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ON FUNCTIONS WITH DERIVATIVE OF BOUNDED VARIATION: AN ANALOGUE OF BANACH'S INDICATRIX THEOREM

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1. Statement of the result

A simple, but nice theorem of Banach states that the variation of a continuous function $F:[a,b]\to\mathbb{R}$ is given by $\int_{-\infty}^{\infty} t(y)\,dy$, where $t(y)$ is defined as the number of $x\in[a,b]$ for which $F(x)=y$ (see, e.g., [**1**], VIII.5, Th. 3). In this paper we essentially derive a similar representation for the variation of F' which also yields a criterion for a function to be an integral of a function of bounded variation. The proof given here is quite elementary, though long and somewhat intriciate.

$$\text{Let } -\infty<a<b<\infty, \quad F\colon\ [a,b]\to\mathbb{R} \text{ be continuous.}$$

For any real function G on $[a,b]$ we denote by $v(G)$ its variation and by $l(G)$ the length of its graph; further $\|G\|:=\sup\{|G(x)|\,|\,x\in[a,b]\}$.

$$\text{Let } a_{nj}:=a+(b-a)j2^{-n}, \quad D_n:=\{a_{nj}\,|\,j=1,\dots,2^n-1\}.$$

For $\alpha>0$ we define $\mathcal{F}_\alpha(F):=\{G:[a,b]\to\mathbb{R}\,|\,\|F-G\|\leqq\alpha,\ G(a)==F(a),\ G(b)=F(b)\}$.

We consider $F_\alpha^n:[a,b]\to\mathbb{R}$ which is defined to be that function H satisfying $H(a)=F(a)$, $H(b)=F(b)$, $F(a_{nj})-\alpha\leqq H(a_{nj})\leqq F(a_{nj})+\alpha$ $(j=1,\dots,2^n-1)$ which has minimal length. Clearly F_α^n is piecewise linear and continuous (see Fig. 1). We shall show that

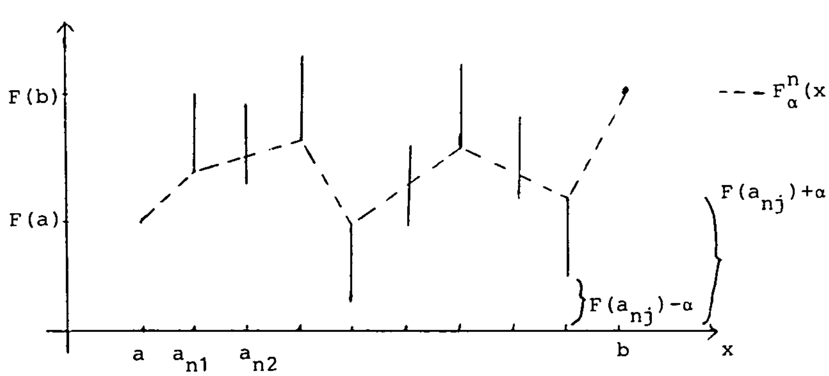

FIGURE 1

$F_\alpha := \lim_{n\to\infty} F_\alpha^n$ exists (pointwise), and F_α is uniquely determined by

$$l(F_\alpha) = \inf\{l(G) \mid G \in \mathscr{F}_\alpha(F)\}. \tag{1}$$

F_α can be visualized as a thread fastened to the point $(a, F(a))$ and drawn as tautly as possible in the region

$$\{(x,y)\in\mathbb{R}^2 \mid x\in[a,b], F(x)-\alpha \leqq y \leqq F(x)+\alpha\}$$

such that it passes through $(b, F(b))$.

