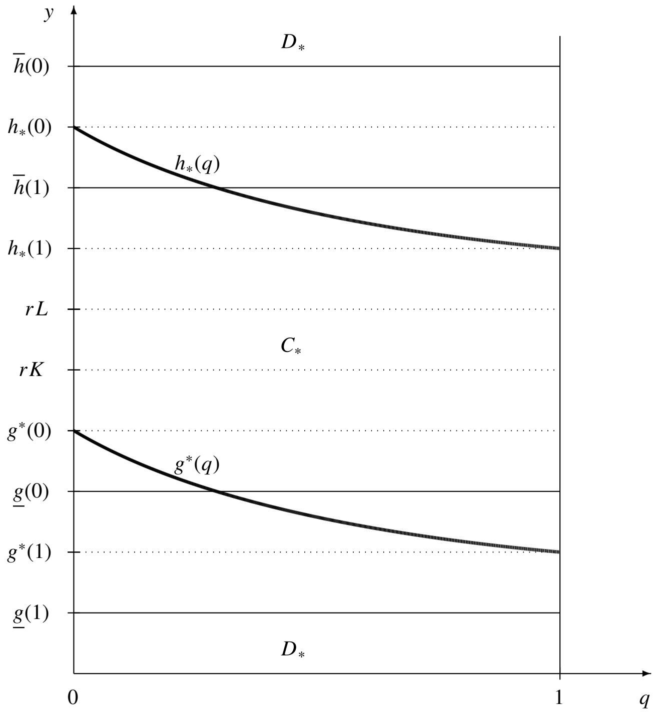

FIGURE 1. Computer plots of the optimal stopping boundaries $g^*(q)$ and $h_*(q)$.

and Q with the representations in (1.5) and (1.6) that the inequality $Y_t^{(y',q')} \le Y_t^{(y'',q'')}$ holds under $\eta_0 > \eta_1 > 0$, for each $0 < y' \le y''$ and $0 < q' \le q'' < 1$ as well as all $t \ge 0$. Then the inequality

$$\mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs} \, Y_s^{(y',q')} \, ds + e^{-r\tau} \, K \, I(\tau < \zeta) + e^{-r\zeta} \, L \, I(\zeta \le \tau) \bigg]$$
$$\le \mathbb{E}\bigg[\int_0^{\tau \wedge \zeta} e^{-rs} \, Y_s^{(y'',q'')} \, ds + e^{-r\tau} \, K \, I(\tau < \zeta) + e^{-r\zeta} \, L \, I(\zeta \le \tau) \bigg] \tag{2.15}$$

holds for any stopping times τ and ζ of the process (Y, Q). Hence, taking the supremum and infimum over all stopping times τ and ζ in either of the order in (2.15), we may conclude that the (coinciding) upper and lower functions $V^*(y, q)$ and $V_*(y, q)$ defined in (2.1) and (2.2) are increasing in the variables y and q on $\mathbb{R}_{++}$ and $[0, 1]$, respectively.

On the one hand, taking any y' such that either $y < y' \le rK$ and $q < q'$, or $rL \le y' < y$ and $q' < q$ holds, and using the property that the functions $V^*(y, q) = V_*(y, q)$ are increasing in y and q on $\mathbb{R}_{++}$ and $[0, 1]$, respectively, we obtain from and (2.14) that either the inequalities $V^*(y', q') - K \ge V^*(y, q) - K > 0$ or $V_*(y', q') - L \le V_*(y, q) - L < 0$ are satisfied, so that $(y', q') \in C_*$ too. On the other hand, if we assume that $(y, q) \in D_*$ such that either $y \le rK$ or $y \ge rL$ holds, then using arguments similar to those above, we obtain that either the inequalities $V^*(y'', q'') - K \le V^*(y, q) - K = 0$ hold, for all $y'' \le y < rK$ and $q'' \le q$, or the ones $V_*(y'', q'') - L \ge V_*(y, q) - L = 0$ hold, for all $rL < y \le y''$ and $q \le q''$, so that $(y'', q'') \in D_*$. Thus, we may conclude that the stopping times τ^* and ζ_* have the form of (2.16), where the left-and right-hand boundary functions $g^*(q)$ and $h_*(q)$ are decreasing on $[0, 1]$.

Observe that, if we suppose that $g^*(j) < \underline{g}(j)$ or $h_*(j) > \overline{h}(j)$ holds, for some $j = 0, 1$, then, for each $y \in (g^*(j), \underline{g}(j))$ or $y \in (\overline{h}(j), h_*(j))$ fixed, we would have $V^*(y, j) > K = \underline{W}(y, j)$ or $V_*(y, j) < L = \overline{W}(y, j)$, respectively, which contradicts the obvious facts mentioned above that the inequalities $V^*(y, j) \leq \underline{W}(y, j)$ and $V_*(y, j) \geq \overline{W}(y, j)$ should hold, for all $y \in \mathbb{R}_{++} \times \{0, 1\}$. Thus, we may conclude that the inequalities $\underline{g}(j) \leq g^*(j)$ and $h_*(j) \leq \overline{h}(j)$ should hold, for every $j = 0, 1$ (see Figure 1 for computer plots of the optimal stopping boundaries $g^*(q)$ and $h_*(q)$, for $q \in [0, 1]$, as well as $\underline{g}(j)$ and $\overline{h}(j)$, for $j = 0, 1$).

Summarizing the arguments shown above, let us formulate the following assertion.

Lemma 2.1. *Suppose that the processes Y and Q are defined by (1.1)–(1.2) and (1.5)–(1.6) with (1.7), for some $r > 0$, $\eta_0 > \eta_1 > 0$, $\sigma > 0$, and $\lambda_j \geq 0$ for $j = 0, 1$. Then the stopping times τ^* and ζ_* from (2.3) forming a Nash equilibrium in the optimal stopping zero-sum game of (2.1) and (2.2), for some $L > K > 0$ fixed, admit the representations*

$$\tau^* = \inf\left\{t \geq 0 \,\middle|\, Y_t \leq g^*(Q_t)\right\} \quad \text{and} \quad \zeta_* = \inf\left\{t \geq 0 \,\middle|\, Y_t \geq h_*(Q_t)\right\} \tag{2.16}$$

so that the continuation and stopping regions C_* and D_* in (2.4) and (2.5) take the form

$$C_* = \left\{(y, q) \in \mathbb{R}_{++} \times [0, 1] \,\middle|\, g^*(q) < y < h_*(q)\right\} \tag{2.17}$$

and

$$D_* = \left\{(y, q) \in \mathbb{R}_{++} \times [0, 1] \,\middle|\, \text{either } y \leq g^*(q) \text{ or } y \geq h_*(q)\right\}, \tag{2.18}$$

respectively. Here, $g^*(q)$ and $h_*(q)$ are functions satisfying the properties

$$g^*(q) : [0, 1] \to (0, rK] \text{ is decreasing and } \underline{g}(j) \leq g^*(j) \leq rK \text{ holds, for } j \in \{0, 1\}, \tag{2.19}$$

and

$$h_*(q) : [0, 1] \to [rL, \infty) \text{ is decreasing and } rL \leq h_*(j) \leq \overline{h}(j) \text{ holds, for } j \in \{0, 1\}, \tag{2.20}$$

where $\underline{g}(j)$ and $\overline{h}(j)$, for $j = 0, 1$, are determined from (5.49)–(5.50) and (5.54)–(5.55) with (5.15) as well as (5.68)–(5.69).

Remark 2.1. It is seen from the results of Lemma 2.1 above, which are further developed in Theorems 4.1 and 5.1 below, that the optimal withdrawal and exercise strategies of the writer and holder of the perpetual redeemable convertible bond change qualitatively when the economic state of the firm expressed by the continuous-time Markov chain $\mathbf{\Theta}$ becomes observable for the parties of the contract, which can then dispose both the processes Y and $\mathbf{\Theta}$ (full information) instead of the process Y and the filtering estimate Q only (partial information) in that case. More precisely, the bond should be withdrawn (redeemed) by the writer when the asset price hits the upper stochastic boundary $h_*(Q)$ under partial information (which is earlier than when the same contract should be withdrawn under full information), when the price rises to the level $\overline{h}(\mathbf{\Theta})$ in the case $\mathbf{\Theta} = 0$ and under the assumed inactivity of the holder. At the same time, the bond should be exercised (converted) by the holder when the asset price hits the lower stochastic boundary $g^*(Q)$ under partial information (which is earlier than when the same contract should be converted under full information), when the price falls to the level $\underline{g}(\mathbf{\Theta})$ in the case $\mathbf{\Theta} = 1$ and under the assumed inactivity of the writer. The same conclusions can be drawn for the lower and upper thresholds $\overline{g}(\mathbf{\Theta})$ and $\underline{h}(\mathbf{\Theta})$ which represent optimal stopping boundaries for the writer and holder of the bond in the model with an observable economic state of the firm expressed by the continuous-time Markov chain $\mathbf{\Theta}$ studied in Section 5 below.

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3. Preliminaries

In this section, we derive analytic properties for the value function of the optimal stopping zero-sum game, which are necessary for the proof of the main results of the paper stated below. For this purpose, we provide essentially non-trivial extensions of the appropriate arguments in relation to the previous literature within the framework of the optimal stopping zero-sum games for two-dimensional non-trivial degenerate diffusion processes with two-sided continuation regions, which has not bern done before, to the best of our knowledge. We start with the formulation of a parabolic-type free-boundary problem which is equivalent to the original optimal stopping zero-sum game.

3.1. The free-boundary problem

By means of standard arguments based on an application of Itô's formula (cf., for example, [61, Theorem 4.4], [52, Chapter V, Section 5.1], [72, Chapter IV, Theorem 3.3] and [64, Theorem 7.5.4]), it is shown that the infinitesimal operator $\mathbb{L}_{(Y,Q)}$ of the process (Y, Q) solving the stochastic differential equations (1.5) and (1.6) has the structure

$$\mathbb{L}_{(Y,Q)} = \big(r - \eta_0 - (\eta_1 - \eta_0)\, q\big)\, y\, \partial_y + \frac{\sigma^2 y^2}{2}\, \partial_{yy} - (\eta_1 - \eta_0)\, y\, q(1-q)\, \partial_{yq} + \big(\lambda_0\,(1-q) - \lambda_1\, q\big)\, \partial_q + \frac{1}{2}\left(\frac{\eta_0 - \eta_1}{\sigma}\right)^2 q^2(1-q)^2\, \partial_{qq} \tag{3.1}$$

for all $(y, q) \in \mathbb{R}_{++} \times (0, 1)$. In order to characterize the unknown value functions $V^*(y, q) = V_*(y, q)$ from (2.1) and (2.2) and the unknown boundaries $g^*(q)$ and $h_*(q)$ from (2.16), we may use the results of general theory of optimal stopping problems for continuous-time Markov processes (cf., for example, [71, Chapter IV, Section 8]) and formulate the associated free-boundary problem

$$(\mathbb{L}_{(Y,Q)}V - rV)(y, q) = -y \quad \text{for } g(q) < y < h(q), \tag{3.2}$$

$$V(y, q)\big|_{y=g(q)} = K \quad \text{and} \quad V(y, q)\big|_{y=h(q)} = L, \tag{3.3}$$

$$V_y(y, q)\big|_{y=g(q)} = 0 \quad \text{and} \quad V_y(y, q)\big|_{y=h(q)} = 0, \tag{3.4}$$

$$V_q(y, q)\big|_{y=g(q)} = 0 \quad \text{and} \quad V_q(y, q)\big|_{y=h(q)} = 0, \tag{3.5}$$

$$V(y, q) = K \quad \text{for } y < g(q) \quad \text{and} \quad V(y, q) = L \quad \text{for } y > h(q), \tag{3.6}$$

$$V(y, q) > K \quad \text{for } y > g(q) \quad \text{and} \quad V(y, q) < L \quad \text{for } y < h(q), \tag{3.7}$$

$$(\mathbb{L}_{(Y,Q)}V - rV)(y, q) < -y \quad \text{for } y < g(q), \tag{3.8}$$

$$(\mathbb{L}_{(Y,Q)}V - rV)(y, q) > -y \quad \text{for } y > h(q), \tag{3.9}$$

for $q \in (0, 1)$. Observe that the semiharmonic characterization of the value function proved in [68, Theorem 2.1] implies that $V^*(y, q) = V_*(y, q)$ forms the smallest function satisfying the left-hand part of the system in (3.2)–(3.3) and (3.6)–(3.7) with the boundary $g^*(q)$ and the largest function satisfying the right-hand part of the system in (3.2)–(3.3) and (3.6)–(3.7) with the boundary $h_*(q)$. Note that inequalities (3.8) and (3.9) follow directly from the assertion of Lemma 2.1 which was proved in parts (i)–(iii) of Section 2.3 above.

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3.2. Continuity of the value function

Let us now show that the lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) is continuous at any point $(y, q) \in \mathbb{R}_{++} \times [0, 1]$. In order to deduce this property, it is enough to prove the following assertion.

Lemma 3.1. *The lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) has the properties*

$$y \mapsto V_*(y, q) \text{ is continuous at } y' \text{ uniformly over } q \in [q' - \delta, q' + \delta], \tag{3.10}$$

$$q \mapsto V_*(y', q) \text{ is continuous at } q', \tag{3.11}$$

for every $(y', q') \in \mathbb{R}_{++} \times (0, 1)$ fixed, with some $\delta > 0$ small enough.

Proof. In order to deduce property (3.10), let us fix some $y_1 \le y_2$ in $[y' - \delta, y' + \delta]$ and $q \in [q' - \delta, q' + \delta]$ such that the associated square belongs to $\mathbb{R}_{++} \times [0, 1]$. We consider $\tau^* = \tau^*(y_2, q)$ and $\zeta_* = \zeta_*(y_1, q)$ the stopping times from the Nash equilibria in (2.1) and (2.2) associated with the starting points (y_2, q) and (y_1, q) of the process (Y, Q), respectively. Then, taking into account the structure of the integrand in (2.2) based on the explicit form of the process Y in (1.1) and (1.2) and the representations for the components of the process (Y, Q) in (1.5) and (1.6) as well as the fact that the left-hand boundary $g^*(q)$ and the right-hand boundary $h_*(q)$ from (2.16) are decreasing functions, under the assumption that $\eta_0 > \eta_1 > 0$, we get

$$\begin{aligned}
0 &\le V_*(y_2, q) - V_*(y_1, q) \\
&\le \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-rs}\, Y_s^{(y_2, q)}\, ds + e^{-r\tau^*} K\, I(\tau^* < \zeta_*) + e^{-r\zeta_*} L\, I(\zeta_* \le \tau^*) \bigg] \\
&\quad - \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-rs}\, Y_s^{(y_1, q)}\, ds + e^{-r\tau^*} K\, I(\tau^* < \zeta_*) + e^{-r\zeta_*} L\, I(\zeta_* \le \tau^*) \bigg] \\
&= \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-rs}\, \big(Y_s^{(y_2, q)} - Y_s^{(y_1, q)}\big)\, ds \bigg] = (y_2 - y_1)\, \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-rs}\, Y_s^{(1, q)}\, ds \bigg],
\end{aligned} \tag{3.12}$$

where the last expectation is finite, for all $y_1 \le y_2$ from $[y' - \delta, y' + \delta]$. Hence, we see that the right-hand side in (3.12) converges monotonically to zero as y_1 approaches y_2, independently of $q \in [q' - \delta, q' + \delta]$, for any $\delta > 0$ fixed, and thus property (3.10) holds.

In order to deduce property (3.11), we fix $q_1 \le q_2$ in $[q' - \delta, q' + \delta]$ such that the associated interval belongs to [0,1]. We now denote by $\tau^* = \tau^*(y', q_2)$ and $\zeta_* = \zeta_*(y', q_1)$ the stopping times from the Nash equilibria in (2.1) and (2.2) associated with the starting points (y', q_2) and (y', q_1) of the process (Y, Q), respectively. Then, taking into account the explicit form of the process Y in (1.1) and (1.2) and the representations for the components of the process (Y, Q)

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in (1.5) and (1.6), we get

$$\begin{aligned}
0 &\le V_*(y', q_2) - V_*(y', q_1) \\
&\le \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-r\,s}\, Y_s^{(y', q_2)}\, ds + e^{-r\tau^*}\, K\, I(\tau^* < \zeta_*) + e^{-r\zeta_*}\, L\, I(\zeta_* \le \tau^*) \bigg] \\
&\quad - \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-r\,s}\, Y_s^{(y', q_1)}\, ds + e^{-r\tau^*}\, K\, I(\tau^* < \zeta_*) + e^{-r\zeta_*}\, L\, I(\zeta_* \le \tau^*) \bigg] \\
&= \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-r\,s}\, \big(Y_s^{(y', q_2)} - Y_s^{(y', q_1)}\big)\, ds \bigg] = y'\, \mathbb{E}\bigg[\int_0^{\tau^* \wedge \zeta_*} e^{-r\,s}\, \big(Y_s^{(1, q_2)} - Y_s^{(1, q_1)}\big)\, ds \bigg],
\end{aligned} \tag{3.13}$$

where the last expectation is finite like tat in (3.12), for all $q_1 \le q_2$ from $[q' - \delta, q' + \delta]$, and, under the assumption that $\eta_0 > \eta_1 > 0$, we have

$$\begin{aligned}
Y_t^{(1, q_2)} - Y_t^{(1, q_1)} &= Y_t^{(1, q_1)} \left(\frac{Y_t^{(1, q_2)}}{Y_t^{(1, q_1)}} - 1 \right) \\
&= Y_t^{(1, q_1)} \left(\exp\left(\int_0^t (\eta_0 - \eta_1)\, \big(Q_s^{(q_2)} - Q_s^{(q_1)}\big)\, ds \right) - 1 \right)
\end{aligned} \tag{3.14}$$

for all $t \ge 0$. Hence, by using the comparison results for strong solutions of stochastic differential equations from [32, Theorem 1] and applying the Lebesgue dominated convergence theorem, we may conclude that the right-hand side in (3.13) converges to zero as q_1 approaches q_2, for any $\delta > 0$ fixed, and thus property (3.11) holds. This fact, together with property (3.10), means in particular that the instantaneous-stopping conditions of (3.3) above are satisfied. □

3.3. Regularity of the optimal stopping boundaries

Let us now prove the probabilistic regularity of the boundary ∂C_* of the continuation region in (2.17).

