INDUCTIVE EXTENSION OF A VECTOR MEASURE UNDER A CONVERGENCE CONDITION

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1. Introduction. Let μ be a vector measure (countably additive set function with values in a Banach space) on a field. If μ is of bounded variation, it extends to a vector measure on the generated σ-field (**2**; **5**; **8**). Arsene and Strătilă (**1**) have obtained a result, which when specialized somewhat in form and context, reads as follows: "A vector measure on a field, majorized in norm by a positive, finite, subadditive increasing set function defined on the generated σ-field, extends to a vector measure on the generated σ-field". This includes the bounded variation case, for it suffices to take the total variation (extended) as the majorizing positive set function. Such results may be looked upon as possible steps toward the attainment of a vector measure extension theorem without any condition, if this is possible, or with a proven minimum condition, if not. In this paper we prove, by an intuitively simple induction, another such extension, under a convergence condition which is sufficient (as is to be proved) and also necessary (as will be obvious). It will include the bounded variation case. The intervals of Euclidean space constitute the simplest class on which it is natural to introduce a measure. Abstraction gives rise to the "semi-field" (semi-ring (**6**) of ring context). Accordingly, we begin with a vector measure defined on a semi-field.

2. Terminology and notation. As in (**3**), *generalized sequence* means Moore-Smith sequence, and "sequence" retains its elementary meaning. Unless the context indicates otherwise, a *set* is a subset of a fixed set S; the null set is denoted $\emptyset$. As in (**6**), a set of sets (subsets of S) is called a *class*. The members of a class Γ are Γ *sets*; a sequence of Γ sets is called a Γ *sequence*. In the context of sequences, or of generalized sequences, of sets, *convergence* means set-theoretical convergence. The convergence of a sequence of sets $\{E_n\}$ to a set E is denoted (as is convergence in other contexts): $\lim_n E_n = E$, or $E_n \rightarrow E$; if, further, the sequence increases: $E_n \subseteq E_{n+1}$ (decreases: $E_n \supseteq E_{n+1}$), the notation $E_n \uparrow E$ ($E_n \downarrow E$) may be employed. With reference to a class Γ, Γ_σ (Γ_δ) is the notation for the class of countable (including finite) unions (intersections) of Γ sets. The notation $E_1 + E_2 + \ldots$ ($\sum E_n$) may be employed to denote the *disjoint* union $E_1 \cup E_2 \cup \ldots$ ($\cup E_n$); in this case the union is referred to as a *sum*. A *set function* is a function μ whose domain $\mathscr{D}(\mu)$ is a class, and whose range is a subset of a Banach space. The condition of finite,

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or countable, additivity on a set function μ is applicable only to those disjoint sequences of $\mathscr{D}(\mu)$ sets whose sums belong to $\mathscr{D}(\mu)$. The term "field" refers to a field of sets whose maximum (with respect to inclusion) element is S. In the context of the lemmas "vector measure" (countably additive vector-valued set function) is abbreviated to "measure".

3. Extension from a semi-field to the generated field.

Definition. A *semi-field* is a class Λ such that (a) $\emptyset$, $S \in \Lambda$; (b) Λ is closed under finite intersections; (c) whenever A and B are Λ sets such that $A \subseteq B$, there exists a finite increasing sequence of Λ sets: $A = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_n = B$, such that $E_i - E_{i-1} \in \Lambda$ for $i = 1, 2, \ldots, n$. A field is a specialized semi-field. Given a semi-field Λ, let $\Sigma(\Lambda)$ denote the class of finite sums of Λ sets; then $\Sigma(\Lambda)$ is the field generated by Λ.

Definition. A vector measure μ on a semi-field Λ is *monotonely convergent* if, for every disjoint Λ sequence $\{E_n\}$, the series $\sum_n \mu(E_n)$ converges.

THEOREM 1. *A vector measure on a semi-field extends uniquely to a vector measure on the generated field. If the original vector measure is monotonely convergent, so is its extension.*

Proof. The first statement, for a positive measure, is essentially Theorem 8, E of (**6**). The generalization to a vector measure does not affect the proof: the association and inversion involving double sums remain valid operations since countable additivity refers to unordered summation. The second statement is obvious.