Let $F^0(x)$ be the straight line joining $(a, F(a))$ and $(b, F(b))$,

$$\alpha_0 := \sup\{|F(x)-F^0(x)| \mid x\in[a,b]\}.$$

Suppose $\alpha_0 > 0$. It will be proved that for $\alpha\in(0,\alpha_0)$ there is a finite number of open intervals $J_{1\alpha}, \ldots, J_{k_\alpha,\alpha} \subset [a,b]$ (ordered from left to right) with the following properties:

(i) $|F_\alpha(x) - F(x)| < \alpha$ for all $x \in \bigcup_{i=1}^{k_\alpha} J_{i\alpha}$

(ii) Let $J_{i\alpha} = (x_{i\alpha}, x'_{i\alpha})$. Then for $i = 2, \ldots, k_\alpha - 1$ either
$F_\alpha(x_{i\alpha}) - F(x_{i\alpha}) = \alpha$ and $F_\alpha(x'_{i\alpha}) - F(x'_{i\alpha}) = -\alpha$ or
$F_\alpha(x_{i\alpha}) - F(x_{i\alpha}) = -\alpha$ and $F_\alpha(x'_{i\alpha}) - F(x'_{i\alpha}) = \alpha$; further
$x_{1\alpha} = a$, $F_\alpha(x'_{1\alpha}) - F(x'_{1\alpha}) = \pm\alpha$ and $x'_{k_\alpha,\alpha} = b$,
$F_\alpha(x_{k_\alpha,\alpha}) - F(x_{k_\alpha,\alpha}) = \pm\alpha$.

Let for $\alpha \geqq \alpha_0$ $s(\alpha) := 0$ and for $\alpha\in(0,\alpha_0)$

$$s(\alpha) := [4(x'_{1\alpha} - a)]^{-1} + (x'_{2\alpha} - x_{2\alpha})^{-1} + \cdots + (x'_{k_\alpha-1,\alpha} - x_{k_\alpha-1,\alpha})^{-1} + [4(b - x_{k_\alpha,\alpha})]^{-1}. \tag{2}$$

$s(\alpha)$ will be seen to be monotone decreasing. It measures how often F_α varies from $F+\alpha$ to $F-\alpha$ and vice versa and how fast this happens.

Now we can formulate the result.

Theorem. *If $\int_0^\varepsilon s(\alpha)\,d\alpha < \infty$ for some $\varepsilon > 0$, there is a $f: [a,b] \to \mathbb{R}$ such that*

$$F(x) = F(a) + \int_a^x f(t)\,dt \quad \text{for all } x\in[a,b] \tag{3}$$

$$v(f) = 4\int_0^\infty s(\alpha)\,d\alpha. \tag{4}$$

If there is a $f:[a,b]\to\mathbb{R}$ *of bounded variation satisfying* (3), *there exists a* $\tilde{f}:[a,b]\to\mathbb{R}$ *such that* $f=\tilde{f}$ *almost everywhere and* $v(\tilde{f})=4\int_0^\infty s(\alpha)\,d\alpha<\infty$.

2. Proof of the Theorem

We have subdivided the proof into a number of separate steps.

(a) It is clear that $\sup_n l(F_\alpha^n)<\infty$ so that also $\sup_n v(F_\alpha^n)<\infty$. By Helly's extraction theorem ([**1**], p. 250), there is a pointwise convergent subsequence $F_\alpha^{n_j}\to F_\alpha$, and we have $l(F_\alpha)\leqq\liminf_{j\to\infty} l(F^{n_j\alpha})$. Each $G:[a,b]\to$ for which $\|F-G\|\leqq\alpha$ and $G(a)=F(a)$, $G(b)=F(b)$ satisfies $l(G)\geqq l(F_\alpha^n)$ for all $n\in\mathbb{N}$ so that $l(G)\geqq l(F_\alpha)$. Thus (1) holds. We shall show that (1) uniquely determines F_α thus getting $F_\alpha^n\to F_\alpha$.

(b) F_α is continuous on $[a,b]$. Indeed, since $v(F_\alpha)<\infty$, $F_\alpha(x+)$ and $F_\alpha(x-)$ exist for all $x\in(a,b)$. Suppose, e.g., $F_\alpha(x_1-)<F_\alpha(x_1+)$ for some $x_1\in(a,b)$. Define $\tilde{F}_\alpha^\varepsilon(x):=F_\alpha(x)$ for $x\notin[x_1,x_1+\varepsilon)$, $\tilde{F}_\alpha^\varepsilon(x_1):=\frac{1}{2}F_\alpha(x_1-)+\frac{1}{2}F_\alpha(x_1+)$, $\tilde{F}_\alpha^\varepsilon$ linear on $[x_1,x_1+\varepsilon)$ and $\lim_{x\to x_1+\varepsilon-}\tilde{F}_\alpha^\varepsilon(x):=F_\alpha(x_1+\varepsilon-)$.