Lemma 3.2. *The boundary ∂C_* of the continuation region in (2.17) is probabilistically regular for the stopping region D_* in (2.18) relative to the process (Y, Q) defined in (1.1)–(1.2) and (1.5)–(1.6) with (1.7), under the assumption $\eta_0 > \eta_1 > 0$.*

Proof. By virtue of sample path and distributional properties of the two-dimensional diffusion process (Y, Q) from (1.5) and (1.6) starting at some point $(y', q') \in \mathbb{R}_{++} \times (0, 1)$ fixed, on an infinitesimally small deterministic time interval of length ε, we observe that the representations

$$Y_\varepsilon = y' + \big(r - \eta_0 - (\eta_1 - \eta_0)\, q'\big)\, y'\, \varepsilon + \sigma\, y'\, \overline{B}_\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0 \text{ ($\mathbb{P}$-almost surely (a.s.))} \tag{3.15}$$

and

$$Q_\varepsilon = q' + \big(\lambda_0\, (1 - q') - \lambda_1\, q'\big)\, \varepsilon + \frac{\eta_0 - \eta_1}{\sigma}\, q'(1 - q')\, \overline{B}_\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0 \text{ ($\mathbb{P}$-a.s.)} \tag{3.16}$$

hold, where $o(\varepsilon)$ denotes a random function satisfying $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \downarrow 0$ ($\mathbb{P}$-a.s.). Recall that $\overline{B}_\varepsilon \sim \sqrt{\varepsilon}\Psi$ as $\varepsilon \downarrow 0$ ($\mathbb{P}$-a.s.), where Ψ is a standard normal random variable. Then, starting

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from the point (y', q') and letting the process (Y, Q) evolve for an infinitesimal amount of time ε, we see that the process moves infinitesimally in the direction

$$\big(Y_\varepsilon - y', Q_\varepsilon - q'\big) \sim \left(\sqrt{\varepsilon}\,\sigma\, y'\,\Psi,\ \sqrt{\varepsilon}\,\frac{\eta_0-\eta_1}{\sigma}\,q'(1-q')\,\Psi\right) \quad \text{as } \varepsilon \downarrow 0 \ (\mathbb{P}\text{-a.s.}) \tag{3.17}$$

for any point $(y', q') \in \mathbb{R}_{++} \times (0, 1)$ fixed. In other words, the two-dimensional process (Y, Q) moves in either the south-west or the north-east direction in the plane, under the assumption that $\eta_0 > \eta_1 > 0$. Hence, combining this fact with the fact that the boundaries $g^*(q)$ and $h_*(q)$ are decreasing and taking into account the fact that the local drift of the process (Y, Q) has the order of ε, we may conclude that the first hitting time $\tau^*(y', q') \wedge \zeta_*(y', q')$ in the stopping set D_* converges to zero ($\mathbb{P}$-a.s.) as the point (y', q') approaches the point (y, q) such that $y = g^*(q)$ or $y = h_*(q)$. This fact means precisely that all the points of the boundary ∂C_* are probabilistically regular for the stopping region D_* relative to the process (Y, Q) (cf., for example, [64, Section 9.2] for an extensive discussion on this point and other references to the related literature). Note that we can also deduce the assertion stated above by means of arguments similar to those applied in the proofs in [49, Section 10] and [40, Theorem 12], with the difference with respect to the arguments used in the latter reference that we have the same standard Brownian motion driving the stochastic differential equations (1.5) and (1.6) for the processes Y and Q, and not the independent ones. More precisely, the desired assertion follows directly from the structure of the regions C_* and D_* in (2.17) and (2.18) having the left- and right-hand decreasing boundary functions $g^*(q)$ and $h_*(q)$ and an application of Blumenthal's zero–one law (cf., for example, [64, Lemma 9.2.6] and subsequent results and examples). □

3.4. Smooth-fit conditions for the value function

Let us now show that the lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) satisfies the smooth-fit conditions of (3.4) above.

Lemma 3.3. *The lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) satisfies the smooth-fit conditions of (3.4).*

Proof. Let us now consider a point $(y, q) \in [rL, \infty) \times (0, 1)$ at the boundary ∂C_*, so that $y = h_*(q)$ and $V_*(y, q) = L$ holds. In order to derive the right-hand property in (3.4), we first observe directly from the structure of the continuation region C_* in (2.4) and (2.17) that the inequality

$$\liminf_{\delta\downarrow 0} \frac{V_*(y-\delta, q) - V_*(y, q)}{-\delta} \ge 0 \tag{3.18}$$

is satisfied, due to the fact that the value function $V_*(y, q)$ from (2.2) is increasing in y on $\mathbb{R}_{++}$ (see part (i) of Section 2.3 above). Let us now denote by $\tau^1_\delta = \tau^*(y-\delta, q)$ and $\zeta^1_\delta = \zeta_*(y-\delta, q)$ the stopping times which form a Nash equilibrium in (2.1) and (2.2) associated with the starting point $(y-\delta, q)$ of the process (Y, Q), with some $\delta > 0$ small enough. Then, taking into account the structure of the integrand in (2.2) based on the explicit form of the process Y in (1.1) and the representations for the components of the process (Y, Q) in (1.5) and (1.6) as well as the fact that the left-hand boundary $g^*(q)$ and the right-hand boundary $h_*(q)$

from (2.16) are decreasing functions, we get

$$\begin{aligned} & V_*(y-\delta, q) - V_*(y, q) \\ & \geq \mathbb{E}\bigg[\int_0^{\tau_\delta^1 \wedge \zeta_\delta^1} e^{-r\,s}\, Y_s^{(y-\delta,q)}\, ds + e^{-r\tau_\delta^1}\, K\, I(\tau_\delta^1 < \zeta_\delta^1) + e^{-r\zeta_\delta^1}\, L\, I(\zeta_\delta^1 \leq \tau_\delta^1) \bigg] \\ & \quad - \mathbb{E}\bigg[\int_0^{\tau_\delta^1 \wedge \zeta_\delta^1} e^{-r\,s}\, Y_s^{(y,q)}\, ds + e^{-r\tau_\delta^1}\, K\, I(\tau_\delta^1 < \zeta_\delta^1) + e^{-r\zeta_\delta^1}\, L\, I(\zeta_\delta^1 \leq \tau_\delta^1) \bigg] \\ & = \mathbb{E}\bigg[\int_0^{\tau_\delta^1 \wedge \zeta_\delta^1} e^{-r\,s} \big(Y_s^{(y-\delta,q)} - Y_s^{(y,q)} \big)\, ds \bigg] = -\delta\, \mathbb{E}\bigg[\int_0^{\tau_\delta^1 \wedge \zeta_\delta^1} e^{-r\,s}\, Y_s^{(1,q)}\, ds \bigg], \end{aligned} \tag{3.19}$$

where the last expectation is positive and finite, for any $\delta > 0$ small enough. Hence, by using the fact that $\tau_\delta^1 \wedge \zeta_\delta^1 \to 0$ ($\mathbb{P}$-a.s.) as $\delta \downarrow 0$, due to the probabilistic regularity of the boundary ∂C_* for the region D_* relative to (Y, Q), and applying the Lebesgue dominated convergence theorem, we obtain that the inequality

$$\limsup_{\delta \downarrow 0} \frac{V_*(y-\delta, q) - V_*(y, q)}{-\delta} \leq 0 \tag{3.20}$$

holds. Thus, taking inequalities (3.18) and (3.20) together, we conclude that the smooth-fit condition in the right-hand part of (3.4) is satisfied, while the condition in the left-hand part is deduced in the same way. □

In order to derive the property in the right-hand part of (3.5), we first observe directly from the structure of the continuation region C_* in (2.4) and (2.17) that the inequality

$$\liminf_{\delta \downarrow 0} \frac{V_*(y, q-\delta) - V_*(y, q)}{-\delta} \geq 0 \tag{3.21}$$

is satisfied, due to the fact, proved in part (ii) of Section 2.3 above, that the value function $V_*(y, q)$ from (2.2) is increasing in q on $(0, 1)$. Let us finally denote by $\tau_\delta^2 = \tau^*(y, q-\delta)$ and $\zeta_\delta^2 = \zeta_*(y, q-\delta)$ the stopping times which form a Nash equilibrium in (2.1) and (2.2) associated with the starting point $(y, q-\delta)$ of the process (Y, Q), with some $\delta > 0$ small enough. Then, taking into account the explicit form of the process Y in (1.1) and the representations for the components of the process (Y, Q) in (1.5) and (1.6), by applying Itô's formula to the process $(Q_t/(1-Q_t))_{t \geq 0}$, we get

$$\begin{aligned} & V_*(y, q-\delta) - V_*(y, q) \\ & \geq \mathbb{E}\bigg[\int_0^{\tau_\delta^2 \wedge \zeta_\delta^2} e^{-r\,s}\, Y_s^{(y,q-\delta)}\, ds + e^{-r\tau_\delta^2}\, K\, I(\tau_\delta^2 < \zeta_\delta^2) + e^{-r\zeta_\delta^2}\, L\, I(\zeta_\delta^2 \leq \tau_\delta^2) \bigg] \\ & \quad - \mathbb{E}\bigg[\int_0^{\tau_\delta^2 \wedge \zeta_\delta^2} e^{-r\,s}\, Y_s^{(y,q)}\, ds + e^{-r\tau_\delta^2}\, K\, I(\tau_\delta^2 < \zeta_\delta^2) + e^{-r\zeta_\delta^2}\, L\, I(\zeta_\delta^2 \leq \tau_\delta^2) \bigg] \\ & = \mathbb{E}\bigg[\int_0^{\tau_\delta^2 \wedge \zeta_\delta^2} e^{-r\,s} \big(Y_s^{(y,q-\delta)} - Y_s^{(y,q)} \big)\, ds \bigg] = y\, \mathbb{E}\bigg[\int_0^{\tau_\delta^2 \wedge \zeta_\delta^2} e^{-r\,s} \big(Y_s^{(1,q-\delta)} - Y_s^{(1,q)} \big)\, ds \bigg], \end{aligned} \tag{3.22}$$

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where the last expectation is negative and finite, for any $q \in (0, 1)$ and $\delta > 0$ small enough, under the assumption that $\eta_0 > \eta_1 > 0$, and we have

$$Y_t^{(1,q-\delta)} - Y_t^{(1,q)} = Y_t^{(1,q)} \left(\exp\left(\int_0^t (\eta_0 - \eta_1) \left(Q_s^{(q-\delta)} - Q_s^{(q)}\right) ds \right) - 1 \right)$$
$$= Y_t^{(1,q)} \left(\exp\left(\int_0^t (\eta_0 - \eta_1)\, Q_s^{(q)} \left(1 - Q_s^{(q-\delta)}\right) \left(\frac{Q_s^{(q-\delta)}(1 - Q_s^{(q)})}{(1 - Q_s^{(q-\delta)})Q_s^{(q)}} - 1 \right) ds \right) - 1 \right) \quad (3.23)$$

and

$$\frac{Q_t^{(q-\delta)}(1 - Q_t^{(q)})}{(1 - Q_t^{(q-\delta)})Q_t^{(q)}} = \frac{(q-\delta)(1-q)}{(1-q+\delta)q} \exp\left(\int_0^t \left(\frac{\lambda_0(1 - Q_s^{(q-\delta)}) - \lambda_1 Q_s^{(q-\delta)}}{Q_s^{(q-\delta)}(1 - Q_s^{(q-\delta)})} - \frac{\lambda_0(1 - Q_s^{(q)}) - \lambda_1 Q_s^{(q)}}{Q_s^{(q)}(1 - Q_s^{(q)})} + \frac{(\eta_0 - \eta_1)^2}{\sigma^2} \left(Q_s^{(q-\delta)} - Q_s^{(q)}\right) \right) ds \right) \quad (3.24)$$

for all $t \geq 0$. Hence, we can take into account the dependence structure of the solution $Q^{(q)}$ of the stochastic differential equation (1.6) on its starting point $q \in (0, 1)$ and provide the appropriate Taylor's expansions for the exponential functions in (3.23) and (3.24) to get that the inequality

$$\int_0^{\tau_\delta^2 \wedge \zeta_\delta^2} e^{-rs} \left(Y_s^{(1,q-\delta)} - Y_s^{(1,q)}\right) ds \geq -\delta\, (\eta_0 - \eta_1)\, (\tau_\delta^2 \wedge \zeta_\delta^2) + o(\delta) \quad \text{as } \delta \downarrow 0 \ (\mathbb{P}\text{-a.s.}) \quad (3.25)$$

holds, where $o(\delta)$ denotes a random bounded function satisfying $o(\delta)/\delta \to 0$ as $\delta \downarrow 0$ ($\mathbb{P}$-a.s.). Thus, by using the fact that $\tau_\delta^2 \wedge \zeta_\delta^2 \to 0$ ($\mathbb{P}$-a.s.) as $\delta \downarrow 0$, due to the probabilistic regularity of the boundary ∂C_* for the region D_* relative to (Y, Q), and applying the Lebesgue dominated convergence theorem, we combine (3.22)–(3.25) to obtain that the inequality

$$\limsup_{\delta \downarrow 0} \frac{V_*(y, q-\delta) - V_*(y, q)}{-\delta} \leq 0 \quad (3.26)$$

is satisfied. Therefore, taking inequalities (3.21) and (3.26) together, we conclude that the smooth-fit condition in the right-hand part of (3.5) above is satisfied, while the condition in the left-hand part is deduced in the same way.

3.5. Continuous differentiability of the value function

Let us now show that the lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) is continuously differentiable on the whole state space $\mathbb{R}_{++} \times [0, 1]$ of the process (Y, Q).

Lemma 3.4. *The lower value function $V_*(y, q)$ (which coincides with the upper value function $V^*(y, q)$) of the optimal stopping zero-sum game in (2.1) and (2.2) belongs to the class $C^{1,1}$ (which actually coincides with the class C^1) on $\mathbb{R}_{++} \times (0, 1)$.*

Proof. We first recall that, by means of the arguments related to the proof of the strong Markov property of the process (Y, Q), which means the subsequent applications of Itô's formula and Doob's optional sampling theorem for the resulting local martingale taken at the

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time of the exit of the process (Y, Q) from a (bounded) open subset of the continuation region C_* from (2.17) followed by a passage to the limit in the resulting quotient as in [71, Chapter III, Sections 7.1–7.3], it is shown that the value function $V_*(y, q)$ solves the parabolic-type partial differential equation of (3.1) and (3.2) (which has degenerate coefficients), and thus the function $V_*(y, q)$ surely belongs at least to the class $C^{1,1}$ (which coincides with the class C^1) on the continuation region C_* in (2.17). In this respect, it remains for us to prove that the partial derivatives $(V_*)_y(y, q)$ and $(V_*)_q(y, q)$ are continuous functions at the boundary ∂C_*. For this purpose, we will show the existence of other directional derivatives of $V_*(y, q)$ along the boundary ∂C_* following the schema of arguments used in [49, Section 11] combined with those of Lemma 3.3 above. Note that the method applied below allow us to prove the assertion for any point (y, q) of the state space $\mathbb{R}_{++} \times (0, 1)$ of the process (Y, Q).

