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Partially well-ordered sets of infinite matrices and closed classes of abelian groups

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We give a necessary and sufficient condition for a class of row-decreasing infinite matrices to be partially well-ordered with regard to the component-wise ordering. Then, using these matrices, we determine all the classes of abelian groups, closed under taking subgroups, direct limits, and isomorphic groups.

1. Introduction

Throughout the paper, by sequences and subsequences, we shall mean infinite sequences and infinite subsequences, respectively. A preordered, or quasi-ordered, set $(A, \leq)$ is a nonempty set A with a reflexive, transitive binary relation $\leq$ on A. Erdös and Rado (see Higman, [4]) called a preordered set $(A, \leq)$ a *partially well-ordered* set if every sequence of elements of A contains an ascending subsequence. Some classes of vectors of nonnegative integers with certain preorderings give natural examples of partially well-ordered sets, and, in addition, they turn out to be very nice tools for characterization of some algebras. For example, Perkins [6] used such a class to show that every commutative semigroup is finitely based, and Cohen [2] used such two classes to prove that every commutative ring is finitely based (for a detailed proof with a generalization to wider classes, see Bang and Mandelberg [1]). The purpose of this paper is to show such other application of a partially well-ordered set. We shall first show that some classes of infinite matrices are

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partially well-ordered, and then apply the result to determine all the classes of abelian groups, closed under taking subgroups, direct limits, and isomorphic groups.

2. Partially well-ordered sets

Higman [4] gave extensive equivalent conditions for a preordered set to be partially well-ordered, from which we list the following for later use.

LEMMA 1. *Let* $(A, \leq)$ *be a preordered set. Then the following conditions are equivalent:*

(a) $(A, \leq)$ *is partially well-ordered;*

(b) if $\{a_1, a_2, \ldots\}$ *is a sequence of elements of* A *, there exist positive integers* i, j *such that* $i < j$ *and* $a_i \leq a_j$ *;*

(c) there exists neither a strictly descending sequence, nor an infinity of mutually incomparable elements in A .

Let $(A, \leq)$ be a partially ordered set. A nonempty subset $B \subseteq A$ will be called an *inductive tower* of A if $\alpha \in B$ and $\beta \leq \alpha$ in A together imply $\beta \in B$ and each chain in B has its supremum in B . For any nonempty subset $C \subseteq A$, let $\max C$ mean the set of all maximal elements of C , and let C^* denote the inductive tower of A generated by C . Denote by $I(A)$ the set of all inductive towers of A . With these terminologies, we can obtain the following lemma.

LEMMA 2. *Let* $(A, \leq)$ *be a partially ordered set which is partially well-ordered. Then the following are true:*

(a) each $B \in I(A)$ *is finitely generated by* $\max B$ *;*

(b) $(I(A), \subseteq)$ *satisfies the descending chain condition;*

(c) no member covers and cocovers infinitely many members in $(I(A), \subseteq)$ *;*

(d) $|I(A)| = |A|$ *if* $|A| = \infty$.

3. Row-decreasing matrices

Let $K^* = \{0, 1, \ldots, \infty\}$ with the obvious ordering. By a *decreasing* vector α_i, let us mean $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots)$ with $\alpha_{ij} \in K^*$ and $\alpha_{i1} \geq \alpha_{i2} \geq \ldots$. Denote by V the set of all such decreasing vectors, and let $\leq$ be the componentwise ordering on V. Then, $(V, \leq)$ is a partially ordered set, in fact, a complete lattice. The following is a key lemma of this paper.

LEMMA 3. *$(V, \leq)$ is a complete lattice that is partially well-ordered.*

Proof. Let $U = \{\alpha_1, \alpha_2, \ldots\}$ be a sequence of vectors of V. Write $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots)$ for all i. For the sake of convenience, let us define the width $W(\alpha_i)$, the level $L(\alpha_i)$, and the divinity $D(\alpha_i)$ of each α_i, respectively, as follows:

$$W(\alpha_i) = \min\{j \mid \alpha_{ij} = \alpha_{i(j+1)} = \ldots\} ,$$

$$L(\alpha_i) = \alpha_{iW(\alpha_i)} ,$$

$$D(\alpha_i) = \max\{j \mid \alpha_{ij} = \infty\} \text{ with } \max\{\ \} = 0 .$$

By the nature of the question, we shall freely, without mention, replace U by any of its subsequences. Thus, we may assume that U satisfies the following conditions:

(1) $L(\alpha_1) \leq L(\alpha_2) \leq \ldots$, and

(2) $D(\alpha_1) \leq D(\alpha_2) \leq \ldots$.