Then $\tilde{F}_\alpha^\varepsilon\in\mathscr{F}_\alpha(F)$ for small $\varepsilon>0$ and $l(\tilde{F}_\alpha^\varepsilon)<l(F_\alpha)$, a contradiction. The continuity of F_α in a and b is proved similarly.

(c) By (b), $A_\alpha:=\{x\in[a,b]\,|\,F_\alpha(x)-F(x)<\alpha\}$ is open. F_α is concave on each component of A_α. To see this, let x_1, $x_2\in D_N$ for some N with the properties $U:=(x_1\cdot x_2)\subset A_\alpha$ and $\sup_U(F(x)-\alpha)<\inf_U(F(x)+\alpha)$. For $n\geqq N$ let G_n be the smallest concave function on U satisfying $G_n(x)\geqq F(x)-\alpha$ for all $x\in D_n\cap(x_1,x_2)$, $G_n(x_1)=F_\alpha(x_1)$, $G_n(x_2)=F_\alpha(x_2)$. Obviously we have $G_n\leqq F+\alpha$ and $G_n\leqq F_\alpha$ (in U); G_n is an increasing sequence, and the limit $G:=\lim_{n\to\infty}G_n$ is a concave function on U for which $G\leqq F_\alpha$, $F-\alpha\leqq G\leqq F+\alpha$, $G(x_1)=F_\alpha(x_1)$, $G(x_2)=F_\alpha(x_2)$. So we must have $G=F_\alpha$ on U.

(d) Similarly as in (c) it is seen that F_α is convex on each component of the (by (a)) open set $B_\alpha:=\{x\in[a,b]\,|\,F_\alpha(x)-F(x)>-\alpha\}$. Let $\mathscr{I}_\alpha(\tilde{\mathscr{I}}_\alpha)$ be the set of all intervals $[x,y]\subset[a,b]$ such that $x,y\in A_\alpha^c$ and $[x,y]\cap B_\alpha^c=\varnothing$ ($x,y\in B_\alpha^c$ and $[x,y]\cap A_\alpha^c=\varnothing$). Let $I_\alpha:=\bigcup\{I\,|\,I\in\mathscr{I}_\alpha\}$, $\tilde{I}_\alpha:=\bigcup\{I\,|\,I\in\tilde{\mathscr{I}}_\alpha\}$, $J_\alpha:=[a,b]\setminus(I_\alpha\cup\tilde{I}_\alpha)$. On the components of $I_\alpha\cup J_\alpha$ resp. $\tilde{I}_\alpha\cup J_\alpha$ resp. $J_\alpha F_\alpha$ is convex resp. concave resp. linear. The number of components of I_α, $\tilde{I}_\alpha$ and J_α is finite (otherwise there exist sequences $x_i\in\partial I_\alpha$, $\tilde{x}_i\in\partial\tilde{I}_\alpha$ such that $x_i-\tilde{x}_i\to0$ $(i\to\infty)$ so that $F_\alpha(x_i)-F_\alpha(\tilde{x}_i)\to0$; this yields a contradiction, because $F_\alpha(x_i)=F(x_i)-\alpha$, $F_\alpha(\tilde{x}_i)=F(\tilde{x}_i)+\alpha$).