On the one hand, we need to show that the property

$$\lim_{n\to\infty} (V_*)_y(y_n, q_n) = 0 \tag{3.27}$$

holds, for any sequence $(y_n, q_n)_{n\in\mathbb{N}}$ tending to (y, q) as $n \to \infty$ such that $y = g^*(q)$ or $y = h_*(q)$. Since we have either $V_*(y_n, q_n) = K$ or $V_*(y_n, q_n) = L$, for $(y_n, q_n) \in D_*$, and the conditions of (3.4) and (3.5) hold at either $y = g^*(q)$ or $y = h_*(q)$, respectively, there is no restriction in fixing some (y, q) such that $y = h_*(q)$ and assuming that $(y_n, q_n) \in C_*$, for every $n \in \mathbb{N}$. Then, because of the fact, proved in part (i) of Section 2.3 above, that the function $V_*(y, q)$ from (2.2) is increasing in y on $\mathbb{R}_{++}$, we may conclude that the inequality

$$\liminf_{n\to\infty} (V_*)_y(y_n, q_n) = \liminf_{n\to\infty} \lim_{\delta\downarrow 0} \frac{V_*(y_n - \delta, q_n) - V_*(y_n, q_n)}{-\delta} \ge 0 \tag{3.28}$$

holds. Thus, it remains for us to show that the inequality

$$\limsup_{n\to\infty} (V_*)_y(y_n, q_n) = \limsup_{n\to\infty} \lim_{\delta\downarrow 0} \frac{V_*(y_n - \delta, q_n) - V_*(y_n, q_n)}{-\delta} \le 0 \tag{3.29}$$

holds too. For this purpose, we observe from the first identity in (3.29) that one can choose subsequences $(y_{n_k}, q_{n_k})_{k\in\mathbb{N}}$ and $(\delta_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_y(y_n, q_n) = \lim_{k\to\infty} \frac{V_*(y_{n_k} - \delta_k, q_{n_k}) - V_*(y_{n_k}, q_{n_k})}{-\delta_k} \tag{3.30}$$

with $(y_{n_k} - \delta_k, q_{n_k})_{k\in\mathbb{N}}$ tending to (y, q) as $k \to \infty$. Let us consider the stopping times $\tau_k^1 = \tau^*(y_{n_k} - \delta_k, q_{n_k})$ and $\zeta_k^1 = \zeta_*(y_{n_k} - \delta_k, q_{n_k})$ which form a Nash equilibrium for the starting points $(y_{n_k} - \delta_k, q_{n_k})$ of the process (Y, Q), for every $k \in \mathbb{N}$. Hence, taking into account the explicit form of the process Y in (1.1) and the representations for the components of the process (Y, Q) in (1.5) and (1.6), we find in the same way as in (3.19) above that

$$V_*(y_{n_k} - \delta_k, q_{n_k}) - V_*(y_{n_k}, q_{n_k}) \ge -\delta_k \, \mathbb{E}\bigg[\int_0^{\tau_k^1 \wedge \zeta_k^1} e^{-r\,s} \, Y_s^{(1, q_{n_k})} \, ds \bigg], \tag{3.31}$$

where the last expectation is positive and finite, for every $k \in \mathbb{N}$. Hence, letting $k \to \infty$ and recalling the fact that $\tau_k^1 \wedge \zeta_k^1 \to 0$ ($\mathbb{P}$-a.s.) as $k \to \infty$, due to the probabilistic regularity of the boundary ∂C_* for the region D_* relative to (Y, Q), we see by the Lebesgue dominated convergence theorem that (3.31), combined with (3.30), implies the desired (3.29). Therefore, taking inequalities (3.28) and (3.29) together, we obtain property (3.27) at $y = h_*(q)$, while the same property at $y = g^*(q)$ is deduced similarly.

On the other hand, we need to show that the property

$$\lim_{n\to\infty} (V_*)_q(y_n, q_n) = 0 \tag{3.32}$$

holds, for any sequence $(y_n, q_n)_{n\in\mathbb{N}}$ tending to (y, q) as $n\to\infty$. Since we have either $V_*(y_n, q_n) = K$ or $V_*(y_n, q_n) = L$, for $(y_n, q_n) \in D_*$, and conditions (3.4) hold at either $y = g^*(q)$ or $y = h_*(q)$, respectively, we may fix some (y, q) such that $y = h_*(q)$ and assume again that $(y_n, q_n) \in C_*$, for every $n \in \mathbb{N}$. Then, because of the fact, proved in part (ii) of Section 2.3 above, that the function $V_*(y, q)$ from (2.2) is decreasing in q on (0, 1), we may conclude that the inequality

$$\liminf_{n\to\infty} (V_*)_q(y_n, q_n) = \liminf_{n\to\infty} \lim_{\delta\downarrow 0} \frac{V_*(y_n, q_n - \delta) - V_*(y_n, q_n)}{-\delta} \geq 0 \tag{3.33}$$

holds. Thus, it remains for us to show that the inequality

$$\limsup_{n\to\infty} (V_*)_q(y_n, q_n) = \limsup_{n\to\infty} \lim_{\delta\downarrow 0} \frac{V_*(y_n, q_n - \delta) - V_*(y_n, q_n)}{-\delta} \leq 0 \tag{3.34}$$

holds too. In this case, we observe from the first identity in (3.34) that one can choose subsequences $(y_{n_l}, q_{n_l})_{l\in\mathbb{N}}$ and $(\delta_l)_{l\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_q(y_n, q_n) = \lim_{l\to\infty} \frac{V_*(y_{n_l}, q_{n_l} - \delta_l) - V_*(y_{n_l}, q_{n_l})}{-\delta_l} \tag{3.35}$$

with $(y_{n_l}, q_{n_l} - \delta_l)_{l\in\mathbb{N}}$ tending to (y, q) as $l\to\infty$. Let us consider the stopping times $\tau_l^2 = \tau^*(y_{n_l}, q_{n_l} - \delta_l)$ and $\zeta_l^2 = \zeta_*(y_{n_l}, q_{n_l} - \delta_l)$ which form a Nash equilibrium for the starting points $(y_{n_l}, q_{n_l} - \delta_l)$ of the process (Y, Q), for every $l \in \mathbb{N}$. Then, taking into account the explicit form of the process Y in (1.1) and the representations for the components of the process (Y, Q) in (1.5) and (1.6), by applying Itô's formula to the process $Q/(1-Q)$, we find in the same way as in (3.22) with (3.23) and (3.24) above that

$$\begin{aligned} V_*(y_{n_l}, q_{n_l} - \delta_l) - V_*(y_{n_l}, q_{n_l}) &\geq y_{n_l}\, \mathbb{E}\bigg[\int_0^{\tau_l^2\wedge\zeta_l^2} e^{-rs} \left(Y_s^{(1,q_{n_k}-\delta_k)} - Y_s^{(1,q_k)}\right) ds \bigg] \\ &\geq -\delta_l\, y_{n_l}\, (\eta_0 - \eta_1)\, \mathbb{E}\big[\tau_l^2 \wedge \zeta_l^2\big] + \mathbb{E}\big[o(\delta_l)\big] \end{aligned} \tag{3.36}$$

holds, where the first term on the last line is positive and finite, under $\eta_0 > \eta_1 > 0$, and $o(\delta)$ denotes a random bounded function $o(\delta)/\delta \to 0$ as $\delta \downarrow 0$ ($\mathbb{P}$-a.s.), for every $l \in \mathbb{N}$. Hence, letting $l\to\infty$ and recalling the fact that $\tau_l^2 \wedge \zeta_l^2 \to 0$ ($\mathbb{P}$-a.s.) as $l\to\infty$, due to the probabilistic regularity of the boundary ∂C_* for the region D_* relative to (Y, Q), by means of arguments similar to those used in the end of the previous subsection, we see by the Lebesgue dominated convergence theorem that (3.36), combined with (3.35), implies the desired (3.34). Therefore, taking inequalities (3.33) and (3.34) together, we obtain property (3.32) at $y = h_*(q)$, while the same property at $y = g^*(q)$ is deduced similarly. □

Remark 3.1. Alternatively, we may observe that (Y, Q) is also a strong Feller process, while it has been proved that each point of ∂C_* is probabilistically regular for the set D_*, since the

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coordinate processes Y and Q solve the stochastic differential equations (1.5) and (1.6) with Lipschitz continuous coefficients (cf., for example, [73, page 170]), while the boundaries $g^*(q)$ and $h_*(q)$ are decreasing. Finally, we see from arguments similar to those leading to the explicit expressions in (3.23) and (3.24) above that the process (Y, Q) can be realized as a continuously differentiable stochastic flow (with probability 1 under $\mathbb{P}$), so that the integrability conditions from [18, Theorem 8] are satisfied (actually, all four of them, because of the structure of the reward functionals in (2.1) and (2.2)), and thus the functions $V^*(y, q) = V_*(y, q)$ satisfying the conditions of (3.4) and (3.5) are continuously differentiable by the result of that assertion.

4. Main results and proofs

In this section, we make the appropriate change of variables and provide the appropriate verification assertion which constitute the proof of the main result of the paper stated below.

4.1. The change of variables

In order to provide the analysis of the free-boundary problem in (3.1) and (3.2)–(3.9) and be able to apply the change-of-variable formula from [67, Theorem 3.1] in the verification assertion below, we introduce an appropriate change of variables to reduce the infinitesimal operator $\mathbb{L}_{(Y,Q)}$ of the process (Y, Q) from (3.1) to the normal (or canonical) form and reformulate the initial optimal stopping problem within the new coordinates. For this purpose, let us define the process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t = \frac{Y_t^{-\rho} Q_t}{1 - Q_t} \quad \text{with} \quad \rho = \frac{\eta_0 - \eta_1}{\sigma^2} \quad \text{and thus} \quad Q_t = \frac{Y_t^{\rho} Z_t}{1 + Y_t^{\rho} Z_t}, \tag{4.1}$$

for all $t \geq 0$. Then, by applying Itô's formula to (4.1), we get from the representations in (1.5) and (1.6) that the process (Y, Z) solves the (two-dimensional) system of stochastic differential equations

$$dY_t = \left(r - \eta_0 - (\eta_1 - \eta_0) \, \frac{Y_t^{\rho} Z_t}{1 + Y_t^{\rho} Z_t} \right) Y_t \, dt + \sigma \, Y_t \, d\overline{B}_t \quad (Y_0 = y) \tag{4.2}$$

and

$$dZ_t = \left(\frac{(\lambda_0 - \lambda_1 Y_t^{\rho} Z_t)(1 + Y_t^{\rho} Z_t)}{Y_t^{\rho} Z_t} - \frac{\rho}{2} \left(2r - \eta_0 - \eta_1 - \sigma^2 \right) \right) Z_t \, dt \quad \left(Z_0 = z \equiv \frac{y^{-\rho} q}{1 - q} \right) \tag{4.3}$$

for any $(y, z) \in \mathbb{R}^2_{++}$ (cf., for example, [35, Section 3], [43, Section 3.5] and [49, Section 6] among others for similar transformations of variables). It is seen from the form of the stochastic differential equation (4.3) that the process Z starting at $z \in \mathbb{R}_{++}$ is of bounded variation. More precisely, if the inequality $Y_t^{\rho} Z_t < \nu$ holds with

$$\nu = \frac{1}{2\lambda_1} \left(\sqrt{\left(\lambda_1 - \lambda_0 + \frac{\rho}{2} \left(2r - \eta_0 - \eta_1 - \sigma^2 \right) \right)^2 + 4\lambda_0 \lambda_1} - \left(\lambda_1 - \lambda_0 + \frac{\rho}{2} \left(2r - \eta_0 - \eta_1 - \sigma^2 \right) \right) \right) > 0 \tag{4.4}$$

for $t \geq 0$, then the process Y is increasing, while if the inequality $Y_t^{\rho} Z_t > \nu$ holds, for $t \geq 0$, then the process Z is decreasing.

Observe that, for any $(y, q) \in \mathbb{R}_{++} \times (0, 1)$ fixed, the value function of the optimal stopping zero-sum game in (2.1) and (2.2) takes the form $V^*(y, q) = V_*(y, q) = U_*(y, y^{-\rho} q/(1-q))$ with

$$U_*(y, z) = \sup_\tau \inf_\zeta \mathbb{E}_{y,z}\bigg[\int_0^{\tau\wedge\zeta} e^{-rs}\, Y_s\, ds + e^{-r\tau}\, K\, I(\tau < \zeta) + e^{-r\zeta}\, L\, I(\zeta \le \tau)\bigg]$$
$$= \inf_\zeta \sup_\tau \mathbb{E}_{y,z}\bigg[\int_0^{\tau\wedge\zeta} e^{-rs}\, Y_s\, ds + e^{-r\tau}\, K\, I(\tau < \zeta) + e^{-r\zeta}\, L\, I(\zeta \le \tau)\bigg] \tag{4.5}$$

for some $\lambda > 0$ and $L > K > 0$ fixed, where $\mathbb{E}_{y,z}$ denotes the expectation taken with respect to the probability measure $\mathbb{P}$ under the assumption that the two-dimensional continuous (time-homogeneous) strong Markov process (Y, Z) satisfying the system of stochastic differential equations (4.2) and (4.3) starts at some $(y, z) \in \mathbb{R}^2_{++}$ (with probability one under $\mathbb{P}$). We note from the relations in (4.1) that there exists a one-to-one correspondence between the processes (Y, Q) and (Y, Z), and thus the suprema and infima in (4.5) are equivalently taken over all stopping times τ and ζ with respect to the natural filtration of (Y, Z) which coincides with $(\mathcal{G}_t)_{t\ge 0}$. Therefore, it follows from (2.16) that the stopping times τ^* and ζ_* in (4.5) can be expressed as

$$\tau^* = \inf\big\{t \ge 0 \,\big|\, Y_t \le a^*(Z_t)\big\} \quad \text{and} \quad \zeta_* = \inf\big\{t \ge 0 \,\big|\, Y_t \ge b_*(Z_t)\big\}, \tag{4.6}$$

and thus the continuation region C_* in (2.4) is associated with the set

$$F_* = \big\{(y, z) \in \mathbb{R}^2_{++} \,\big|\, a^*(z) < y < b_*(z)\big\} \tag{4.7}$$

with some functions $a^*(z)$ and $b_*(z)$ such that the inequalities $0 < a^*(z) \le rK < rL \le b_*(z)$ hold, for $z \in \mathbb{R}_{++}$. In order to provide relations between the functions $g^*(q)$, $h_*(q)$ and $a^*(z)$, $b_*(z)$, we follow the sequence of arguments around [49, formula (11.20)] to see that, for each $z \in \mathbb{R}_{++}$ fixed, there exists a unique $q \in (0, 1)$ such that the equalities

$$y = g^*(q) = \left(\frac{q}{z(1-q)}\right)^{1/\rho} = a^*(z) \quad \text{and} \quad y = h_*(q) = \left(\frac{q}{z(1-q)}\right)^{1/\rho} = b_*(z) \tag{4.8}$$

hold. In this case, we observe that the first equalities in (4.8) take the form $q = (g^*)^{-1}(y)$ and $q = (h_*)^{-1}(y)$, where $(g^*)^{-1}(y)$ and $(h_*)^{-1}(y)$ are understood as the generalized inverses of the functions $g^*(q)$ and $h_*(q)$. Then it follows from the change of variables introduced in (4.1) that the latter equalities are equivalent to

$$\frac{y^\rho z}{1 + y^\rho z} = (g^*)^{-1}(y), \quad \text{and thus} \quad z = \frac{(g^*)^{-1}(y)}{y^\rho(1 - (g^*)^{-1}(y))} \equiv \widetilde{g}(y), \tag{4.9}$$

and

$$\frac{y^\rho z}{1 + y^\rho z} = (h_*)^{-1}(y), \quad \text{and thus} \quad z = \frac{(h_*)^{-1}(y)}{y^\rho(1 - (h_*)^{-1}(y))} \equiv \widetilde{h}(y), \tag{4.10}$$

for each $z \in \mathbb{R}_{++}$ fixed. Hence, we may conclude that the functions $\widetilde{g}(y)$ and $\widetilde{h}(y)$ from (4.9) and (4.10) are positive and decreasing, and thus we can define $a^*(z) = \widetilde{g}^{-1}(z)$ and $b_*(z) = \widetilde{h}^{-1}(z)$, for each $z \in \mathbb{R}_{++}$ fixed, where $\widetilde{g}^{-1}(z)$ and $\widetilde{h}^{-1}(z)$ are the generalized inverses of the functions $\widetilde{g}(y)$ and $\widetilde{h}(y)$. In this view, by virtue of the fact, proved in Lemma 2.1, that the boundaries $g^*(q)$ and $h_*(q)$ in (2.16) and (2.17)–(2.18) are decreasing, we may conclude that the boundaries $a^*(z)$

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and $b_*(z)$ in (4.6) and (4.7) are of bounded variation. On the other hand, we obtain from the change of variables in (4.1) that the last equality in (4.8) takes the form

$$\frac{y^{-\rho} q}{1-q} = (a^*)^{-1}(y), \quad \text{and thus} \quad q = \frac{y^{\rho}(a^*)^{-1}(y)}{1 + y^{\rho}(a^*)^{-1}(y)} \equiv \widehat{g}(y), \tag{4.11}$$

and

$$\frac{y^{-\rho} q}{1-q} = (b_*)^{-1}(y), \quad \text{and thus} \quad q = \frac{y^{\rho}(b_*)^{-1}(y)}{1 + y^{\rho}(b_*)^{-1}(y)} \equiv \widehat{h}(y), \tag{4.12}$$

for each $q \in (0, 1)$ fixed. Therefore, we may conclude that the functions $\widehat{a}(y)$ and $\widehat{b}(y)$ from (4.11) and (4.12) are positive and decreasing, and thus we can define $g^*(q) = \widehat{g}^{-1}(q)$ and $h_*(q) = \widehat{h}^{-1}(q)$, for each $q \in (0, 1)$ fixed, where $\widehat{g}^{-1}(q)$ and $\widehat{h}^{-1}(q)$ are the generalized inverses of the functions $\widehat{g}(y)$ and $\widehat{h}(y)$, respectively.