Case 1. $D(\alpha_1) < D(\alpha_2) < \ldots$. Since $D(\alpha_i) \leq W(\alpha_i) < \infty$ for all i in this case, we may assume $D(\alpha_1) \leq W(\alpha_1) \leq D(\alpha_2) \leq W(\alpha_2) \leq \ldots$. Then, by (1), we have $\alpha_1 < \alpha_2 < \ldots$.

Case 2. $D(\alpha_1) = D(\alpha_2) = \ldots$. If $D(\alpha_1) = D(\alpha_2) = \ldots = \infty$, $\alpha_1 = \alpha_2 = \ldots$, and we are done. Otherwise, without loss of generality, we may assume $D(\alpha_1) = D(\alpha_2) = \ldots = 0$, that is, all vectors α_i do not have

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components which are ∞ .

Assume first that the components of all α_i are bounded above, or equivalently, $\alpha_{i1} = k$ for all i . We shall use induction on k . If $k = 0$, $\alpha_1 = \alpha_2 = \ldots$. Suppose $k \geq 1$, and let m_i be the number of components $= k$ in α_i . We may assume either $m_1 < m_2 < \ldots$ or $m_1 = m_2 = \ldots$. In the former case, since $m_i \leq W(\alpha_i) < \infty$, we have $m_1 \leq W(\alpha_1) \leq m_2 \leq W(\alpha_2) \leq \ldots$. Then, by (1), we have $\alpha_1 < \alpha_2 < \ldots$. In the latter case, if $m_1 = m_2 = \ldots = \infty$, we have $\alpha_1 = \alpha_2 = \ldots$. Otherwise, without loss of generality, we may assume $m_1 = m_2 = \ldots = 0$. Then, since $\alpha_{i1} < k$ for all i , we have $\alpha_1 \leq \alpha_2 \leq \ldots$ by the induction assumption.

Suppose next that the components of all α_i are not bounded above. Let m_i be the number of components $\geq \alpha_{11}$ in each α_i . We may assume either $m_2 < m_3 < \ldots$ or $m_2 = m_3 = \ldots$. In the former case, there exists an integer i such that $W(\alpha_1) < m_i$. Then, by (1), $\alpha_1 \leq \alpha_i$ and we are finished because of Lemma 1 *(b)*. In the latter case, if we have $m_2 = m_3 = \ldots = \infty$, we have $\alpha_1 \leq \alpha_2$ and we are done. Otherwise, without loss of generality, we may assume $m_1 = m_2 = \ldots = 0$. Then all the components are $< \alpha_{11}$ and hence, using the preceding result of the bounded case, we obtain $\alpha_1 \leq \alpha_2 \leq \ldots$. This completes the proof.

By a *row-decreasing* matrix α , let us mean an infinite matrix $\alpha = [\alpha_{ij}]_{i,j=1,2,\ldots}$ where $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots) \in V$ for all i . Let $K = \{1, 2, \ldots\}$. For each $J \subseteq K$, denote M_J the set of all row-decreasing matrices α such that $\alpha_i = (0, 0, \ldots)$ for all $i \in K-J$, and let $\leq$ be the componentwise ordering on M_J .

THEOREM 4. $(M_J, \leq)$ *is a complete lattice. Furthermore, it is partially well-ordered if, and only if,* $|J| < \infty$.

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Proof. The *if* part follows immediately from Lemma 3. For the *only if* part, assume $|J| = \infty$. For each $j \in J$, let β^j be the matrix $[\alpha_{ij}]$ consisting of all zero components except $\alpha_{j1} = 1$. Then, the sequence $\{\beta^j \mid j \in J\}$ contains no ascending subsequence and, hence, $(M_J, \leq)$ is not partially well-ordered. This finishes the proof.