(e) By (d), F_α is absolutely continuous. If $\bar{F}_\alpha$ is a solution of (1) for which $F_\alpha\not\equiv\bar{F}_\alpha$, $\bar{F}_\alpha$ is also absolutely continuous, and

$$
\begin{aligned}
l(\tfrac{1}{2}F_\alpha+\tfrac{1}{2}\bar{F}_\alpha)&=\int_a^b[1+(\tfrac{1}{2}F_\alpha'(x)+\tfrac{1}{2}\bar{F}_\alpha'(x))^2]^{1/2}\,dx\\
&<\int_a^b[\tfrac{1}{2}[1+F_\alpha'(x)^2]^{1/2}+\tfrac{1}{2}(1+\bar{F}_\alpha'(x)^2)^{1/2}]\,dx\\
&=\tfrac{1}{2}l(F_\alpha)+\tfrac{1}{2}l(\bar{F}_\alpha)=l(F_\alpha).
\end{aligned}
\tag{5}
$$

Since $\frac{1}{2}F_\alpha+\frac{1}{2}\tilde{F}_\alpha\in\mathscr{F}_\alpha(F)$, (5) contradicts (1). So F_α is uniquely determined by (1), and $F_\alpha^n\to F_\alpha$ $(n\to\infty)$.

(f) J_α is increasing, I_α and $\tilde{I}_\alpha$ are decreasing with respect to α. Consider for example A_α^c. Suppose on the contrary that for some $\alpha<\beta$ and $x_0\in(a,b)$ we have $F_\beta(x_0)=F(x_0)+\beta$ and $F_\alpha(x_0)<F(x_0)+\alpha$. Let $U=(x_1,x_2)$ be the maximal interval in $[a,b]$ containing x_0 such that $F_\beta(x)-\beta>F_\alpha(x)-\alpha$ for $x\in U$. Clearly $a<x_1<x_0<x_2<b$, and F_α is concave on U (otherwise there is a $x_3\in U$ for which $F_\alpha(x_3)=F(x_3)+\alpha$, but then $F_\beta(x_3)-\beta>F(x_3)$, a contradiction). Note that for $x\in U$

$$F_\beta(x)-\beta>F_\alpha(x)-\alpha\geqq F(x)-2\alpha$$

$$>F(x)-2\beta \tag{6}$$

so that $F_\beta(x)>F(x)-\beta$. Consequently $U\subset B_\beta$, and F_β is convex on U. The convex F_β coincides with the concave $F_\alpha+\beta-\alpha$ at x_1 and at x_2, so $F_\beta=F_\alpha+\beta-\alpha$ on U. This is a contradiction to the definition of U.

(g) It follows from (f) that s, as defined by (2), is a monotone decreasing function (note that the construction of F_α^n shows that $J_\alpha\neq\varnothing$ for $\alpha<\alpha_0$).

(h) For the rest of the proof we assume without restriction of generality that

$$\inf I_\alpha<\inf\tilde{I}_\alpha.$$

Thus F_α "has a convex start".

(i) J_α is a finite disjoint union of open intervals $J_{1,\alpha},\dots,J_{k_\alpha,\alpha}$ (ordered from left to right). Denote their lengths by $l_{1,\alpha},\dots,l_{k_\alpha,\alpha}$ and set

$$\begin{aligned} h_\alpha &:= -l_{1,\alpha}^{-1}1_{J_{1,\alpha}}+2l_{2,\alpha}^{-1}1_{J_{2,\alpha}}-\cdots+(-1)^{k_\alpha-1}2l_{k_\alpha-1,\alpha}^{-1}1_{J_{k_\alpha-1,\alpha}} \\ &\quad +(-1)^{k_\alpha}l_{k_\alpha,\alpha}^{-1}1_{J_{k_\alpha,\alpha}}, \qquad \alpha\in(0,\alpha_0) \qquad (7)\\ h_\alpha &:=0, \quad \alpha\geqq\alpha_0. \end{aligned}$$

If s is defined by (2), it is easy to verify that

$$4\,s(\alpha)=v(h_\alpha). \tag{8}$$

Further, for all $\varepsilon>0$,

$$4\int_\varepsilon^\infty s(\alpha)\,d\alpha=\int_\varepsilon^\infty v(h_\alpha)\,d\alpha=v\left(\int_\varepsilon^\infty h_\alpha\,d\alpha\right). \tag{9}$$

To establish the second equation in (9), consider an arbitrary partition $a=x_0<x_1<\cdots<x_N=b$ with the property that no interval $[x_{j-1},x_j]$ contains points from