4. Extension from a field to the generated σ-field. For a vector measure μ on a field Σ, "monotone convergence" is equivalent to the following property:

"For every monotone Σ sequence $\{E_n\}$, $\{\mu(E_n)\}$ converges".
Without effect on the property, we may read "increasing" or "decreasing" for "monotone".

Definition. Let Γ be a class closed under finite unions and finite intersections, and let E be a set. The class $\Gamma^{(+)}(E) = \{A: A \in \Gamma, A \supseteq E\}$, if not empty, is directed by inclusion:

$$A \geqq B \quad \Leftrightarrow \quad A \subseteq B \qquad (A, B \in \Gamma^{(+)}(E)).$$

Let μ be a set function such that $\Gamma \subseteq \mathscr{D}(\mu)$ and $E \in \mathscr{D}(\mu)$. If $\Gamma^{+}(E)$ is non-empty and the generalized sequence $\{\mu(A)\}$, $A \in \Gamma^{(+)}(E)$, converges to $\mu(E)$, we will say that μ is *upper* Γ *continuous* at E. The set function μ will be called "upper Γ continuous" if it is so at every $\mathscr{D}(\mu)$ set. Similarly, with $\Gamma^{(-)}(E) = \{A: A \in \Gamma, A \subseteq E\}$, reversing inclusions, we define "lower Γ continuity".

The symbol (Σ, μ) will serve as abbreviation for the recurring hypothesis: "Σ is a field and μ is a monotonely convergent vector measure on Σ".

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LEMMA 1. *Under hypothesis* (Σ, μ), μ *extends to a finitely additive set function* λ *on* Σ_σ, *uniquely characterized by lower* Σ *continuity. For every increasing* Σ *sequence* $\{E_n\}$, $\lim_n \lambda(E_n) = \lambda(\lim_n E_n)$.

Proof. For $E \in \Sigma_\sigma$, the generalized sequence $\{\mu(A)\}$, $A \in \Sigma^{(-)}(E)$, converges. In fact, if this were not so, there would exist $\epsilon > 0$ such that, whenever E contains a Σ set A, there is also a Σ set B such that $A \subset B \subset E$,

$$||\mu(A) - \mu(B)|| \geqq \epsilon.$$

Accordingly, whenever a sequence of n Σ sets A_i has been established such that $A_1 \subset A_2 \subset \ldots \subset A_n \subset E$, $||\mu(A_i) - \mu(A_{i+1})|| \geqq \epsilon$ $(1 \leqq i < n)$, there is a Σ set A_{n+1} such that $A_n \subset A_{n+1} \subset E$, $||\mu(A_n) - \mu(A_{n+1})|| \geqq \epsilon$. This would prove inductively the existence of an increasing Σ sequence $\{A_n\}$ such that $||\mu(A_n) - \mu(A_{n+1})|| \geqq \epsilon$, which would contradict the monotone convergence. Defining $\lambda(E)$ to be the limit of the above generalized sequence, we obtain the lower Σ continuous extension λ of μ, of domain Σ_σ. The finite additivity is preserved in the passage to the limit, and uniqueness is clear. Suppose that $E_n \uparrow E$ $(E_n \in \Sigma)$; by the lower Σ continuity, for arbitrary $\epsilon > 0$ there exists a Σ set A contained in E such that

$$A \subseteq B \subseteq E, \qquad B \in \Sigma \Rightarrow ||\lambda(E) - \lambda(B)|| < \epsilon.$$

Since μ is a measure, we have that

$$\begin{aligned} ||\lambda(E) - \lambda(E_n)|| &\leqq ||\lambda(E) - \lambda(A)|| + ||\lambda(A) - \lambda(E_n)|| \\ &\leqq ||\lambda(E) - \lambda(A)|| + ||\mu(E_n - A)|| + ||\mu(A - E_n)|| \\ &< \epsilon + 2\epsilon + ||\mu(A - E_n)|| \to 3\epsilon \qquad (n \to \infty). \end{aligned}$$

LEMMA 1 (*equivalent dual form*). *Under hypothesis* (Σ, μ), μ *extends to a finitely additive set function* ν *on* Σ_δ, *uniquely characterized by upper* Σ *continuity. For every decreasing* Σ *sequence* $\{E_n\}$, $\lim_n \nu(E_n) = \nu(\lim_n E_n)$.