By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(Y,Z)}$ of the process (Y, Z) solving the stochastic differential equations (4.2) and (4.3) has the structure

$$\begin{aligned} \mathbb{L}_{(Y,Z)} = &\left(r - \eta_0 - (\eta_1 - \eta_0) \frac{y^{\rho} z}{1 + y^{\rho} z} \right) y \, \partial_y + \frac{\sigma^2 y^2}{2} \, \partial_{yy} \\ &+ \left(\frac{(\lambda_0 - \lambda_1 y^{\rho} z)(1 + y^{\rho} z)}{y^{\rho} z} - \frac{\rho}{2} \left(2r - \eta_0 - \eta_1 - \sigma^2 \right) \right) z \, \partial_z \end{aligned} \tag{4.13}$$

for all $(y, z) \in \mathbb{R}^2_{++}$. In this case, according to system (3.2)–(3.8) above, the unknown value function $U_*(y, z)$ from (4.5) and the unknown boundaries $a^*(z)$ and $b_*(z)$ from (4.6) can be characterized by the associated free-boundary problem

$$(\mathbb{L}_{(Y,Z)} U - rU)(y, z) = -y \quad \text{for } a(z) < y < b(z), \tag{4.14}$$

$$U(y, z)\big|_{y=a(z)} = K \quad \text{and} \quad U(y, z)\big|_{y=b(z)} = L, \tag{4.15}$$

$$U_y(y, z)\big|_{y=a(z)} = 0 \quad \text{and} \quad U_y(y, z)\big|_{y=b(z)} = 0, \tag{4.16}$$

$$U_z(y, z)\big|_{y=a(z)} = 0 \quad \text{and} \quad U_z(y, z)\big|_{y=b(z)} = 0, \tag{4.17}$$

$$U(y, z) = K \quad \text{for } y < a(z) \quad \text{and} \quad U(y, z) = L \quad \text{for } y > b(z), \tag{4.18}$$

$$U(y, z) > K \quad \text{for } y > a(z) \quad \text{and} \quad U(y, z) < L \quad \text{for } y < b(z), \tag{4.19}$$

$$(\mathbb{L}_{(Y,Z)} U - rU)(y, z) < -y \quad \text{for } y < a(z), \tag{4.20}$$

$$(\mathbb{L}_{(Y,Z)} U - rU)(y, z) > -y \quad \text{for } y > b(z), \tag{4.21}$$

for $z \in \mathbb{R}_{++}$. We recall that the existence of the value of the optimal stopping zero-sum game in (4.5) follows from the results of [24, Theorem 2.1] and [68, Theorem 2.1] and the one-to-one

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correspondence between the processes (Y, Q) and (Y, Z). Then, by means of the arguments of the proof of the strong Markov property of the process (Y, Z), which means the subsequent standard applications of Itô's formula and Doob's optional sampling theorem for the resulting local martingale taken at the time of the exit of the process (Y, Z) from a (bounded) open subset of F_* from (4.7) followed by a passage to the limit in the resulting quotient as in [71, Chapter III, Sections 7.1–7.3], it is shown that the value function $U_*(y, z)$ from (4.5) solves the parabolic-type partial differential equation in (4.13) and (4.14). Hence, taking into account the probabilistic regularity of the points of the optimal exercise boundary ∂C_* for the stopping region D_* relative to the process (Y, Q), we may conclude from the results on parabolic-type partial differential equations from [60, Chapter V], combined with standard techniques related to the proof of the strong Markov property of the process (Y, Z) mentioned above, that the value function $U_*(y, z)$ belongs to the class $C^{2,1}$ in the regions $F_* \cap (\mathbb{R}^2_{++} \setminus E)$, which naturally coincides with the closure of F_*, because the set $E = \{(y, z) \in \mathbb{R}^2_{++} \mid y^\rho z = \nu\}$ is thin, where the region F_* has the form of (4.7) and ν is given by (4.4). Moreover, by virtue of the regularity of the value function proved in Lemmas 3.1–3.4 above and the bijective and smooth change of variables introduced in (4.1), it follows that the instantaneous-stopping and smooth-fit conditions of (4.15) and (4.16)–(4.17) hold for the value function $U_*(y, z)$ too.

4.2. The verification lemma

In order to formulate and prove the main results of the paper, taking into account the structure of the partial differential equation in (4.13) and (4.14) as well as the instantaneous-stopping and smooth-fit conditions in (4.15) and (4.16)–(4.17), we observe that the second-order derivative $(U_*)_{yy}(y, z)$ admits a continuous extension to the closure of the appropriate continuation region F_* from (4.7). This fact means that, by virtue of the probabilistic regularity of the boundary ∂C_* for D_* relative to (Y, Q) proved in Lemma 3.2 above, the function $U_*(y, z)$ admits a natural extension in the class $C^{2,1}$ in the closure of the region $F_* \cap (\mathbb{R}^2_{++} \setminus E)$, which naturally coincides with the closure of F_*, and in the class $C^{2,0}$ in the closure of the appropriate region $F_* \cap E$, where $E = \{(y, z) \in \mathbb{R}^2_{++} \mid y^\rho z = \nu\}$ and ν is given by (4.4). Moreover, by virtue of the results of [70, Theorem 3], it follows from the regularity of the value function $U_*(y, z)$ and the expressions (4.9) and (4.11) that the boundaries $a^*(z)$ and $b_*(z)$ in (4.7) are continuous functions (of bounded variation). Recall the property that the process Y is monotone outside the curve $E = \{(y, z) \in \mathbb{R}^2_{++} \mid y^\rho z = \nu\}$ with ν given by (4.4). In this case, it can be shown by means of arguments similar to those applied in part 1 of the proof of [49, Theorem 19] that there exists a sequence of piecewise-monotone processes $Z^n = (Z^n_t)_{t \geq 0}$, for $n \in \mathbb{N}$, which converges to Z in $\mathbb{P}$-probability on compact time intervals from $\mathbb{R}_+ \equiv [0, \infty)$, and the sequence of total variations of Z^n, for $n \in \mathbb{N}$, also converge in $\mathbb{P}$-probability to that of Z, on each interval $[0, T]$, for any $T > 0$ fixed, as $n \to \infty$. Note that, without loss of generality, the processes Z^n, for $n \in \mathbb{N}$, can be assumed to be continuous, by virtue of possible applications of standard straight-line approximations. Since each of the resulting continuous processes $a^*(Z^n)$ and $b_*(Z^n)$ are of bounded variation, so that they are continuous semimartingales, the change-of-variable formula from [67, Theorem 3.1] can be applied to the process $e^{-rt} U(Y, Z^n)$, for every $n \in \mathbb{N}$, and thus to the process $e^{-rt} U(Y, Z)$, by virtue of the appropriate convergence relations mentioned above and the assumed regularity of the candidate value function $U(y, z)$ which is defined by the right-hand side of (4.27) below (see part 1 of the proof of [49, Theorem 19] for further arguments).

In order to formulate the assertion below, let us denote by $p(t; y, z; y', z')$ the transition density of the continuous (time-homogeneous) strong Markov process (Y, Z) under $\mathbb{P}_{y,z}$ in the sense

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that

$$\mathbb{P}_{y,z}(Y_t \le y'', Z_t \le z'') = \int_0^{y''} \int_0^{z''} p(t;y, z;y', z')\, dy'\, dz' \tag{4.22}$$

for $t > 0$ with (y, z) and $(y'', z'') \in \mathbb{R}^2_{++}$. The function $p(t;y, z;y', z')$ is characterized as the unique non-negative solution to the Kolmogorov backward equation

$$p_t(t;y, z;y', z') = (\mathbb{L}_{(Y,Z)}p)(t;y, z;y', z') \tag{4.23}$$

with the initial condition

$$p(0+;y, z;y', z') = \delta_{(y,z)}(y', z') \quad \text{(weakly)} \tag{4.24}$$

satisfying

$$\int_0^{\infty} \int_0^{\infty} p(t;y, z;y', z')\, dy' dz' = 1 \tag{4.25}$$

for $t > 0$ and (y, z) as well as (y', z') in $\mathbb{R}^2_{++}$ (cf. [30]), where we recall that the infinitesimal operator $\mathbb{L}_{(Y,Z)}$ of the process (Y, Z) is given in (4.13) above and $\delta_{(y,z)}$ denotes the Dirac measure at the point $(y, z) \in \mathbb{R}^2_{++}$. The initial value problem (4.22)–(4.25) can be used to determine the transition density $p(t;y, z;y', z')$. Having found the function $p(t;y, z;y', z')$, we can evaluate the expression

$$\mathbb{E}_{y,z}\big[Y_t\, I(a(Z_t) < Y_t < b(Z_t)\big] = \int_0^{\infty} \int_{a(z')}^{b(z')} y'\, p(t;y, z;y', z')\, dy'\, dz', \tag{4.26}$$

for $t > 0$ and $(y, z) \in \mathbb{R}^2_{++}$, which appears in the formulation of the verification assertion below.

We now formulate the following verification assertion related to the free-boundary problem in (4.13) and (4.14)–(4.21). The proof is based on the essentially non-trivial development of the arguments from [49, Theorem 19] to the case of the considered optimal stopping zero-sum game of (4.5) for the two-dimensional diffusion process (4.2) and (4.3).

Lemma 4.1. *Suppose that the processes Y and Z solve the stochastic differential equations (4.2) and (4.3) with $r > 0$, $\eta_0 > \eta_1 > 0$, $\sigma > 0$, and $\lambda_j \ge 0$ for $j = 0, 1$. Then the value function $U_*(y, z)$ of the optimal stopping zero-sum game in (4.5), with some $L > K > 0$ fixed, admits the representation*

$$U_*(y, z) = \int_0^{\infty} \int_0^{\infty} \int_{a^*(z')}^{b_*(z')} e^{-rt}\, y'\, p(t;y, z;y', z')\, dy'\, dz'\, dt \tag{4.27}$$

for $(y, z) \in \mathbb{R}^2_{++}$, where the continuous boundaries of bounded variation $a^*(z)$ and $b_*(z)$ such that $0 < a^*(z) \le rK < rL \le b_*(z)$ holds, for $z \in \mathbb{R}_{++}$, are determined as a unique solution to the coupled system of nonlinear Fredholm integral equations

$$K = \int_0^{\infty} \int_0^{\infty} \int_{a(z')}^{b(z')} e^{-rt}\, y'\, p(t;a(z), z;y', z')\, dy'\, dz'\, dt \tag{4.28}$$

and

$$L = \int_0^{\infty} \int_0^{\infty} \int_{a(z')}^{b(z')} e^{-rt}\, y'\, p(t;b(z), z;y', z')\, dy'\, dz'\, dt \tag{4.29}$$

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for $z \in \mathbb{R}_{++}$, among the couples of continuous functions of bounded variation a(z) and b(z) satisfying the inequalities $0 < a(z) \le rK < rL \le b(z)$, while the stopping times τ^* and ζ_* from (4.6) form a Nash equilibrium.

Proof.

(i) Let us denote by $U(y, z)$ the right-hand side of (4.27) and continue the sequence of arguments started above. We observe that the function $U(y, z)$ belongs to the class $C^{1,1}$ at the appropriate boundaries $\partial F_* \setminus E$ and to $C^{1,0}$ in $\partial F_* \cap E$, and thus to the class $C^{2,1}$ in the closure of the appropriate region $F_* \cap (\mathbb{R}^2_{++} \setminus E)$ and to $C^{2,0}$ in the closure of $F_* \cap E$, where we have $E = \{(y, z) \in \mathbb{R}^2_{++} \mid y^\rho z = \nu\}$ and ν is given by (4.4), and the region F_* is given by (4.7). Hence, we can apply the change-of-variable formula with local time on surfaces from [67, Theorem 3.1] (see also [71, Chapter II, Section 3.5] for a summary of the related results and further references) to the process $(e^{-rt} U(Y_t, Z_t))_{t \ge 0}$ to obtain

$$\begin{aligned} e^{-rt}\, U(Y_t, Z_t) &= U(y, z) \\ &+ \int_0^t e^{-rs}\, (\mathbb{L}_{(Y,Z)} U - rU)(Y_s, Z_s)\, I\big(Y_s \ne a^*(Z_s),\, Y_s \ne b_*(Z_s)\big)\, ds \\ &+ \int_0^t e^{-rs}\, U_y(Y_s, Z_s)\, I\big(Y_s \ne a^*(Z_s),\, Y_s \ne b_*(Z_s)\big)\, \sigma\, Y_s\, d\overline{B}_s, \end{aligned} \tag{4.30}$$

where the stochastic integral process is a continuous local martingale with respect to the probability measure $\mathbb{P}_{y,z}$. Observe that, since the time spent by the process Y at the surfaces $a^*(Z)$ and $b_*(Z)$ of bounded variation is of Lebesgue measure zero (cf., for example, [11, Chapter II, Section 1]), the indicators which appear in the integrals in (4.30) can be set to 1. We also note that, since the candidate value function $U(y, z)$ defined in (4.27) is continuous and bounded, and the Lebesgue integral in (4.30) taken up to infinity has a finite expectation, we may conclude that the stochastic integral is a uniformly integrable martingale. Therefore, taking the expectation under $\mathbb{P}_{y,z}$ on the both sides of (4.30) and letting t go to infinity, by means of the Lebesgue dominated convergence theorem, after changing the order of expectation and integration with respect to the time variable, we obtain the expression

$$U(y, z) = \int_0^\infty e^{-rt}\, \mathbb{E}_{y,z}\big[Y_t\, I\big(a^*(Z_t) < Y_t < b_*(Z_t)\big)\big]\, dt \tag{4.31}$$

for all $(y, z) \in \mathbb{R}^2_{++}$.

(ii) It follows from the arguments related to the proof of the strong Markov property of the process (Y, Z) mentioned above (see [71, Chapter III, Sections 7.1–7.3]) that the function $U(y, z)$ defined by the right-hand side of (4.27) satisfies the parabolic-type partial differential equation in (4.13) and (4.14). This fact, together with conditions (4.15)–(4.17) and equalities (4.18) meaning directly that the inequalities in (4.20) and (4.21) hold, implies that the inequality $(\mathbb{L}_{(Y,Z)} U - rU)(y, z) \le -y$ is satisfied, for all $y < b_*(z)$ such that $y \ne a^*(z)$, and $(\mathbb{L}_{(Y,Z)} U - rU)(y, z) \ge -y$ is satisfied, for all $y > a^*(z)$ such that $y \ne b_*(z)$. Moreover, by virtue of inequalities (4.19), which follow from the semiharmonic characterization of the value function (cf., for example, [68, Theorem 2.1]), we see from equalities (4.18) and inequalities (4.19) that the inequalities $K \le U(y, z) \le L$

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hold, for all $(y, z) \in \mathbb{R}^2_{++}$, too. Then, inserting $\tau \wedge t$ or $\zeta \wedge t$ instead of t and taking expectations with respect to the probability measure $\mathbb{P}_{y,z}$ on both sides in (4.30), by means of Doob's optional sampling theorem (cf., for example, [61, Chapter III, Theorem 3.6], [52, Chapter I, Theorem 3.22] and [72, Chapter II, Theorem 3.2]), we get that the inequalities

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau\wedge\zeta_*\wedge t} e^{-rs}\,Y_s\,ds + e^{-r(\tau\wedge t)}\,K\,I(\tau\wedge t<\zeta_*) + e^{-r\zeta_*}\,L\,I(\zeta_*\le\tau\wedge t)\Big]$$
$$\le \mathbb{E}_{y,z}\Big[\int_0^{\tau\wedge\zeta_*\wedge t} e^{-rs}\,Y_s\,ds + e^{-r(\tau\wedge\zeta_*\wedge t)}\,U(Y_{\tau\wedge\zeta_*\wedge t}, Z_{\tau\wedge\zeta_*\wedge t})\Big] \le U(y,z) \qquad (4.32)$$

and

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau^*\wedge\zeta\wedge t} e^{-rs}\,Y_s\,ds + e^{-r\tau^*}\,K\,I(\tau^*<\zeta\wedge t) + e^{-r(\zeta\wedge t)}\,L\,I(\zeta\wedge t\le\tau^*)\Big]$$
$$\ge \mathbb{E}_{y,z}\Big[\int_0^{\tau^*\wedge\zeta\wedge t} e^{-rs}\,Y_s\,ds + e^{-r(\tau^*\wedge\zeta\wedge t)}\,U(Y_{\tau^*\wedge\zeta\wedge t}, Z_{\tau^*\wedge\zeta\wedge t})\Big] \ge U(y,z) \qquad (4.33)$$

hold, for any stopping times τ and ζ of the process (Y, Z). Observe that the expectations in (4.32) and (4.33) are bounded from below and above, because the process $(e^{-rt}Y_t)_{t\ge 0}$ is a strict supermartingale closed at zero, under the assumption $\eta_0 > \eta_1 > 0$. Hence, letting t go to infinity and using Fatou's lemma, we obtain that the inequalities

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau\wedge\zeta_*} e^{-rs}\,Y_s\,ds + e^{-r\tau}\,K\,I(\tau<\zeta_*) + e^{-r\zeta_*}\,L\,I(\zeta_*\le\tau)\Big] \le U(y,z) \qquad (4.34)$$

and

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau^*\wedge\zeta} e^{-rs}\,Y_s\,ds + e^{-r\tau^*}\,K\,I(\tau^*<\zeta) + e^{-r\zeta}\,L\,I(\zeta\le\tau^*)\Big] \ge U(y,z) \qquad (4.35)$$

are satisfied, for any stopping times τ and ζ and all $(y, z) \in \mathbb{R}^2_{++}$. Thus, taking into account the fact that the function $U(y, z)$ and the continuous boundaries $a^*(z)$ and $b_*(z)$ of bounded variation solve the parabolic-type partial differential equation in (4.13) and (4.14) and satisfy conditions (4.15), inserting τ^* in place of τ into (4.34) and ζ_* in place of ζ into (4.35), we obtain that the equality

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau^*\wedge\zeta_*} e^{-rs}\,Y_s\,ds + e^{-r\tau^*}\,K\,I(\tau^*<\zeta_*) + e^{-r\zeta_*}\,L\,I(\zeta_*\le\tau^*)\Big] = U(y,z) \qquad (4.36)$$

holds, for all $(y, z) \in \mathbb{R}^2_{++}$. Therefore, we may conclude from (4.36) that the candidate function $U(y, z)$ coincides with the value function $U_*(y, z)$ of the optimal stopping game in (4.5), so that the stopping times τ^* and ζ_* form a Nash equilibrium of the game.