4. Closed classes of abelian groups

Let us call C a *closed class* [5] if it is a nonempty class of abelian groups, closed under taking subgroups, direct limits, and isomorphic groups. Fuchs [3, p. 71] asked to determine all closed classes, and Hill [5] gave a solution in a group-theoretic argument. We shall redo this question in a combinatoric way using the results in the preceding sections. We feel our method is very different and much easier. Some terminologies are from [5] as indicated.

Let $p_2, p_3, p_4, \ldots$ be all the distinct positive prime integers with $p_1 = \infty$. For a closed class C, let $p(C)$ be the set of all p_i's such that there is a group in C containing an element of order p_i. $p(C)$ may be called the *associated primes* of C. To each row-decreasing matrix $\alpha = [\alpha_{ij}] \in M_J$ with $\alpha_{11} = \alpha_{12} = \ldots = \alpha_1$, assign an abelian group A given by

$$A = \left(\oplus_{\alpha_1} Z\right) \oplus \left(\oplus_{i\geq 2}\left(\oplus_{j\geq 1} Z\left(p_i^{\alpha_{ij}}\right)\right)\right) ,$$

and call A a *decreasing* group. Here, A is a direct sum of α_1 copies of Z and p_i-groups $Z\left(p_i^{\alpha_{ij}}\right)$, where $Z\left(p_i^{\alpha_{ij}}\right)$ is a cyclic group of order $p_i^{\alpha_{ij}}$ if α_{ij} is finite and a p_i-quasicyclic group if $\alpha_{ij} = \infty$, and $\alpha_1 = \infty$ should be understood as $\alpha_1 = \aleph_0$. Note that every finitely generated abelian group is a decreasing group and, hence, has a matrix representation in the above sense. If C is generated by a single group as a closed class, C is said to be *cyclic* [5]. We show the following theorem which determines all closed classes.

THEOREM 5. *Let* C *be any closed class of abelian groups. Then the following are true:*

(a) *C is generated by its subclass of all finitely generated groups;*

(b) *C is finitely generated if* $|p(C)| < \infty$;

(c) *C is the union of finitely many cyclic closed subclasses if* $|p(C)| < \infty$;

(d) *all closed classes of torsion-free groups are cyclic, and form a countable chain with regard to the inclusion* $\subseteq$;

(e) *a family* F *of closed classes satisfies the descending chain condition in* $(F, \subseteq)$ *if* $|\bigcup_{C \in F} p(C)| < \infty$;

(f) *a family* F *of closed classes does not contain a member which covers or cocovers infinitely many members in* $(F, \subseteq)$ *if* $|\bigcup_{C \in F} p(C)| < \infty$;

(g) *let* P *be a set of primes* p_i . *Then there are exactly countably (respectively, continuously) many closed classes* C *with* $p(C) \subseteq P$ *if* $|P| < \infty$ *(respectively,* $= \infty$*)* .

Proof. *(a)* This is obvious since a group is a direct limit of its finitely generated subgroups.

(b) Take all finitely generated groups of C and let M be the set of corresponding matrices of these groups. Clearly, $M \subseteq M_J$ with $|J| < \infty$ because $|p(C)| < \infty$. Then, $\max(M^*)$ is a finite set by Lemma 2 *(a)*. It is now easy to see that C is generated by the finitely many decreasing groups corresponding to $\max(M^*)$.

(c) C is the union of cyclic closed subclasses, each of which is generated by one of the finitely many decreasing groups corresponding to $\max(M^*)$.

(d) Note that, in this case, $\max(M^*)$ is a singleton set of $\alpha = [\alpha_{ij}]$ with $\alpha_{11} = \alpha_{12} = \ldots = \alpha_1$ and $\alpha_{ij} = 0$ for all $i \geq 2$.

Therefore, C is generated by a free group $\oplus_{\alpha_1} Z$. The rest is obvious.

(e), *(f)*, and *(g)* are obvious by Theorem 4 and Lemma 2 *(b)*, 2 *(c)*, and 2 *(d)*, respectively. This completes the proof.

It is easy to give counterexamples if we change the *if* condition in each of *(b)*, *(c)*, *(e)*, and *(f)*.

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