Proof. The duality is immediate by complementation, except for the finite additivity of ν. Let E and F be disjoint Σ_δ sets and limits, respectively, of decreasing Σ sequences $\{E_n\}$ and $\{F_n\}$. Since $(E_n \cap F_n) \downarrow \emptyset$, we have that

$$\nu(E + F) = \lim_n \mu(E_n \cup F_n) = \lim_n[\mu(E_n) + \mu(F_n) - \mu(E_n \cap F_n)] = \nu(E) + \nu(F).$$

Under hypothesis (Σ, μ), if $E \in \Sigma_\sigma \cap \Sigma_\delta$, then there are Σ sequences, $\{E_n\}$, $\{F_n\}$, increasing, decreasing, respectively, to E, therefore

$$\nu(E) - \lambda(E) = \lim_n[\mu(F_n) - \mu(E_n)] = \lim_n \mu(F_n - E_n) = 0.$$

Thus λ and ν combine to form a single extension (denoted by μ), whose domain is $\Sigma_\sigma \cup \Sigma_\delta$:

$$\mu(E) = \begin{cases} \lambda(E) & \text{if } E \in \Sigma_\sigma, \\ \nu(E) & \text{if } E \in \Sigma_\delta. \end{cases}$$

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This extension, which is finitely additive on each of the parts Σ_σ and Σ_δ of its domain, will henceforth be understood, in the context of the hypothesis (Σ, μ). Some of the lemmas to follow also have dual forms; these will not be stated, but appealed to as needed in proofs. Under the hypothesis (Σ, μ), Lemma 1 (with its dual) describes the behaviour of μ with respect to monotone Σ sequences. Under the same hypothesis, Lemmas 2 and 3 (with their duals) will describe the behaviour of μ (extended) with respect to monotone Σ_σ and Σ_δ sequences.

LEMMA 2. *Under hypothesis (Σ, μ), if a Σ_σ sequence $\{E_n\}$ decreases (decreases to $\emptyset$), then $\{\mu(E_n)\}$ converges (converges to 0).*

Proof. By the lower Σ continuity, for arbitrary $\epsilon > 0$, there exists a Σ sequence $\{F_n\}$ such that $F_n \subseteq E_n$ and

$$F_n \subseteq A \subseteq E_n, \qquad A \in \Sigma \Rightarrow ||\mu(E_n) - \mu(A)|| < \epsilon/2^n,$$

so that

$$A \subseteq E_n - F_n, \qquad A \in \Sigma \Rightarrow ||\mu(A)|| < \epsilon/2^n.$$

Write

$$A_n = \bigcap_1^n F_i,$$

then for $n > 1$,

$$F_n - A_n = [F_n - (F_n \cap F_1)] + [(F_n \cap F_1) - (F_n \cap F_1 \cap F_2)] + \ldots \\ + [(F_n \cap F_1 \cap \ldots \cap F_{n-2}) - (F_n \cap F_1 \cap \ldots \cap F_{n-1})].$$

The kth term of the sum is a Σ set contained in $E_k - F_k$, thus the norm of its measure is less than $2\epsilon/2^k$, hence $||\mu(F_n - A_n)|| < 2\epsilon$. Therefore

$$||\mu(E_n) - \mu(A_n)|| \leqq ||\mu(E_n) - \mu(F_n)|| + ||\mu(F_n - A_n)|| < 3\epsilon.$$

Since ϵ is arbitrary and $\{\mu(A_n)\}$ converges (converges to 0), the conclusions follow.

LEMMA 3. *Under hypothesis (Σ, μ), for every increasing Σ_σ sequence $\{E_n\}$,* $\lim_n \mu(E_n) = \mu(\lim_n E_n)$.