(iii) In order to prove the uniqueness of the candidate value function $U(y, z)$ and the boundaries $a^*(z)$ and $b_*(z)$ as a solution to the free-boundary problem (4.14)–(4.21), let us assume that there exist another candidate value function

$$\widetilde{U}(y,z) = \int_0^\infty e^{-rt}\,\mathbb{E}_{y,z}\big[Y_t\,I\big(\widetilde{a}(Z_t) < Y_t < \widetilde{b}(Z_t)\big)\big]\,dt \qquad (4.37)$$

which is another solution to the system in (4.14)–(4.17) and (4.19) with the candidate boundaries $\widetilde{a}(z)$ and $\widetilde{b}(z)$ of bounded variation solving the system of integral equations in (4.28) and (4.29) such that $0 < \widetilde{a}(z) \le rK < rL \le \widetilde{b}(z)$ holds, so that the inequalities in (4.20) and (4.21) are satisfied, for all $z \in \mathbb{R}_{++}$. Note that, since the integral in (4.37) taken up to infinity is of finite expectation, the function $\widetilde{U}(y, z)$ is also continuous and bounded. We can also follow the arguments from part (i) of the proof and use the facts that the function $\widetilde{U}(y, z)$ belongs to the class $C^{2,1}$ in the closure of the region $\widetilde{F} \cap (\mathbb{R}^2_{++} \setminus E)$ with $\widetilde{F}$ defined by

$$\widetilde{F} = \big\{(y, z) \in \mathbb{R}^2_{++} \,\big|\, \widetilde{a}(z) < y < \widetilde{b}(z)\big\} \tag{4.38}$$

and to $C^{2,0}$ in the closure of $\widetilde{F} \cap E$, where we recall that $E = \{(y, z) \in \mathbb{R}^2_{++} \mid y^\rho z = \nu\}$ and ν is given by (4.4). Moreover, we observe that the function $\widetilde{U}(y, z)$ solves the partial differential equation in (4.13) and (4.14) as well as satisfies conditions (4.15)–(4.17) at $\widetilde{a}(z)$ and $\widetilde{b}(z)$ instead of $a^*(z)$ and $b_*(z)$, by construction. Hence, we can apply the result of [67, Theorem 3.1] to get

$$\begin{aligned} e^{-rt}\, \widetilde{U}(Y_t, Z_t) = \widetilde{U}(y, z) \\ + \int_0^t e^{-rs}\, (\mathbb{L}_{(Y,Z)}\widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds + \int_0^t e^{-rs}\, \widetilde{U}_y(Y_s, Z_s)\, \sigma\, Y_s\, d\overline{B}_s \end{aligned} \tag{4.39}$$

where the stochastic integral is a continuous uniformly integrable martingale with respect to the probability measure $\mathbb{P}_{y,z}$, by virtue of the arguments from part (i) of this proof. Let us now consider the stopping times

$$\widetilde{\tau} = \inf\big\{t \ge 0 \,\big|\, Y_t \le \widetilde{a}(Z_t)\big\} \quad \text{and} \quad \widetilde{\zeta} = \inf\big\{t \ge 0 \,\big|\, Y_t \ge \widetilde{b}(Z_t)\big\}. \tag{4.40}$$

Then, inserting $\widetilde{\tau} \wedge t$ and $\widetilde{\zeta} \wedge t$ instead of t into (4.39) and applying Doob's optional sampling theorem, letting t go to infinity and using Lebesgue's dominated convergence theorem, we obtain the equalities

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\Big[e^{-r\widetilde{\tau}}\, \widetilde{U}(Y_{\widetilde{\tau}}, Z_{\widetilde{\tau}}) - \int_0^{\widetilde{\tau}} e^{-rs}\, (\mathbb{L}_{(Y,Z)}\widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\Big] \tag{4.41}$$

and

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\Big[e^{-r\widetilde{\zeta}}\, \widetilde{U}(Y_{\widetilde{\zeta}}, Z_{\widetilde{\zeta}}) - \int_0^{\widetilde{\zeta}} e^{-rs}\, (\mathbb{L}_{(Y,Z)}\widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\Big] \tag{4.42}$$

for all $(y, z) \in \mathbb{R}^2_{++}$. Now, if we take a point (y, z) such that either $0 < y \le \widetilde{a}(z)$ or $y \ge \widetilde{b}(z)$ holds, for any $z \in \mathbb{R}_{++}$, then we see that, since the equalities $\widetilde{U}(\widetilde{a}(z), z) = K$ and $\widetilde{U}(\widetilde{b}(z), z) = L$ are satisfied by construction, we may conclude from (4.14) and (4.18) that the expectations on the right-hand sides of (4.41) and (4.42) are equal to K and L, respectively. Therefore, we may conclude that $\widetilde{U}(y, z) = K$ when $0 < y \le \widetilde{a}(z)$, and $\widetilde{U}(y, z) = L$ when $y \ge \widetilde{b}(z)$, for all $z \in \mathbb{R}_{++}$.

(iv) We need to prove the fact that the inequalities $a^*(z) \le \widetilde{a}(z)$ and $\widetilde{b}(z) \le b_*(z)$ hold, for all $z \in \mathbb{R}_{++}$. For this purpose, we recall the fact that τ^* and ζ_* are the stopping times forming a Nash equilibrium in (2.1), where the couples of boundaries $a^*(z)$ and $b_*(z)$ as

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well as $\widetilde{a}(z)$ and $\widetilde{b}(z)$ satisfy the system of integral equations (4.28) and (4.29). We first observe that the arguments of parts (i) and (ii) above yield that inequalities (4.34) and (4.35) are also satisfied with $\widetilde{U}(y, z)$ from (4.37) instead of $U(y, z)$ from (4.31), for all $0 < y \le a^*(z)$ and $y \ge b_*(z)$, for any $z \in \mathbb{R}_{++}$, respectively. This fact directly implies the property that the inequalities $\widetilde{U}(y, z) \le U_*(y, z) = K$ should hold, for all $0 < y \le a^*(z)$, and the inequalities $\widetilde{U}(y, z) \ge U_*(y, z) = L$ should hold, for all $y \ge b_*(z)$, for any $z \in \mathbb{R}_{++}$. We now assume that either there exists some $z \in \mathbb{R}_{++}$ such that $\widetilde{a}(z) < a^*(z)$ holds, or there exists some $z \in \mathbb{R}_{++}$ such that $\widetilde{b}(z) > b_*(z)$ holds, and consider the stopping times

$$\tau' = \inf\big\{t \ge 0 \,\big|\, Y_t \ge a^*(Z_t)\big\} \quad \text{and} \quad \zeta' = \inf\big\{t \ge 0 \,\big|\, Y_t \le b_*(Z_t)\big\}. \tag{4.43}$$

Then, inserting $\tau' \wedge t$ and $\zeta' \wedge t$ instead of t into (4.39) and applying Doob's optional sampling theorem, letting t go to infinity and using Lebesgue's dominated convergence theorem, we obtain the equalities

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\bigg[e^{-r\tau'}\, \widetilde{U}(Y_{\tau'}, Z_{\tau'}) - \int_0^{\tau'} e^{-rs}\, (\mathbb{L}_{(Y,Z)}\widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\bigg] \tag{4.44}$$

and

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\bigg[e^{-r\zeta'}\, \widetilde{U}(Y_{\zeta'}, Z_{\zeta'}) - \int_0^{\zeta'} e^{-rs}\, (\mathbb{L}_{(Y,Z)}\widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\bigg] \tag{4.45}$$

for all $(y, z) \in \mathbb{R}^2_{++}$. Hence, if we take a point (y, z) such that either $0 < y \le \widetilde{a}(z)$ or $y \ge \widetilde{b}(z)$ holds, for any $z \in \mathbb{R}_{++}$, then we see from (4.44) and (4.45) and the facts that $U_*(Y_{\tau'}, Z_{\tau'}) = L$ and $U_*(Y_{\zeta'}, Z_{\zeta'}) = K$ as well as $\widetilde{U}(Y_{\tau'}, Z_{\tau'}) \le K$ and $\widetilde{U}(Y_{\zeta'}, Z_{\zeta'}) \ge L$ ($\mathbb{P}_{y,z}$-a.s.) that the inequalities

$$\mathbb{E}_{y,z}\bigg[\int_0^{\tau'} e^{-rs}\, (Y_s - rK)\, I\big(Y_s > \widetilde{a}(Z_s)\big)\, ds\bigg] \ge 0 \tag{4.46}$$

and

$$\mathbb{E}_{y,z}\bigg[\int_0^{\zeta'} e^{-rs}\, (Y_s - rL)\, I\big(Y_s < \widetilde{b}(Z_s)\big)\, ds\bigg] \le 0 \tag{4.47}$$

should hold, for $0 < y \le \widetilde{a}(z)$ and $y \ge \widetilde{b}(z)$, for any $z \in \mathbb{R}_{++}$, respectively. However, the assumptions that $\widetilde{a}(z) < a^*(z)$ and $\widetilde{b}(z) > b_*(z)$, together with the continuity of the boundaries $\widetilde{a}(z)$ and $\widetilde{b}(z)$, imply that the process (Y, Z) starting either below $\widetilde{a}(z)$ or above $\widetilde{b}(z)$ spends a strictly positive amount of time (with respect to Lebesgue measure) before hitting the appropriate boundaries $a^*(z)$ and $b_*(z)$ with a strictly positive probability under $\mathbb{P}_{y,z}$. Thus, taking into account the directly implied fact that the expectations in (4.46) and (4.47) are actually strictly negative and positive, for either $y < \widetilde{a}(z) \le rK$ or $y > \widetilde{b}(z) \ge rL$, for all $z \in \mathbb{R}_{++}$, respectively, we may conclude that the assumption of this part fails to hold, so that the inequalities $a^*(z) \le \widetilde{a}(z)$ and $\widetilde{b}(z) \le b_*(z)$ should be satisfied, for all $z \in \mathbb{R}_{++}$.

(v) Finally, we show that the boundaries $\widetilde{a}(z)$ and $\widetilde{b}(z)$ should coincide with $a^*(z)$ and $b_*(z)$. For this purpose, we take a point (y, z) such that either $y \in (a^*(z), \widetilde{a}(z))$ or $y \in (\widetilde{b}(z), b_*(z))$ holds, for some $z \in \mathbb{R}_{++}$, and set

$$\tau'' = \inf\big\{t \ge 0 \,\big|\, U_*(Y_t, Z_t) < \widetilde{U}(Y_t, Z_t)\big\} \tag{4.48}$$

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and

$$\zeta'' = \inf\big\{t \geq 0 \,\big|\, U_*(Y_t, Z_t) > \widetilde{U}(Y_t, Z_t)\big\}, \tag{4.49}$$

respectively. Then, inserting $\tau'' \wedge t$ and $\zeta'' \wedge t$ instead of t into (4.39) and applying Doob's optional sampling theorem, letting t go to infinity and using Lebesgue's dominated convergence theorem, we obtain the equalities

$$U_*(y, z) = \mathbb{E}_{y,z}\Big[e^{-r(\tau^* \wedge \tau'')}\, U_*(Y_{\tau^*\wedge\tau''}, Z_{\tau^*\wedge\tau''}) - \int_0^{\tau^*\wedge\tau''} e^{-rs}\, (\mathbb{L}_{(Y,Z)} U_* - rU_*)(Y_s, Z_s)\, ds\Big] \tag{4.50}$$

and

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\Big[e^{-r(\tau^* \wedge \tau'')}\, \widetilde{U}(Y_{\tau^*\wedge\tau''}, Z_{\tau^*\wedge\tau''}) - \int_0^{\tau^*\wedge\tau''} e^{-rs}\, (\mathbb{L}_{(Y,Z)} \widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\Big] \tag{4.51}$$

as well as

$$U_*(y, z) = \mathbb{E}_{y,z}\Big[e^{-r(\zeta_* \wedge \zeta'')}\, U_*(Y_{\zeta_*\wedge\zeta''}, Z_{\zeta_*\wedge\zeta''}) - \int_0^{\zeta_*\wedge\zeta''} e^{-rs}\, (\mathbb{L}_{(Y,Z)} U_* - rU_*)(Y_s, Z_s)\, ds\Big] \tag{4.52}$$

and

$$\widetilde{U}(y, z) = \mathbb{E}_{y,z}\Big[e^{-r(\zeta_* \wedge \zeta'')}\, \widetilde{U}(Y_{\zeta_*\wedge\zeta''}, Z_{\zeta_*\wedge\zeta''}) - \int_0^{\zeta_*\wedge\zeta''} e^{-rs}\, (\mathbb{L}_{(Y,Z)} \widetilde{U} - r\widetilde{U})(Y_s, Z_s)\, ds\Big] \tag{4.53}$$

for all $(y, z) \in \mathbb{R}^2_{++}$. Recall that the functions $U_*(y, z)$ and $\widetilde{U}(y, z)$ satisfy conditions (4.18) and (4.20)–(4.21) with the boundaries $a^*(z)$ and $b_*(z)$ as well as $\widetilde{a}(z)$ and $\widetilde{b}(z)$, respectively. Hence, if we take a point (y, z) such that either $a^*(z) < y < \widetilde{a}(z)$ or $\widetilde{b}(z) < y < b_*(z)$ holds, for any $z \in \mathbb{R}_{++}$, then we see from (4.44) and (4.45) and the facts that $\widetilde{U}(y, z) = K$ for all $0 < y < \widetilde{a}(z)$, and $\widetilde{U}(y, z) = L$ for all $y > \widetilde{b}(z)$, that $U_*(Y_{\tau^*\wedge\tau''}, Z_{\tau^*\wedge\tau''}) = \widetilde{U}(Y_{\tau^*\wedge\tau''}, Z_{\tau^*\wedge\tau''})$ and $U_*(Y_{\zeta_*\wedge\zeta''}, Z_{\zeta_*\wedge\zeta''}) = \widetilde{U}(Y_{\zeta_*\wedge\zeta''}, Z_{\zeta_*\wedge\zeta''})$ ($\mathbb{P}_{y,z}$-a.s.) holds, and thus, because of the facts that $U_*(y, z) > K$ for all $y > a^*(z)$, and $U_*(y, z) < L$ for all $0 < y < b_*(z)$, the inequalities

$$\mathbb{E}_{y,z}\Big[\int_0^{\tau^*\wedge\tau''} e^{-rs}\, (Y_s - rK)\, I\big(Y_s \leq \widetilde{a}(Z_s)\big)\, ds\Big] > 0 \tag{4.54}$$

and

$$\mathbb{E}_{y,z}\Big[\int_0^{\zeta_*\wedge\zeta''} e^{-rs}\, (Y_s - rL)\, I\big(Y_s \geq \widetilde{b}(Z_s)\big)\, ds\Big] < 0 \tag{4.55}$$

should be satisfied. However, using arguments similar to those used in part (iv) of the proof, we see that the strict inequalities (4.54) and (4.55) cannot be satisfied due to the continuity of the boundaries $a^*(z) \leq \widetilde{a}(z)$ and $\widetilde{b}(z) \leq b_*(z)$ as well as the inequalities $\widetilde{a}(z) \leq rK$ and $\widetilde{b}(z) \geq rL$, which leads to a contradiction with the assumption of this part. We may therefore conclude that the equalities $a^*(z) = \widetilde{a}(z)$ and $b_*(z) = \widetilde{b}(z)$ hold, so that $\widetilde{U}(y, z)$ coincides with $U_*(y, z)$, for all $(y, z) \in \mathbb{R}^2_{++}$. □

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4.3. The result

Taking the assertions of Lemmas 2.1, 3.1–3.4, and 4.1 together, we conclude this paper by formulating its main result concerning the optimal stopping problem related to the pricing of perpetual redeemable convertible bonds in the hidden Markov model considered.

Theorem 4.1. *Let the processes Y and Q be defined by (1.1)–(1.2) and (1.5)–(1.6) with (1.7), for $r > 0$, $\eta_0 > \eta_1 > 0$, $\sigma > 0$, and $\lambda_j \geq 0$ for $j = 0, 1$. Then the value functions of the optimal stopping game in (2.1) and (2.2), with some $L > K > 0$ fixed, take the form $V^*(y, q) = V_*(y, q) = U_*(y, y^{-\rho} q/(1-q))$,and the stopping times τ^* and ζ_* from (2.16) form a Nash equilibrium with the decreasing boundaries $g^*(q) = \widehat{g}^{-1}(q)$ and $h_*(q) = \widehat{h}^{-1}(q)$, for $(y, q) \in \mathbb{R}_{++} \times [0, 1]$, where $\widehat{g}^{-1}(q)$ and $\widehat{h}^{-1}(q)$ are generalized inverses of the functions $\widehat{g}(y)$ and $\widehat{h}(y)$ from (4.11) and (4.12). Here, the function $U_*(y, z)$ admits the representation in (4.27), where the continuous boundaries of bounded variation $a^*(z)$ and $b_*(z)$ provide a unique solution to the coupled system of nonlinear Fredholm integral equations (4.28) and (4.29) among the pairs of continuous functions of bounded variation a(z) and b(z) satisfying the inequalities $0 < a(z) \leq rK < rL \leq b(z)$, for $z \in \mathbb{R}_{++}$.*

4.4. The solution in a particular case

In order to underline the complexity in the structure of solutions to the optimal stopping zero-sum game of (2.1) and (2.2) in the two-dimensional diffusion model defined by (1.1)–(1.2) and (1.5)–(1.6) with (1.7), we present a specific choice of parameters under which the original problem admits an analytic solution. More precisely, let us assume for the rest of this section that $\lambda_0 = \lambda_1 = 0$ and $\eta_0 + \eta_1 = 2r - \sigma^2$ hold in (4.3). The first equality means that $\mathbf{\Theta}_t = \mathbf{\Theta}_0$, for all $t \geq 0$, where $\mathbb{P}(\mathbf{\Theta}_0 = 1) = q$ and $\mathbb{P}(\mathbf{\Theta}_0 = 0) = 1 - q$, for $q \in [0, 1]$ (cf., for example, [71, Chapter VI, Section 21] and [35, Section 4]). Such a situation occurs when the unknown economic state of the firm is not changed during the whole infinite time interval (see also [19, 23, 25, 35] for solutions of other optimal stopping problems in the models with random drift rates). In this case, the parabolic-type partial differential equation in (4.13) and (4.14) is degenerated into an ordinary one, and the general solution to that equation which is finite at the infinity takes the form

$$U(y, z) = G_1(z) \, H_1(y, z) + G_2(z) \, H_2(y, z) + H_0(y, z) \tag{4.56}$$

where $G_k(z)$, for $k = 1, 2$, are arbitrary continuous functions. Here, we assume that the functions $H_k(y, z)$, for $k = 1, 2$, represent fundamental solutions to the homogeneous second-order ordinary differential equation related to that of (4.13) and (4.14) under $\lambda_0 = \lambda_1 = 0$ and $\eta_0 + \eta_1 = 2r - \sigma^2$ given by

$$H_k(y, z) = y^{\chi_{0,k}} \, F(\psi_{k,1}, \psi_{k,2}; \varphi_k; -y^{\rho} z) \tag{4.57}$$

for every $k = 1, 2$ and each $z \in \mathbb{R}_{++}$ fixed. We denote by $F(\alpha, \beta; \gamma; x)$ the Gauss hypergeometric function which is defined by means of the expansion

$$F(\alpha, \beta; \gamma; x) = 1 + \sum_{m=1}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \, \frac{x^m}{m!} \tag{4.58}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma \neq 0, -1, -2, \ldots$, where $(\alpha)_m = \alpha(\alpha+1) \cdots (\alpha+m-1)$, for $m \in \mathbb{N}$, and $(\alpha)_0 = 1$ (cf., for example, [1, Chapter XV] and [2, Chapter II]), and additionally set

$$\psi_{k,l} = \frac{\chi_{0,k} - \chi_{1,l}}{\rho} \quad \text{and} \quad \varphi_k = 1 + \frac{2}{\rho} \left(\chi_{0,k} - \frac{1}{2} + \frac{r - \eta_0}{\sigma^2} \right) \tag{4.59}$$

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as well as

$$\chi_{j,i} = \frac{1}{2} - \frac{r-\eta_j}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\eta_j}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{4.60}$$

so that $\chi_{j,2} < 0 < 1 < \chi_{j,1}$ holds, for every k, l, $i = 1, 2$ and $j = 0, 1$. Thus, $H_k(y, z)$, for $k = 1, 2$, in (4.57) are (strictly) increasing and decreasing (convex) functions satisfying the properties $H_1(0+, z) = +0$, $H_1(\infty, z) = \infty$ and $H_2(0+, z) = \infty$, $H_2(\infty, z) = +0$, for each $z \in \mathbb{R}_{++}$ fixed (cf., for example, [73, Chapter V, Section 50] for further details). Moreover, the function $H_0(y, z)$ in (4.56) represents a particular solution to the equation in (4.13) and (4.14) given by

$$H_0(y, z) = H_1(y, z) \int^y \frac{2H_2(x, z)}{\sigma^2 x G_0(x, z)}\, dx + H_2(y, z) \int_y \frac{2H_1(x, z)}{\sigma^2 x G_0(x, z)}\, dx \tag{4.61}$$

where $G_0(y, z)$ is the appropriate Wronskian determinant which admits the representation

$$G_0(y, z) = H_1(y, z)\, \partial_y H_2(y, z) - \partial_y H_1(y, z)\, H_2(y, z) \tag{4.62}$$

for each $z \in \mathbb{R}_{++}$ fixed.

Finally, by applying conditions (4.15) and (4.16) to the function in (4.56), we obtain that the equalities

$$G_1(z)\, H_1(a(z), z) + G_2(z)\, H_2(a(z), z) + H_0(a(z), z) = K, \tag{4.63}$$

$$G_1(z)\, \partial_y H_1(a(z), z) + G_2(z)\, \partial_y H_2(a(z), z) + \partial_y H_0(a(z), z) = 0, \tag{4.64}$$

$$G_1(z)\, H_1(b(z), z) + G_2(z)\, H_2(b(z), z) + H_0(b(z), z) = L, \tag{4.65}$$

$$G_1(z)\, \partial_y H_1(b(z), z) + G_2(z)\, \partial_y H_2(b(z), z) + \partial_y H_0(b(z), z) = 0 \tag{4.66}$$

should hold, for each $z \in \mathbb{R}_{++}$ fixed. Then, solving the system of equations in (4.63) and (4.65), we obtain that the function

$$U(y, z; a(z), b(z)) = G_1(z; a(z), b(z))\, H_1(y, z) + G_2(z; a(z), b(z))\, H_2(y, z) + H_0(y, z) \tag{4.67}$$

for $a(z) < y < b(z)$, satisfies the system in (4.13) and (4.14)–(4.15), when we set

$$G_i(z; a(z), b(z)) = \frac{(K - H_0(a(z), z)) H_{3-i}(b(z), z) - (L - H_0(b(z), z)) H_{3-i}(a(z), z)}{H_i(a(z), z) H_{3-i}(b(z), z) - H_i(b(z), z) H_{3-i}(a(z), z)} \tag{4.68}$$

for each $z \in \mathbb{R}_{++}$ fixed, and every $i = 1, 2$. Hence, putting the expressions from (4.68) into the system of equations in (4.64) and (4.66), we obtain that the function in (4.67) satisfies conditions (4.16) when the equalities

$$\frac{(K - H_0(a(z), z)) \partial_y H_i(a(z), z) + \partial_y H_0(a(z), z) H_i(a(z), z)}{(L - H_0(b(z), z)) \partial_y H_i(b(z), z) + \partial_y H_0(h(z), z) H_i(b(z), z)} = \frac{G_0(a(z), z)}{G_0(b(z), z)} \tag{4.69}$$

hold, for each $z \in \mathbb{R}_{++}$ fixed, and every $i = 1, 2$, where the function $G_0(y, z)$ is given by (4.62). Observe that it can be shown by means of straightforward computations based on applications of the implicit function theorem that the candidate value function in (4.67) with (4.68) and

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such that the equalities in (4.69) hold inherently satisfies the conditions of (4.17) (see also [35, Remark 4.1] for similar arguments applied for another optimal stopping and free-boundary problem in this model). Note that the uniqueness of the pair of solutions $a^*(z)$ and $b_*(z)$ to the system of higher transcendental equations of (4.69) follows from the uniqueness of the solution to the system in (4.13) and (4.14)–(4.17) and (4.19)–(4.21) which is proved in Lemma 4.1 above. Recall that the existence of the solution to the system in (4.13) and (4.14)–(4.17) and (4.19)–(4.21) follows from the results of [15, Theorem 4.1].

Let us now summarize the above results in the following assertion.

Corollary 4.1. *Suppose that the assumptions of Lemma 4.1 are satisfied with $\lambda_0 = \lambda_1 = 0$ and $\eta_0 + \eta_1 = 2r - \sigma^2$. Then the value function $U_*(y, z)$ of the optimal stopping game in (4.5), with some $r > 0$ and $L > K > 0$ fixed, admits the representation of (4.27), where the function $U(y, z; a^*(z), b_*(z))$ is given by (4.67) with (4.68), and the system of equations in (4.69) admits a unique pair of solutions $a^*(z)$ and $b_*(z)$ such that $0 < a^*(z) \le rK < rL \le b_*(z)$ holds, for each $z \in \mathbb{R}_{++}$ fixed.*

Remark 4.1 It is seen from the results of Corollary 4.1 above as well as of Corollary 5.3 below that the optimal exercise strategies of both the writer and holder of the perpetual redeemable convertible bond are still different in the models in which the degenerate continuous-time Markov chain $\boldsymbol{\Theta}$ with intensities $\lambda_0 = \lambda_1 = 0$ is either unobservable or observable for the both parties of the contract. However, in that case, the equivalent optimal stopping boundaries $a^*(Z)$ and $b_*(Z)$ are constant over the infinite time horizon, under the additional condition $\eta_0 + \eta_1 = 2r - \sigma^2$, like the thresholds $\overline{g}(\boldsymbol{\Theta})$ and $\underline{h}(\boldsymbol{\Theta})$, for the optimal stopping zero-sum games in the model with unobservable and observable Markov chain $\boldsymbol{\Theta}$, respectively. This property can be explained by the fact that $\boldsymbol{\Theta}_t = \boldsymbol{\Theta}_0$ holds, for all $t \ge 0$, under the assumption $\lambda_0 = \lambda_1 = 0$, and thus $\boldsymbol{\Theta}_0$ represents a Bernoulli random variable which can take the values 0 and 1 in that case.

5. Auxiliary results

In this section, we derive a closed-form solution of an auxiliary optimal stopping zero-sum game which is related to the problem of pricing perpetual redeemable convertible bonds in the underlying diffusion-type model with random dividends and an *observable* economic state of the firm described by a continuous-time Markov chain with two states.

5.1. The optimal stopping game and free-boundary problem

Let us now consider an optimal stopping zero-sum game with the upper and value functions $W^*(y, j) \ge W_*(y, j)$ given by

$$W^*(y, j) = \inf_{\zeta'} \sup_{\tau'} \mathbb{E}_{y,j}\bigg[\int_0^{\tau' \wedge \zeta'} e^{-rs}\, Y_s\, ds + e^{-r\tau'}\, K\, I(\tau' < \zeta') + e^{-r\zeta'}\, L\, I(\zeta' \le \tau') \bigg] \quad (5.1)$$

and

$$W_*(y, j) = \sup_{\tau'} \inf_{\zeta'} \mathbb{E}_{y,j}\bigg[\int_0^{\tau' \wedge \zeta'} e^{-rs}\, Y_s\, ds + e^{-r\tau'}\, K\, I(\tau' < \zeta') + e^{-r\zeta'}\, L\, I(\zeta' \le \tau') \bigg] \quad (5.2)$$

for some $r > 0$ and $L > K > 0$ fixed, where the suprema and infima are taken over all stopping times τ' and ζ' with respect to the natural filtration $(\mathcal{H}_t)_{t \ge 0}$ of the process $(Y, \boldsymbol{\Theta})$ with the first component given by (1.1) and (1.2). Here, $\mathbb{E}_{y,j}$ denotes the expectation taken with

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respect to the probability measure $\mathbb{P}_{y,j}$ under which the two-dimensional right-continuous (time-homogeneous) strong Markov process $(Y, \mathbf{\Theta})$, which is also left-continuous over stopping times with respect to $(\mathcal{H}_t)_{t\,\ge\,0}$, that is, quasi-left-continuous (cf., for example, [72, Chapter III, Section 2, Exercise 2.33]), starts at some $(y, j) \in \mathbb{R}_{++} \times \{0, 1\}$. Then, by referring to arguments similar to those used in Section 2.2 above and applying the results of [24, Theorem 2.1] and [68, Theorem 2.1], we conclude that the optimal stopping game of (5.1) and (5.2) admits both a Stackelberg and a Nash equilibrium, so that in particular the equality $W^*(y, j) = W_*(y, j)$ holds, for each $(y, j) \in \mathbb{R}_{++} \times \{0, 1\}$ fixed. Since the continuous-time Markov chain $\mathbf{\Theta}$ is observable in this formulation, being based on arguments similar to those presented in parts (ii) and (iii) of Section 2.3 and taking into account the arguments used in [36, Appendix] as well as in [44, Section 2] and [45, Section 2], we may assume that the stopping times forming a Nash equilibrium in the game of (5.1) and (5.2) are of the form

$$\overline{\tau} = \inf\left\{t \,\ge\, 0 \,\middle|\, Y_t \le \overline{g}(\mathbf{\Theta}_t)\right\} \quad \text{and} \quad \underline{\zeta} = \inf\left\{t \,\ge\, 0 \,\middle|\, Y_t \ge \underline{h}(\mathbf{\Theta}_t)\right\} \tag{5.3}$$

for some numbers $0 < \overline{g}(j) \le rK < rL \le \underline{h}(j)$, for $j = 0, 1$, to be determined.

In order to specify the location of the optimal stopping boundaries $\overline{g}(j)$ and $\underline{h}(j)$, for $j = 0, 1$, we use comparison arguments similar to those applied in part (iii) of Section 2.3 above. More precisely, if we suppose that $\overline{g}(j) < \underline{g}(j)$ or $\underline{h}(j) > \overline{h}(j)$ holds, for some $j = 0, 1$, then, for each $y \in (\overline{g}(j), \underline{g}(j))$ or $y \in (\overline{h}(j), \underline{h}(j))$ fixed, we would have $W^*(y, j) > K = \underline{W}(y, j)$ or $W_*(y, j) < L = \overline{W}(y, j)$, respectively, contradicting the obvious facts that the equalities $W^*(y, j) \le \underline{W}(y, j)$ and $W_*(y, j) \ge \overline{W}(y, j)$ hold, for all $(y, j) \in \mathbb{R}_{++} \in \{0, 1\}$. Here, we recall that the functions $\underline{W}(y, j)$ and $\overline{W}(y, j)$ from (2.6) and (2.7) admit the explicit expressions in (5.56) and (5.57) and the associated optimal stopping times $\underline{\tau}$ and $\overline{\zeta}$ are given by (2.8), where the numbers $\underline{g}(j)$ and $\overline{h}(j)$, for $j = 0, 1$, are determined from the expressions (5.49)–(5.50) and (5.54)–(5.55) with (5.15), respectively. Thus, we may conclude that the inequalities $\underline{g}(j) \le \overline{g}(j)$ and $\underline{h}(j) \le \overline{h}(j)$ should hold, for every $j = 0, 1$.

By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(Y,\mathbf{\Theta})}$ of the process $(Y, \mathbf{\Theta})$ from (1.1)–(1.2) acts on an arbitrary locally bounded function $F(y, j)$, which is of the class C^2 in y on $\mathbb{R}_{++}$ under $\mathbf{\Theta} = j$, for any $j = 0, 1$ fixed, according to the rule

$$\begin{aligned}(\mathbb{L}_{(Y,\mathbf{\Theta})}F)(y, j) = \big(r - \eta_0 - (\eta_1 - \eta_0)\, j\big)\, y\, F_y(y, j)& \\ + \frac{\sigma^2 y^2}{2}\; F_{yy}(y, j) + \eta_j \,\big(F(y, 1 - j) - F(y, j)\big)&\end{aligned} \tag{5.4}$$

for all $(y, j) \in \mathbb{R}_{++} \times \{0, 1\}$. Following the way of arguments from [44, Section 2] and [45, Section 2] (see also [47] for the case of optimal stopping problems in more general models with regime switching), we conclude that the unknown value functions $W^*(y, j) = W_*(y, j)$ from (5.1) and (5.2) and the unknown numbers $\overline{g}(j)$ and $\underline{h}(j)$, for $j = 0, 1$, from (5.3) solve the following free-boundary problem for a coupled system of second-order ordinary differential equations:

$$(\mathbb{L}_{(Y,\mathbf{\Theta})}W - rW)(y, j) = -y \quad \text{for } g(j) < y < h(j), \tag{5.5}$$

$$W(y, j)\big|_{y=g(j)} = 0 \quad \text{and} \quad W(y, j)\big|_{y=h(j)} = 0, \tag{5.6}$$

$$W_y(y, j)\big|_{y=g(j)} = 0 \quad \text{and} \quad W_y(y, j)\big|_{y=h(j)} = 0, \tag{5.7}$$

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$$W(y, j) = K \quad \text{for } y < g(j) \quad \text{and} \quad W(y, j) = L \quad \text{for } y > h(j), \tag{5.8}$$
$$W(y, j) > K \quad \text{for } y > g(j) \quad \text{and} \quad W(y, j) < L \quad \text{for } x < h(j), \tag{5.9}$$
$$(\mathbb{L}_{(Y,\Theta)} W - rW)(y, j) < -y \quad \text{for } y < g(j), \tag{5.10}$$
$$(\mathbb{L}_{(Y,\Theta)} W - rW)(y, j) > -y \quad \text{for } y > h(j), \tag{5.11}$$

for $j = 0, 1$.