Proof. For arbitrary $\epsilon > 0$ let $\{F_n\}$ be a Σ sequence such that $F_n \subseteq E_n$ and $F_n \subseteq A \subseteq E_n$, $A \in \Sigma \Rightarrow ||\mu(E_n) - \mu(A)|| < \epsilon$. Each E_n is the limit of an increasing Σ sequence $\{A_{nm}\}$, $m = 1, 2, \ldots$, such that $F_n \subseteq A_{nm} \subseteq E_n$. Write

$$B_{nm} = \bigcup_{i=1}^n A_{im}, \qquad D_n = \bigcup_{i=1}^n B_{ii}.$$

Then $F_n \subseteq D_n \subseteq E_n$ and $r \leqq s \leqq m \Rightarrow B_{rm} \subseteq B_{sm}$. If p is an arbitrary point of $E = \lim_n E_n$, there is an index r such that $p \in E_r$, then there is an index $s > r$ such that $p \in A_{rs} \subseteq B_{rs} \subseteq B_{ss} \subseteq D_s$. This proves that $D_n \uparrow E$, then,

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by Lemma 1, $\mu(D_n) \to \mu(E)$, and the conclusion follows from the following inequality:

$$\begin{aligned} ||\mu(E) - \mu(E_n)|| &\leqq ||\mu(E) - \mu(D_n)|| + ||\mu(D_n) - \mu(E_n)|| \\ &< ||\mu(E) - \mu(D_n)|| + \epsilon. \end{aligned}$$

The following weak additivity relation will suffice for present purposes.

LEMMA 4. *Under hypothesis* (Σ, μ), *if* $E \in \Sigma$, $F \in \Sigma_\sigma$, *and* $E \supseteq F$, *then* $\mu(E - F) + \mu(F) = \mu(E)$.

Proof. Let $F_n \uparrow F$ $(F_n \in \Sigma)$. It suffices to consider the equation

$$\mu(E - F_n) + \mu(F_n) = \mu(E)$$

in the limit $(n \to \infty)$, applying Lemma 1 and its dual.

Definition. A vector measure μ on a field Σ is *null convergent* if $E_n \to \emptyset$, $(E_n \in \Sigma) \Rightarrow \mu(E_n) \to 0$.

LEMMA 5. *Let* μ *be a null convergent measure on a field* Σ, *then if* $\{E_n\}$ *is a convergent* Σ *sequence (whose limit need not be a* Σ *set), then* $\{\mu(E_n)\}$ *converges.*

Proof. The double sequence $\{\mu(E_m - E_n)\}$, $m, n = 1, 2, \ldots$, converges to 0; for if not, for some $\epsilon > 0$ and some pair of strictly increasing sequences of integers $\{m_k\}$, $\{n_k\}$, we have that $||\mu(E_{m_k} - E_{n_k})|| \geqq \epsilon$ $(k = 1, 2, \ldots)$, and since $E_{m_k} - E_{n_k} \to \emptyset$, this would contradict the null convergence. Now the convergence of $\{\mu(E_n)\}$ follows from the following inequality:

$$||\mu(E_m) - \mu(E_n)|| \leqq ||\mu(E_m - E_n)|| + ||\mu(E_n - E_m)||.$$

THEOREM 2. *For a vector measure on a field, null convergence and monotone convergence are equivalent properties.*

Proof. It remains to show that monotone convergence implies null convergence, since the converse is contained in Lemma 5. We shall apply the following part of the dual of Lemma 2 (obtained by complementation, with the aid of Lemma 4):

"Under hypothesis (Σ, μ), if a Σ_δ sequence $\{E_n\}$ increases, then $\{\mu(E_n)\}$ converges".

Let μ be a vector measure on a field Σ which is not null convergent. Then for some $\epsilon > 0$ there exists a Σ sequence $\{E_n\}$ such that $||\mu(E_n)|| \geqq \epsilon$ $(n = 1, 2, \ldots)$ and $E_n \to \emptyset$. Write $A_n = \bigcup_{i=n}^{\infty} E_i$. Since the Σ_σ sequence $\{E_1 \cap A_n\}$ decreases to $\emptyset$, we have that $\mu(E_1 \cap A_n) \to 0$ (Lemma 2). Let n_1 be an index such that $||\mu(E_1 \cap A_{n_1})|| < \frac{1}{2}\epsilon$. Then $F_1 = E_1 - (E_1 \cap A_{n_1})$ is a Σ_δ set such that $||\mu(F_1)|| > \frac{1}{2}\epsilon$ (Lemma 4). By the same argument, for some $n_2 > n_1$, $A_{n_1} - A_{n_2}$ contains a Σ_δ set F_2 such that $||\mu(F_2)|| > \frac{1}{2}\epsilon$, and so on, inductively. Hence there exists a disjoint Σ_δ sequence $\{F_n\}$ such that $||\mu(F_n)|| > \frac{1}{2}\epsilon$ $(n = 1, 2, \ldots)$, in which case, the series $\sum_1^\infty \mu(F_n)$ diverges. Writing $G_n = \sum_1^n F_i$, we have