5.2. Solutions to the free-boundary problems

By means of straightforward computations, we obtain that the general solutions to the two-dimensional system of second-order ordinary differential equations in (5.4) and (5.5) are given by

$$W(y, 1) = D_1(1)\, y^{\alpha_1} + D_2(1)\, y^{\alpha_2} + B(y, 1) \tag{5.12}$$

for $g(1) < y \le g(0)$, and

$$W(y, j) = C_1(j)\, y^{\beta_1} + C_2(j)\, y^{\beta_2} + C_3(j)\, y^{\beta_3} + C_4(j)\, y^{\beta_4} + A(y, j) \tag{5.13}$$

for $g(0) < y < h(1)$ and $j = 0, 1$, as well as

$$W(y, 0) = D_1(0)\, y^{\gamma_1} + D_2(0)\, y^{\gamma_2} + B(y, 0) \tag{5.14}$$

for $h(1) \le y < h(0)$, with

$$A(y, j) = \frac{(\eta_{1-j} + \lambda_0 + \lambda_1) y}{\eta_0 \eta_1 + \eta_0 \lambda_1 + \eta_1 \lambda_0} \quad \text{and} \quad B(y, j) = \frac{y}{\eta_j + \lambda_j} \tag{5.15}$$

for $y \in \mathbb{R}_{++}$ and $j = 0, 1$, where $C_k(j)$ and $D_i(j)$, for $j = 0, 1$, $k = 1, 2, 3, 4$, and $i = 1, 2$, are arbitrary constants. Here, the α_i, for $i = 1, 2$, are explicitly given by

$$\alpha_i = \frac{1}{2} - \frac{r - \eta_1}{\sigma^2} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r - \eta_1}{\sigma^2}\right)^2 + \frac{2(r + \lambda_1)}{\sigma^2}} \tag{5.16}$$

so that $\alpha_2 < 0 < 1 < \alpha_1$, β_k, for $k = 1, 2, 3, 4$, are the roots of the appropriate characteristic equation

$$P_0(\beta)\, P_1(\beta) = \lambda_0 \lambda_1, \quad \text{with } P_j(\beta) = r + \lambda_j - (r - \eta_j)\, \beta - \frac{\sigma^2}{2}\, \beta(\beta - 1), \tag{5.17}$$

so that $\beta_4 < \beta_3 < 0 < 1 < \beta_2 < \beta_1$ holds, and the γ_i, for $i = 1, 2$, are explicitly given by

$$\gamma_i = \frac{1}{2} - \frac{r - \eta_0}{\sigma^2} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r - \eta_0}{\sigma^2}\right)^2 + \frac{2(r + \lambda_0)}{\sigma^2}} \tag{5.18}$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds (cf., for example, [44, Section 2] and [45, Section 2] for similar arguments related to other optimal stopping problems in models with observable continuous-time Markov chains).

Then, using the structure of the coupled system of ordinary differential equations in (5.4) and (5.5) and applying the conditions of (5.6) and (5.7) to the functions in (5.12)–(5.14) with (5.15), we obtain that the equalities

$$D_1(1)\, g^{\gamma_1}(1) + D_2(1)\, g^{\gamma_2}(1) + B(g(1), 1) = K, \tag{5.19}$$

$$D_1(1)\, \gamma_1\, g^{\gamma_1}(1) + D_2(1)\, \gamma_2\, g^{\gamma_2}(1) + B_y(g(1), 1)\, g(1) = 0, \tag{5.20}$$

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$$C_1(0)\, g^{\beta_1}(0) + C_2(0)\, g^{\beta_2}(0) + C_3(0)\, g^{\beta_3}(0) + C_4(0)\, g^{\beta_4}(0) + A(g(0), 0) = K, \tag{5.21}$$

$$C_1(0)\, \beta_1\, g^{\beta_1}(0) + C_2(0)\, \beta_2\, g^{\beta_2}(0) + C_3(0)\, \beta_3\, g^{\beta_3}(0) + C_4(0)\, \beta_4\, g^{\beta_4}(0), + A_y(g(0), 0)\, g(0) = 0, \tag{5.22}$$

$$C_k(0)\, P_0(\beta_k) = C_k(1)\, \eta_1 \quad \text{for} \quad k = 1, 2, 3, 4 \tag{5.23}$$

$$C_1(1)\, h^{\beta_1}(1) + C_2(1)\, h^{\beta_2}(1) + C_3(1)\, h^{\beta_3}(1) + C_4(1)\, h^{\beta_4}(1) + A(h(1), 1) = L, \tag{5.24}$$

$$C_1(1)\, \beta_1\, h^{\beta_1}(1) + C_2(1)\, \beta_2\, h^{\beta_2}(1) + C_3(1)\, \beta_3\, h^{\beta_3}(1) + C_4(1)\, \beta_4\, h^{\beta_4}(1) + A_y(h(1), 1)\, h(1) = 0, \tag{5.25}$$

$$D_1(0)\, h^{\alpha_1}(0) + D_2(0)\, h^{\alpha_2}(0) + B(h(0), 0) = L, \tag{5.26}$$

$$D_1(0)\, \alpha_1\, h^{\alpha_1}(0) + D_2(0)\, \alpha_2\, h^{\alpha_2}(0) + B_y(h(0), 0)\, h(0) = 0 \tag{5.27}$$

should hold, where we have $P_0(\beta_k)/\lambda_1 = \lambda_0/P_1(\beta_k)$, for $k = 1, 2, 3, 4$, according to (5.17). Observe that, since the points $\overline{g}(0)$ and $\underline{h}(1)$ belong to the appropriate continuation region when $\boldsymbol{\Theta} = 1$ and $\boldsymbol{\Theta} = 0$, respectively, the function in (5.13) should be (at least) continuously differentiable, and thus the equalities

$$\begin{aligned} &C_1(1)\, g^{\beta_1}(0) + C_2(1)\, g^{\beta_2}(0) + C_3(1)\, g^{\beta_3}(0) + C_4(1)\, g^{\beta_4}(0) + A(g(0), 1), \\ &\quad = D_1(1)\, g^{\gamma_1}(0) + D_2(1)\, g^{\gamma_2}(0) + B(g(0), 1), \end{aligned} \tag{5.28}$$

$$\begin{aligned} &C_1(1)\, \beta_1\, g^{\beta_1}(0) + C_2(1)\, \beta_2\, g^{\beta_2}(0) + C_3(1)\, \beta_3\, g^{\beta_3}(0) + C_4(1)\, \beta_4\, g^{\beta_4}(0), \\ &\quad = D_1(1)\, \gamma_1\, g^{\gamma_1}(0) + D_2(1)\, \gamma_2\, g^{\gamma_2}(0) + B_y(g(0), 1)\, g(0) - A_y(g(0), 1)\, g(0) \end{aligned} \tag{5.29}$$

$$\begin{aligned} &C_1(0)\, h^{\beta_1}(1) + C_2(0)\, h^{\beta_2}(1) + C_3(0)\, h^{\beta_3}(1) + C_4(0)\, h^{\beta_4}(1) + A(h(1), 0), \\ &\quad = D_1(0)\, h^{\alpha_1}(1) + D_2(0)\, h^{\alpha_2}(1) + B(h(1), 0), \end{aligned} \tag{5.30}$$

$$\begin{aligned} &C_1(0)\, \beta_1\, h^{\beta_1}(1) + C_2(0)\, \beta_2\, h^{\beta_2}(1) + C_3(0)\, \beta_3\, h^{\beta_3}(1) + C_4(0)\, \beta_4\, h^{\beta_4}(1), \\ &\quad = D_1(0)\, \alpha_1\, h^{\alpha_1}(1) + D_2(0)\, \alpha_2\, h^{\alpha_2}(1) + B_y(h(1), 0)\, h(1) - A_y(h(1), 0)\, h(1) \end{aligned} \tag{5.31}$$

should be satisfied. Hence, solving the systems of arithmetic equations in (5.19)–(5.20) and (5.26)–(5.27) as well as those of the Vandermonde type in (5.21)–(5.22) and (5.28)–(5.29), and (5.24)–(5.25) and (5.24)–(5.25) with (5.23), by means of straightforward computations, we obtain that the functions

$$W(y, 1; g(1)) = \sum_{j=1}^{2} \frac{B_y(g(1), 1) g(1) - \gamma_{3-j}(B(g(1), 1) - K)}{\gamma_{3-j} - \gamma_j} \left(\frac{y}{g(1)}\right)^{\gamma_j} + B(y, 1) \tag{5.32}$$

for $g(1) < y \le g(0)$, and

$$W(y, j; g(0), g(1), h(0), h(1)) = \sum_{k=1}^{4} C_k(j; g(0), g(1), h(0), h(1))\, y^{\beta_k} + A(y, j) \tag{5.33}$$

for $g(0) < y < h(1)$ and $j = 0, 1$, as well as

$$W(y, 0; h(0)) = \sum_{j=1}^{2} \frac{B_y(h(0), 0) h(0) - \alpha_{3-j}(B(h(0), 0) - L)}{\alpha_{3-j} - \alpha_j} \left(\frac{y}{h(0)}\right)^{\alpha_j} + B(y, 0) \tag{5.34}$$

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for $h(1) \le y < h(0)$, satisfy the system in (5.4) and (5.5) and (5.6)–(5.7), when the equalities

$$\sum_{k=1}^{4} C_k(1;g(0),\, g(1),\, h(0),\, h(1))\, h^{\beta_k}(1) + A(h(1),\, 1) = L \tag{5.35}$$

and

$$\sum_{k=1}^{4} C_k(1;g(0),\, g(1),\, h(0),\, h(1))\, \beta_k\, h^{\beta_k}(1) + A_y(h(1),\, 1)\, h(1) = 0 \tag{5.36}$$

as well as

$$\begin{aligned} &\sum_{k=1}^{4} C_k(0;g(0),\, g(1),\, h(0),\, h(1))\, h^{\beta_k}(1) + A(h(1),\, 0), \\ &= \sum_{j=1}^{2} \frac{B_y(h(0),\, 0)h(0) - \alpha_{3-j}(B(h(0),\, 0) - L)}{\alpha_{3-j} - \alpha_j} \left(\frac{h(1)}{h(0)}\right)^{\alpha_j} + B(h(1),\, 0) \end{aligned} \tag{5.37}$$

and

$$\begin{aligned} &\sum_{k=1}^{4} C_k(0;g(0),\, g(1),\, h(0),\, h(1))\, \beta_k\, h^{\beta_k}(1) + A_y(h(1),\, 0)\, h(1) \\ &= \sum_{j=1}^{2} \frac{B_y(h(0),\, 0)h(0) - \alpha_{3-j}(B(h(0),\, 0) - L)}{\alpha_{3-j} - \alpha_j}\, \alpha_j \left(\frac{h(1)}{h(0)}\right)^{\alpha_j} + B_y(h(1),\, 0)\, h(1) \end{aligned} \tag{5.38}$$

hold. Here, the coefficients $C_k(0;g(0),\, g(1),\, h(0),\, h(1))$, for $k = 1, 2, 3, 4$, are explicitly given by

$$\begin{aligned} C_1(0) = \Big(&\big((W_y(g(0),\, 1) - A_y(g(0),\, 1))\, g(0) - \beta_4\, (W(g(0),\, 1) - A(g(0),\, 1)) \\ &+ \delta_3\, \big(A_y(g(0),\, 0)\, g(0) + \beta_4\, (K - A(g(0),\, 0))\big)\big)\, (\beta_2 - \beta_3)(\delta_2 - \delta_4) \\ &- \big((W_y(g(0),\, 1) - A_y(g(0),\, 1))\, g(0) + \delta_4\, A_y(g(0),\, 0)\, g(0) \\ &- \beta_3\, \big(W(g(0),\, 1) - A(g(0),\, 1) - \delta_4\, (K - A(g(0),\, 0))\big)\big)\, (\beta_2 - \beta_4)(\delta_2 - \delta_3)\Big) \Big/ \\ \Big(&g^{\beta_1}(0)\, \big((\beta_1 - \beta_4)(\beta_2 - \beta_3)(\delta_1 - \delta_3)(\delta_2 - \delta_4) - (\beta_1 - \beta_3)(\beta_2 - \beta_4)(\delta_1 - \delta_4)(\delta_2 - \delta_3)\big)\Big), \end{aligned} \tag{5.39}$$

$$\begin{aligned} C_2(0) = \Big(&\big((W_y(g(0),\, 1) - A_y(g(0),\, 1))\, g(0) - \beta_4\, (W(g(0),\, 1) - A(g(0),\, 1)) \\ &+ \delta_3\, \big(A_y(g(0),\, 0)\, g(0) + \beta_4\, (K - A(g(0),\, 0))\big)\big)\, (\beta_1 - \beta_3)(\delta_1 - \delta_4) \\ &- \big((W_x(g(0),\, 1) - A_x(g(0),\, 1))\, g(0) + \delta_4\, A_y(g(0),\, 0)\, g(0) \\ &- \beta_3\, \big(W(g(0),\, 1) - A(g(0),\, 1) - \delta_4\, (K - A(g(0),\, 0))\big)\big)\, (\beta_1 - \beta_4)(\delta_1 - \delta_3)\Big) \Big/ \\ \Big(&g^{\beta_2}(0)\, \big((\beta_1 - \beta_3)(\beta_2 - \beta_4)(\delta_1 - \delta_4)(\delta_2 - \delta_3) - (\beta_1 - \beta_4)(\beta_2 - \beta_3)(\delta_1 - \delta_3)(\delta_2 - \delta_4)\big)\Big), \end{aligned} \tag{5.40}$$

$$\begin{aligned} C_3(0) = \Big(&\big((W_y(g(0),\, 1) - A_y(g(0),\, 1))\, g(0) - \beta_1\, (W(g(0),\, 1) - A(g(0),\, 1)) \\ &+ \delta_2\, \big(A_y(g(0),\, 0)\, g(0) + \beta_1\, (K - A(g(0),\, 0))\big)\big)\, (\beta_2 - \beta_4)(\delta_1 - \delta_4) \\ &- \big((W_y(g(0),\, 1) - A_y(g(0),\, 1))\, g(0) + \delta_1\, A_y(g(0),\, 0)\, g(0) \\ &- \beta_2\, \big(W(g(0),\, 1) - A(g(0),\, 1) - \delta_1\, (K - A(g(0),\, 0))\big)\big)\, (\beta_1 - \beta_4)(\delta_2 - \delta_4)\Big) \Big/ \\ \Big(&g^{\beta_3}(0)\, \big((\beta_1 - \beta_3)(\beta_2 - \beta_4)(\delta_1 - \delta_4)(\delta_2 - \delta_3) - (\beta_1 - \beta_4)(\beta_2 - \beta_3)(\delta_1 - \delta_3)(\delta_2 - \delta_4)\big)\Big), \end{aligned} \tag{5.41}$$

and

$$C_4(0) = \Big(\big((W_y(g(0), 1) - A_y(g(0), 1))\, g(0) - \beta_1\, (W(g(0), 1) - A(g(0), 1)) \\ + \delta_2\, \big(A_y(g(0), 0)\, g(0) + \beta_1\, (K - A(g(0), 0))\big)\big)\, (\beta_2 - \beta_3)(\delta_1 - \delta_3) \\ - \big((W_y(g(0), 1) - A_y(g(0), 1))\, g(0) + \delta_1\, A_y(g(0), 0)\, g(0) \\ - \beta_2\, \big(W(g(0), 1) - A(g(0), 1) - \delta_1\, (K - A(g(0), 0))\big)\big)\, (\beta_1 - \beta_3)(\delta_2 - \delta_3)\Big) \Big/ \\ \Big(g^{\beta_4}(0)\, \big((\beta_1 - \beta_4)(\beta_2 - \beta_3)(\delta_1 - \delta_3)(\delta_2 - \delta_4) - (\beta_1 - \beta_3)(\beta_2 - \beta_4)(\delta_1 - \delta_4)(\delta_2 - \delta_3)\big)\Big), \tag{5.42}$$

with $\delta_k = P_0(\beta_k)/\lambda_1$, for $k = 1, 2, 3, 4$, as well as the relations

$$C_k(j; g(0), g(1), h(0), h(1))\, P_j(\beta_k) = C_k(1 - j; g(0), g(1), h(0), h(1))\, \lambda_{1-j} \tag{5.43}$$

are satisfied, for $j = 0, 1$ and $k = 1, 2, 3, 4$ (see also [44, Section 2] and [45, Section 2] for similar calculations related to other optimal stopping problems in models with observable continuous-time Markov chains).

5.3. The result

Summarizing the facts proved above, let us formulate the following result which can be proved by means of the same arguments as the appropriate verification assertions from [36; Corollary A.1], [44, Theorem 2.1] and [45, Theorem 2] related to the optimal stopping problems in the models with observable continuous-time Markov chains with two states.

Theorem 5.1. *Suppose that the process Y is defined by (1.1) and (1.2), with some $r > 0$, $\eta_0 > \eta_1 > 0$, and $\sigma > 0$, while $\boldsymbol{\Theta}$ is a continuous-time Markov chain with the state space $\{0, 1\}$ and transition intensities $\lambda_j > 0$, for $j = 0, 1$, which is independent of standard Brownian motion B. Then the value functions $W^*(y, j) = W_*(y, j)$ of the optimal stopping zero-sum game in (5.1) and (5.2), with $L > K > 0$ fixed, are given by*

$$W^*(y, 0) = W_*(y, 0) = \begin{cases} K, & \text{if } y \le \overline{g}(0) \\ W(y, 0; \overline{g}(0), \overline{g}(1), \underline{h}(0), \underline{h}(1)), & \text{if } \overline{g}(0) < y < \underline{h}(1) \\ W(y, 0; \underline{h}(0)), & \text{if } \underline{h}(1) \le y < \underline{h}(0) \\ L, & \text{if } y \ge \underline{h}(0) \end{cases} \tag{5.44}$$

and

$$W^*(y, 1) = W_*(y, 1) = \begin{cases} K, & \text{if } y \le \overline{g}(1) \\ W(y, 1; \overline{g}(1)), & \text{if } \overline{g}(1) < y \le \overline{g}(0) \\ W(y, 1; \overline{g}(0), \overline{g}(1), \underline{h}(0), \underline{h}(1)), & \text{if } \overline{g}(0) < y < \underline{h}(1) \\ L, & \text{if } y \ge \underline{h}(1) \end{cases} \tag{5.45}$$

and the stopping times $\overline{\tau}$ and $\underline{\zeta}$ forming a Nash equilibrium have the form of (5.3), where the function $W(y, 1; p(1))$ is given by (5.32), the functions $W(y, j; g(0), g(1), h(0), h(1))$, for $j = 0, 1$, are given by (5.33) with (5.39)–(5.43), the function $W(y, 0; h(0))$ is given by (5.34), and the numbers $0 < \overline{g}(1) \le \overline{g}(0) \le rK < rL \le \underline{g}(1) \le \underline{g}((0)$ are uniquely determined by the system of arithmetic equations in (5.35)–(5.38).