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the increasing Σ_δ sequence $\{G_n\}$ such that $\{\mu(G_n)\}$ diverges. Thus, by the dual of Lemma 2, μ is not monotonely convergent.

Definition. Let Σ be a field; the class of limits of convergent Σ sequences, which is obviously a field containing Σ, is called the *limit field* of Σ, and is denoted by $\bar{\Sigma}$. We note that the field Σ is a σ-field if and only if $\Sigma = \bar{\Sigma}$.

LEMMA 6. *Under hypothesis* (Σ, μ), μ *extends to a finitely additive set function* $\bar{\mu}$ *on* $\bar{\Sigma}$, *uniquely determined by the following condition*:

$$E_n \to E, \qquad (E_n \in \Sigma) \Rightarrow \mu(E_n) \to \bar{\mu}(E).$$

Proof. Let E be an arbitrary $\bar{\Sigma}$ set and let $\{E_n\}$ be a Σ sequence converging to E. Then $\{\mu(E_n)\}$ converges (Theorem 2, Lemma 5), and it follows from the null convergence that the limit is independent of the particular Σ sequence converging to E. The required extension is therefore defined by the following formula:

$$\bar{\mu}(E) = \lim_n \mu(E_n), \quad \text{where } E_n \to E \quad (E_n \in \Sigma,\ E \in \bar{\Sigma}).$$

It remains to verify the finite additivity of $\bar{\mu}$, but this follows as in the proof of the dual of Lemma 1, applying, this time, the null convergence. The extension $\bar{\mu}$ extends the previous extensions λ and ν. Henceforth, in the context (Σ, μ), the extension $\bar{\mu}$ will be understood, and we will write μ instead of $\bar{\mu}$.

LEMMA 7. *Under hypothesis* (Σ, μ),

$$E_n \to \emptyset, \qquad (E_n \in \Sigma_\sigma) \Rightarrow \mu(E_n) \to 0.$$

Proof. For arbitrary $\epsilon > 0$, there is a Σ sequence $\{F_n\}$ such that $F_n \subseteq E_n$ and $||\mu(F_n) - \mu(E_n)|| < \epsilon$ (Lemma 1), and the conclusion follows from the null convergence.

LEMMA 8. *Under hypothesis* (Σ, μ), *every* $\bar{\Sigma}$ *set* E *is the limit of a decreasing* Σ_σ *sequence* $\{E_n\}$ *such that* $\mu(E_n) \to \mu(E)$.

Proof. Let $\{F_n\}$ be a Σ sequence converging to E and write $E_n = \bigcup_{i=n}^{\infty} F_i$. Then $E_n \downarrow E$ $(E_n \in \Sigma_\sigma)$, $\mu(F_n) \to \mu(E)$, and, by Lemma 7,

$$\mu(E_n) - \mu(F_n) = \mu(E_n - F_n) \to 0.$$

LEMMA 9. *Under hypothesis* (Σ, μ),

$$E_n \downarrow E, \qquad (E_n \in \Sigma_\sigma,\ E \in \bar{\Sigma}) \Rightarrow \mu(E_n) \to \mu(E).$$

Proof. By the dual of Lemma 8, E is the limit of an increasing Σ_δ sequence $\{F_n\}$ such that $\mu(F_n) \to \mu(E)$, and the Σ_σ sequence $\{E_n - F_n\}$ converges to $\emptyset$, thus by Lemma 7 (or Lemma 2),

$$\mu(E_n) - \mu(F_n) = \mu(E_n - F_n) \to 0.$$

LEMMA 10. *Under hypothesis* (Σ, μ), *the additive extension* μ *on* $\bar{\Sigma}$ *is upper* Σ_σ *continuous.*