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5.4. Solutions to the auxiliary optimal stopping problems

By means of straightforward computations, we obtain that the function $\underline{W}(y, j)$ in (2.6) and the boundary $\underline{g}(j)$ in (2.8) solve the left-hand system in (5.4)+(5.5)–(5.9)+(5.10), while the function $\overline{W}(y, j)$ in (2.7) and the boundary $\overline{h}(j)$ in (2.8) solve the right-hand system in (5.4)+(5.5)–(5.9)+(5.11). In this case, the candidate value function for $\underline{W}(y, j)$ should be of the form of (5.12) for $g(1) < y \le g(0)$, and of (5.13) for $y > g(0)$, while the candidate value function for $\overline{W}(y, j)$ should be of the form of (5.13) for $y < h(1)$, and of (5.14) for $h(1) \le y < h(0)$, with (5.15). Moreover, on the one hand, we observe that $C_1(j) = C_2(j) = 0$ should hold in the representation related to (5.13) for $\underline{W}(y, j)$, since otherwise, we would have $W(y, j) \to \pm\infty$ with more than a linear growth as $y \uparrow \infty$, which must be excluded by virtue of the obvious fact that the value function in (2.6) is of a linear growth under $y \uparrow \infty$, for any $j = 0, 1$ fixed. On the other hand, we observe that $C_3(j) = C_4(j) = 0$ should hold in the representation related to (5.13) for $\overline{W}(y, j)$, since otherwise we would have $W(y, j) \to \pm\infty$ with less than linear growth as $y \downarrow 0$, which must be excluded by virtue of the obvious fact that the value function in (2.7) is of a linear growth under $y \downarrow 0$, for any $j = 0, 1$ fixed.

Hence, following arguments similar to those applied in Section 5.2 above (see also [36, Appendix]) and solving the systems of arithmetic equations in (5.19)–(5.23) and (5.24)–(5.25), under $C_1(j) = C_2(j) = 0$, we obtain that the functions

$$W(y,\, 1; g(1)) = \sum_{j=1}^{2} \frac{B_y(g(1),\, 1)g(1) - \gamma_{3-j} B(g(1),\, 1)}{\gamma_{3-j} - \gamma_j} \left(\frac{y}{g(1)} \right)^{\gamma_j} + B(y,\, 1) \tag{5.46}$$

for $g(1) < y \le g(0)$, as well as

$$W(y,\, 0; g(0)) = \sum_{k=3}^{4} \frac{A_y(g(0),\, 0)g(0) - \beta_{7-k} A(g(0),\, 0)}{\beta_{7-k} - \beta_k} \left(\frac{y}{g(0)} \right)^{\beta_k} + A(y,\, 0) \tag{5.47}$$

and

$$W(y,\, 1; g(0)) = \sum_{k=3}^{4} \frac{A_y(g(0),\, 0)g(0) - \beta_{7-k} A(g(0),\, 0)}{\beta_{7-k} - \beta_k} \, \frac{P_0(\beta_k)}{\lambda_1} \left(\frac{y}{g(0)} \right)^{\beta_k} + A(y,\, 1) \tag{5.48}$$

for $y > g(0)$, satisfy the left-hand system in (5.4)–(5.5) and (5.6), while the left-hand condition of (5.7) is also satisfied when the equalities

$$\begin{aligned} &\sum_{k=3}^{4} \frac{A_y(g(0),\, 0)g(0) - \beta_{7-k} A(g(0),\, 0)}{\beta_{7-k} - \beta_k} \, \frac{P_0(\beta_k)}{\eta_1} + A(g(0),\, 1) \\ &= \sum_{j=1}^{2} \frac{B_y(g(1),\, 1)g(1) - \gamma_{3-j} B(g(1),\, 1)}{\gamma_{3-j} - \gamma_j} \left(\frac{g(0)}{g(1)} \right)^{\gamma_j} + B(g(0),\, 1) \end{aligned} \tag{5.49}$$

and

$$\begin{aligned} &\sum_{k=3}^{4} \frac{A_y(g(0),\, 0)g(0) - \beta_{7-k} A(g(0),\, 0)}{\beta_{7-k} - \beta_k} \, \frac{P_0(\beta_k)}{\eta_1} \, \beta_k + A_y(g(0),\, 1)\, g(0) \\ &= \sum_{j=1}^{2} \frac{B_y(g(1),\, 1)g(1) - \gamma_{3-j} B(g(1),\, 1)}{\gamma_{3-j} - \gamma_j} \, \gamma_j \left(\frac{g(0)}{g(1)} \right)^{\gamma_j} + B_y(g(0),\, 1)\, g(0) \end{aligned} \tag{5.50}$$

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hold. Thus, following arguments similar to those applied in Section 5.2 above and solving the systems of arithmetic equations in (5.19)–(5.23) and (5.24)–(5.25), under $C_3(j) = C_4(j) = 0$, we obtain that the functions

$$W(y, 0; h(0)) = \sum_{j=1}^{2} \frac{B_y(h(0), 0)h(0) - \alpha_{3-j}B(h(0), 0)}{\alpha_{3-j} - \alpha_j} \left(\frac{y}{h(0)}\right)^{\alpha_j} + B(y, 0) \tag{5.51}$$

for $h(1) \le y < h(0)$, as well as

$$W(y, 1; h(1)) = \sum_{k=1}^{2} \frac{A_y(h(1), 1)h(1) - \beta_{3-k}A(h(1), 1)}{\beta_{3-k} - \beta_k} \left(\frac{y}{h(1)}\right)^{\beta_k} + A(y, 1) \tag{5.52}$$

and

$$W(y, 0; h(1)) = \sum_{k=1}^{2} \frac{A_y(h(1), 1)h(1) - \beta_{3-k}A(h(1), 1)}{\beta_{3-k} - \beta_k} \frac{P_1(\beta_k)}{\eta_0} \left(\frac{y}{h(1)}\right)^{\beta_k} + A(y, 0) \tag{5.53}$$

for $y < h(1)$, satisfy the right-hand system in (5.4)–(5.5) and (5.6), while the right-hand condition of (5.7) is also satisfied when the equalities

$$\begin{aligned} &\sum_{k=1}^{2} \frac{A_y(h(1), 1)h(1) - \beta_{3-k}A(h(1), 1)}{\beta_{3-k} - \beta_k} \frac{P_1(\beta_k)}{\lambda_0} + A(h(1), 0) \\ &= \sum_{j=1}^{2} \frac{B_y(h(0), 0)h(0) - \alpha_{3-j}B(h(0), 0)}{\alpha_{3-j} - \alpha_j} \left(\frac{h(1)}{h(0)}\right)^{\alpha_j} + B(h(1), 0) \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} &\sum_{k=1}^{2} \frac{A_y(h(1), 1)h(1) - \beta_{3-k}A(h(1), 1)}{\beta_{3-k} - \beta_k} \frac{P_1(\beta_k)}{\lambda_0} \beta_k + A_y(h(1), 0)\, h(1) \\ &= \sum_{j=1}^{2} \frac{B_y(h(0), 0)h(0) - \alpha_{3-j}B(h(0), 0)}{\alpha_{3-j} - \alpha_j} \alpha_j \left(\frac{h(1)}{h(0)}\right)^{\alpha_j} + B_y(h(1), 0)\, h(1) \end{aligned} \tag{5.55}$$

hold.

Summarizing the facts shown above, let us formulate the following result which can be proved by means of the same arguments as Theorem 5.1 above.

Corollary 5.1. *Suppose that the process Y is defined by (1.1) and (1.2), with some $r > 0$, $\eta_0 > \eta_1 > 0$, and $\sigma > 0$, while $\boldsymbol{\Theta}$ is a continuous-time Markov chain with state space $\{0, 1\}$ and transition intensities $\lambda_j > 0$, for $j = 0, 1$, which is independent of standard Brownian motion B. Then the value function $\underline{W}(y, j)$ of the optimal stopping problem in (2.6), with some $K > 0$ fixed, admits the representations*

$$\underline{W}(y, 0) = \begin{cases} W(y, 0; \underline{g}(0)), & \text{if } y > \underline{g}(0), \\ K, & \text{if } y \le \underline{g}(0), \end{cases} \tag{5.56}$$

and

$$\underline{W}(y, 1) = \begin{cases} W(y, 1; \underline{g}(0)), & \text{if } y > \underline{g}(0), \\ W(y, 1; \underline{g}(1)), & \text{if } \underline{g}(1) < y \le \underline{g}(0), \\ K, & \text{if } y \le \underline{g}(1), \end{cases} \tag{5.57}$$

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and the optimal stopping time $\underline{\tau}$ has the form of (2.8), where the functions $W(y, j; g(0))$, for $j = 0, 1$, are given by (5.47) and (5.48), the function $W(y, 1; g(1))$ is given by (5.46), and the numbers $\underline{g}(j)$, for $j = 0, 1$, are determined by the system of arithmetic equations in (5.49) and (5.50). Also, the value function $\overline{W}(y, j)$ of the optimal stopping problem in (2.7), with some $L > 0$ fixed, admits the representations

$$\overline{W}(y, 0) = \begin{cases} W(y, 0; \overline{h}(1)), & \text{if } y < \overline{h}(1), \\ W(y, 0; \overline{h}(0)), & \text{if } \overline{h}(1) \le y < \overline{h}(0), \\ L, & \text{if } y \ge \overline{h}(0), \end{cases} \tag{5.58}$$

and

$$\overline{W}(y, 1) = \begin{cases} W(y, 1; \overline{h}(1)), & \text{if } y < \overline{h}(1), \\ L, & \text{if } y \ge \overline{h}(1), \end{cases} \tag{5.59}$$

and the optimal stopping time $\overline{\zeta}$ has the form of (2.8), where the functions $W(y, j; h(1))$, for $j = 0, 1$, are given by (5.52) and (5.53), the function $W(y, 0; h(0))$ is given by (5.51), and the numbers $\overline{h}(j)$, for $j = 0, 1$, are uniquely determined by the system of arithmetic equations in (5.54) and (5.55).

5.5. The solution in a particular case

Finally, let us present explicit solutions of the optimal stopping problems in (5.2) and (2.6)–(2.7) under the assumption $\lambda_0 = \lambda_1 = 0$. In this case, the general solutions to the second-order ordinary differential equations in (5.4) and (5.5) are given by

$$W(y, j) = C_1(j)\, y^{\chi_{j,1}} + C_2(j)\, y^{\chi_{j,2}} + y/\eta_j \tag{5.60}$$

where $C_i(j)$, for $i = 1, 2$ and $j = 0, 1$, are arbitrary constants. Then, by applying the conditions (5.6) and (5.7) to the function in (5.60), we obtain that the equalities

$$C_1(j)\, g^{\chi_{i,1}}(j) + C_2(j)\, g^{\chi_{j,2}}(j) + g(j)/\eta_j = K, \tag{5.61}$$

$$C_1(j)\, \chi_{j,1}\, g^{\chi_{j,1}}(i) + C_2(j)\, \chi_{j,2}\, g^{\chi_{j,2}}(i) + g(j)/\eta_j = 0, \tag{5.62}$$

$$C_1(j)\, h^{\chi_{j,1}}(j) + C_2(j)\, h^{\chi_{j,2}}(j) + h(j)/\eta_j = L, \tag{5.63}$$

$$C_1(j)\, \chi_{j,1}\, h^{\chi_{j,1}}(j) + C_2(j)\, \chi_{j,2}\, h^{\chi_{j,2}}(j) + h(j)/\eta_j = 0 \tag{5.64}$$

should hold, where the $\chi_{j,i}$, for $i = 1, 2$ and $j = 0, 1$, are given by (4.60) above. Hence, solving the system of equations in (5.61) and (5.63), we obtain that the function

$$W(y, j; g(j), h(j)) = C_1(j; g(j), h(j))\, y^{\chi_{j,1}} + C_2(j; g(j), h(j))\, y^{\chi_{j,2}} + y/\eta_j, \tag{5.65}$$

for $g(j) < y < h(j)$, satisfies the system in (5.4) and (5.5)–(5.6) under $\eta_0 = \eta_1 = 0$, when we set

$$C_i(j; g(j), h(j)) = \frac{(K - g(j)/\eta_j) h^{\chi_{j,3-i}}(j) - (L - h(j)/\eta_j) g^{\chi_{j,3-i}}(j)}{g^{\chi_{j,i}}(j) h^{\chi_{j,3-i}}(j) - h^{\chi_{j,i}}(j) g^{\chi_{j,3-i}}(j)} \tag{5.66}$$

for $i = 1, 2$ and $j = 0, 1$. Thus, putting (5.66) into the system of equations in (5.62) and (5.64), we obtain that the function in (5.65) satisfies the conditions of (5.7), when the equalities

$$\frac{\chi_{j,i}(K - g(j)/\eta_j) + g(j)/\eta_j}{\chi_{j,i}(L - h(j)/\eta_j) + h(j)/\eta_j} = \left(\frac{g(j)}{h(j)}\right)^{\chi_{j,3-i}} \tag{5.67}$$

hold, for $i = 1, 2$ and $j = 0, 1$. Therefore, following the arguments similar to those used above

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and solving the systems of equations in (5.61)–(5.62) and (5.63)–(5.64), under $C_1(j) = 0$ and $C_2(j) = 0$, respectively, we obtain that the functions

$$W(y, j; \underline{g}(j)) = \left(K - \frac{\underline{g}(j)}{\eta_j}\right)\left(\frac{y}{\underline{g}(j)}\right)^{\chi_{j,2}} + \frac{y}{\eta_j}, \quad \text{with} \quad \underline{g}(j) = \frac{\chi_{j,2}\eta_j}{\chi_{j,2} - 1} K, \tag{5.68}$$

for $y > \underline{g}(j)$, and

$$W(y, j; \overline{h}(j)) = \left(L - \frac{\overline{h}(j)}{\eta_j}\right)\left(\frac{y}{\overline{h}(j)}\right)^{\chi_{j,1}} + \frac{y}{\eta_j}, \quad \text{with} \quad \overline{h}(j) = \frac{\chi_{j,1}\eta_j}{\chi_{j,1} - 1} L, \tag{5.69}$$

for $0 < y < \overline{h}(j)$, and $j = 0, 1$, satisfy the left-hand and right-hand systems of (5.4) and (5.5) with (5.6)–(5.7), respectively.

Let us summarize the facts shown above in the following assertion.

Corollary 5.2. *Suppose that the process Y is defined by (1.1) and (1.2), with some $r > 0$, $\eta_0 > \eta_1 > 0$, and $\sigma > 0$, while $\boldsymbol{\Theta}_t \equiv \boldsymbol{\Theta}_0$, for $t \geq 0$, is a Bernoulli random variable with the values in $\{0, 1\}$, so that $\lambda_0 = \lambda_1 = 0$ holds, which is independent of standard Brownian motion B. Then the value functions of the optimal stopping problems in (5.1) and (5.2) as well as in (2.6) and (2.7), for some $L > K > 0$ fixed, admit the representations*

$$W^*(y, j) = W_*(y, j) = \begin{cases} K, & \text{if } y \leq \overline{g}(j), \\ W(y, j; \overline{g}(j), \underline{h}(j)), & \text{if } \overline{g}(j) < y < \underline{h}(j), \\ L, & \text{if } y \geq \underline{h}(j), \end{cases} \tag{5.70}$$

as well as

$$\underline{W}(y, j) = \begin{cases} W(y, j; \underline{g}(j)), & \text{if } y > \underline{g}(j), \\ K, & \text{if } y \leq \underline{g}(j), \end{cases} \tag{5.71}$$

and

$$\overline{W}(y, j) = \begin{cases} W(y, j; \overline{h}(j)), & \text{if } y < \overline{h}(j), \\ L, & \text{if } y \geq \overline{h}(j), \end{cases} \tag{5.72}$$

and the stopping times $\overline{\tau}$ and $\underline{\zeta}$ forming a Nash equilibrium as well as $\underline{\tau}$ and $\overline{\zeta}$ have the form of (5.3) and (2.8), respectively, where the functions $W(y, j; \overline{g}(j), \underline{h}(j))$, $W(y, j; \underline{g}(j))$ and $W(y, j; \overline{h}(j))$, as well as the numbers $\underline{g}(j)$ and $\overline{h}(j)$ are given by (5.65)–(5.66) and (5.68)–(5.69), respectively, while the numbers $\overline{g}(j) \leq rK$ and $\underline{h}(j) \geq rL$ are uniquely determined from the system of arithmetic equations in (5.67), for $i = 1, 2$ and $j = 0, 1$.

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