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Proof. Let E be a given $\bar{\Sigma}$ set. The class $\Sigma_\sigma^{(+)}(E) = \{A\colon A \in \Sigma_\sigma, A \supseteq E\}$ is not empty, since Σ is a field and $S \in \Sigma_\sigma^{(+)}(E)$ and the generalized sequence $\{\mu(A)\}$, $A \in \Sigma_\sigma^{(+)}(E)$, converges to a vector x. For if this were not so, there would exist a decreasing Σ_σ sequence $\{F_n\}$ such that $\{\mu(F_n)\}$ diverges, contrary to Lemma 2. Let $\{E_n\}$ be a Σ_σ sequence decreasing to E (Lemma 8); we can construct inductively a decreasing Σ_σ sequence $\{A_n\}$ such that $E_n \supseteq A_n \supseteq E$ and $||\mu(A_n) - x|| < n^{-1}$. By Lemma 9, $\mu(A_n) \to \mu(E)$, thus $x = \mu(E)$.

LEMMA 11. *Under hypothesis* (Σ, μ), *suppose that* μ *has been extended to a finitely additive, upper* Σ_σ *continuous set function (denoted by* μ*) on a field* Γ *containing* Σ. *Then the extension* μ *is a monotonely convergent measure on* Γ.

Proof. The conclusion is equivalent to the conjunction of the following two affirmations:

(a) If $\{E_n\}$ is an increasing Γ sequence, then $\{\mu(E_n)\}$ converges.

(b) If $E_n \uparrow E$ $(E_n \in \Gamma, E \in \Gamma)$, then $\mu(E_n) \to \mu(E)$.

Proof of (a). By the upper Σ_σ continuity, for $\epsilon > 0$ arbitrary, there is a Σ_σ sequence $\{F_n\}$ such that

$$F_n \supseteq E_n \quad \text{and} \quad F_n \supseteq A \supseteq E_n, \qquad A \in \Sigma_\sigma \Rightarrow ||\mu(E_n) - \mu(A)|| < \epsilon/2^n.$$

Write $G_n = \bigcup_1^n F_i$, $G = \bigcup_1^\infty F_n$, so that, by Lemma 3, $\mu(G_n) \to \mu(G)$. For $n > 1$ $(F_0 = \emptyset)$, we have that

$$G_n - F_n = [(F_n \cup F_1) - F_n] + [(F_n \cup F_1 \cup F_2) - (F_n \cup F_1)] + \ldots$$
$$+ [(F_n \cup F_1 \cup \ldots \cup F_{n-1}) - (F_n \cup F_1 \cup \ldots \cup F_{n-2})].$$

The kth term of the sum is a proper difference of Σ_σ sets of the form $D_k = A_k - B_k$, where $B_k \supseteq E_k$ and $D_k \subseteq F_k - E_k$. Replacing A_k and B_k by their intersections with F_k (which does not effect D_k), we may suppose, further, that $E_k \subseteq B_k \subseteq A_k \subseteq F_k$. It follows that $||\mu(D_k)|| < 2\epsilon/2^k$, and therefore $||\mu(G_n - F_n)|| < 2\epsilon$. The affirmation (a) then follows from the following inequality:

$$||\mu(E_n) - \mu(G_n)|| \leqq ||\mu(E_n) - \mu(F_n)|| + ||\mu(G_n - F_n)|| < 3\epsilon,$$

with ϵ arbitrary and $\{\mu(G_n)\}$ convergent.

Proof of (b). We may apply the proof of (a) with the added hypothesis that there exists a Σ_σ set F containing E such that:

$$E \subseteq A \subseteq F, \qquad A \in \Sigma_\sigma \Rightarrow ||\mu(E) - \mu(A)|| < \epsilon.$$

We may then further suppose that $F_n \subseteq F$ $(n = 1, 2, \ldots)$ so that $E \subseteq G \subseteq F$, and therefore $||\mu(G) - \mu(E)|| < \epsilon$, and thus, finally,

$$||\lim_n \mu(E_n) - \mu(E)|| < 4\epsilon.$$

THEOREM 3. *A monotonely convergent vector measure on a field extends uniquely to a monotonely convergent vector measure on its limit field.*

Proof. Under hypothesis (Σ, μ), because of Lemma 10, the hypothesis of Lemma 11 is satisfied for $\Gamma = \bar{\Sigma}$, thus the finitely additive extension μ on $\bar{\Sigma}$ (Lemma 6) is a monotonely convergent measure. If ν is any monotonely convergent measure on $\bar{\Sigma}$, because of the null convergence, we have that

$$E_n \to E, \ (E_n \in \Sigma, E \in \bar{\Sigma}) \Rightarrow \nu(E_n) \to \nu(E).$$

This implies the uniqueness of the measure extension.

In the context (Σ, μ), we understand the extension of μ to a finitely additive function (denoted by μ) on $\bar{\Sigma}$; following Theorem 3, we understand this extension as a monotonely convergent measure on $\bar{\Sigma}$. The final induction will be carried out with the aid of Lemmas 11 and 13, below.

LEMMA 12. *Under hypothesis (Σ, μ), suppose that μ has been extended to a monotonely convergent measure (denoted by μ) on a field Γ containing Σ. Suppose, further, that μ is upper Σ_σ continuous at every Γ set. Then μ is upper Σ_σ continuous at every Γ_σ set.*

Proof. Suppose that $E_n \uparrow E$ $(E_n \in \Gamma)$. Since μ is upper Σ_σ continuous at every E_n, as in the first part of the proof of Lemma 11, there exists, for arbitrary $\epsilon > 0$, an increasing Σ_σ sequence $\{G_n\}$ such that $E_n \subseteq G_n$ and $||\mu(E_n) - \mu(G_n)|| < \epsilon$. Write $G = \lim_n G_n$. Since μ is also a measure on $\bar{\Gamma}$ (extension of Theorem 3) we may pass to the limit to obtain

$$||\mu(E) - \mu(G)|| \leqq \epsilon.$$

Let A be any Σ_σ set such that $E \subseteq A \subseteq G$. It is clear from the proof of Lemma 11, that if we replace each G_n of the above argument by $G_n' = G_n \cap A$, we still have $||\mu(E_n) - \mu(G_n')|| < \epsilon$. Since G is replaced by A, the conclusion is $||\mu(E) - \mu(A)|| \leqq \epsilon$.

LEMMA 13. *Under the hypothesis of Lemma 12, μ is upper Σ_σ continuous at every $\bar{\Gamma}$ set.*

Proof. Let E be a given $\bar{\Gamma}$ set and let $\epsilon > 0$ be arbitrary. Since μ is upper Γ_σ continuous at E (Lemma 10), there is a Γ_σ set F such that

$$E \subseteq F \quad \text{and} \quad E \subseteq A \subseteq G, \qquad A \in \Gamma_\sigma \Rightarrow ||\mu(E) - \mu(A)|| < \epsilon.$$

By the conclusion and Lemma 12, μ is upper Σ_σ continuous at F; thus there is a Σ_σ set B such that

$$F \subseteq B \quad \text{and} \quad F \subseteq C \subseteq B, \qquad C \in \Sigma_\sigma \Rightarrow ||\mu(F) - \mu(C)|| < \epsilon.$$

Let D be any Σ_σ set such that $E \subseteq D \subseteq B$. Then

$$\begin{aligned}||\mu(D) - \mu(E)|| &\leqq ||\mu(D) - \mu(D \cap F)|| + ||\mu(D \cap F) - \mu(E)|| \\ &< ||\mu(D \cup F) - \mu(F)|| + \epsilon.\end{aligned}$$

Since μ is upper Σ_σ continuous at the Γ_σ set $D \cup F$ (Lemma 12) there is a Σ_σ

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set H such that $B \supseteq H \supseteq D \cup F$ and $||\mu(H) - \mu(D \cup F)|| < \epsilon$. But then $B \supseteq H \supseteq F$, $H \in \Sigma_\sigma$, thus

$$||\mu(H) - \mu(F)|| < \epsilon,$$

$$||\mu(D \cup F) - \mu(F)|| \leqq ||\mu(D \cup F) - \mu(H)|| + ||\mu(H) - \mu(F)|| < 2\epsilon,$$

and we have, finally, that $||\mu(D) - \mu(E)|| < 3\epsilon$.

THEOREM 4. *A monotonely convergent vector measure on a semi-field extends uniquely to a vector measure on the generated σ-field.*

Proof. Because of Theorem 1, we may start with hypothesis (Σ, μ). Let Σ' denote the σ-field generated by Σ. Let Φ be the set of all pairs (Γ, λ) such that

(a) Γ is a field such that $\Sigma \subseteq \Gamma \subseteq \Sigma'$,

(b) λ is a monotonely convergent, upper Σ_σ continuous vector measure on Γ, which extends μ, and

(c) λ is the only measure on Γ extending μ.

The non-null set Φ $((\Sigma, \mu) \in \Phi)$ is partially ordered: for (Γ, λ), $(\Gamma', \lambda') \in \Phi$, $(\Gamma, \lambda) \leqq (\Gamma', \lambda')$ means that $\Gamma \subseteq \Gamma'$ and λ' extends λ. In order to apply Zorn's lemma, we will show that Φ is inductive, that is, an arbitrary totally ordered non-null subset Ψ is bounded above in Φ. If $(\Gamma, \lambda) \in \Psi$ we will say that Γ is a "Ψ-field" and that λ is a "Ψ-measure". The union of the Ψ-fields is a field Γ_0; and the upper Σ_σ continuous finitely additive set function λ_0 on Γ_0, extending μ, is well-defined if we set, for $E \in \Gamma_0$, $\lambda_0(E) = \lambda(E)$, where (Γ, λ) is any element of Ψ such that $E \in \Gamma$. But then λ_0 is a monotonely convergent measure (Lemma 11). If ν is any measure on Γ_0 extending μ, then by the definition of Φ, ν extends every Ψ-measure, thus $\nu = \lambda_0$. Hence (Γ_0, λ_0) is an element of Φ, and is the required upper bound of Ψ. The inductivity established, let (Λ, ν) be a maximal element of Φ. Suppose that $\Lambda \subset \Sigma'$. Then $\Lambda \subset \bar{\Lambda} \subseteq \Sigma'$, and by Theorem 3, ν extends uniquely to a monotonely convergent vector measure $\bar{\nu}$ on $\bar{\Lambda}$. Since $(\Lambda, \nu) \in \Phi$, $\bar{\nu}$ is upper Σ_σ continuous at every Λ set, so also at every $\bar{\Lambda}$ set (Lemma 13). Let λ be any vector measure on $\bar{\Lambda}$ extending μ; by the uniqueness of ν, the restriction of λ to Λ is ν, and therefore $\lambda = \bar{\nu}$ (uniqueness condition of Theorem 3). Hence $(\bar{\Lambda}, \bar{\nu}) \in \Phi$, $(\bar{\Lambda}, \bar{\nu}) > (\Lambda, \nu)$, this contradiction showing that $\Lambda = \Sigma'$, and therefore ν is the required extension.

Theorem 4 includes the following known bounded variation case:

COROLLARY. *A vector measure of bounded variation on a semi-field extends uniquely to a vector measure of bounded variation on the generated σ-field.*

Proof. "Bounded variation" is a property of a vector measure on a semi-field (defined there as on a field (**3**)) if and only if it is a property of its extension (Theorem 1) to the generated field. Thus we may suppose the vector measure μ of bounded variation defined on a field Σ. The total variation v of μ is a positive (finite) measure, thus obviously monotonely convergent, hence null convergent (Theorem 2). But then, since $v(E) \geqq ||\mu(E)||$ for all $E \in \Sigma$, μ is null convergent, hence monotonely convergent (Theorem 2), thus Theorem 4 applies to μ. Let $\bar{\mu}$ and $\bar{v}$ be the extensions of μ and v, respectively, to measures on the generated σ-field Σ'; it is seen inductively that $\bar{v}(E) \geqq ||\bar{\mu}(E)||$ for all $E \in \Sigma'$, thus $\bar{\mu}$ is of bounded variation.

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