I_ε *and* $\tilde{I}_\varepsilon$. We notice that for each $\alpha \geqq \varepsilon$ the set of points at which h_α jumps upwards (resp. downwards), is contained in I_ε (resp. $\tilde{I}_\varepsilon$). Thus the sign of $h_\alpha(x_j) - h_\alpha(x_{j-1})$ only depends on j (not on $\alpha \geqq \varepsilon$). Therefore the approximating sums for $\int_\varepsilon^\infty v(h_\alpha)\, d\alpha$ and $v(\int_\varepsilon^\infty h_\alpha\, d\alpha)$ belonging to the above partition are equal:

$$\sum_{j=1}^{N} \left| \int_\varepsilon^\infty h_\alpha(x_j)\, d\alpha - \int_\varepsilon^\infty h_\alpha(x_{j-1})\, d\alpha \right| = \sum_{j=1}^{N} \int_\varepsilon^\infty \left| h_\alpha(x_j) - h_\alpha(x_{j-1}) \right| d\alpha$$

$$= \int_\varepsilon^\infty \left(\sum_{j=1}^{N} \left| h_\alpha(x_j) - h_\alpha(x_{j-1}) \right| \right) d\alpha.$$

This proves the second equation of (9).

(j) We next derive the equation

$$F_\alpha(x) = \int_\alpha^\infty \int_a^x h_\beta(u)\, du\, d\beta + F^0(x). \tag{10}$$

First note that F_α is absolutely continuous with respect to α. Indeed, it follows from the proof in (f) that $F_\beta(x) - \beta \leqq F_\alpha(x) - \alpha$ for $\alpha \leqq \beta$, and a similar argument shows that $F_\alpha(x) + \alpha \leqq F_\beta(x) + \beta$ for $\alpha \leqq \beta$.

Thus the limit of

$$H_\alpha^\varepsilon(x) := \varepsilon^{-1}(F_\alpha(x) - F_{\alpha-\varepsilon}(x)),$$

as $\varepsilon \to 0+$, exists almost everywhere. By the Lipschitz continuity of $\alpha \to F_\alpha(x)$ we have $|H_\alpha^\varepsilon| \leqq 1$. Further for $x \in A_\alpha^c \cup B_\alpha^c$

$$H_\alpha^\varepsilon(x) = -\int_a^x h_\alpha(u)\, du = \begin{cases} 1, & x \in A_\alpha^c \\ -1, & x \in B_\alpha^c. \end{cases} \tag{11}$$

For if $x \in A_\alpha^c$, $F_\alpha(x) = F(x) + \alpha$ and, by (f), $F_{\alpha-\varepsilon}(x) = F(x) + \alpha - \varepsilon$; the assertion for h_α follows from the definition (7) and (h). As F_α is linear on the components of $A_\alpha \cap B_\alpha$, $F_{\alpha-\varepsilon}$ is concave on the components of $A_{\alpha-\varepsilon}$ and (by (f)) $A_{\alpha-\varepsilon} \subset A_\alpha$, we can conclude that H_α^ε is convex on the components of $(A_\alpha \cap B_\alpha) \cap A_{\alpha-\varepsilon} = A_{\alpha-\varepsilon} \cap B_\alpha$. On B_α^c we have $H_\alpha^\varepsilon \equiv -1$, so that H_α^ε is convex on the components of $A_{\alpha-\varepsilon}$. Similarly it is seen that H_α^ε is concave on the components of $B_{\alpha-\varepsilon}$.

It is easily seen that $A_{\alpha-\varepsilon} \uparrow A_\alpha$, $B_{\alpha-\varepsilon} \uparrow B_\alpha$, $J_{\alpha-\varepsilon} \uparrow J_\alpha$, as $\varepsilon \downarrow 0$. If $x \in I_\alpha$, there are $x_1, x_2 \in A_\alpha^c$ such that $x \in [x_1, x_2]$. Since $A_\alpha^c \subset B_\alpha = \bigcup_{0<\varepsilon<\alpha} B_{\alpha-\varepsilon}$, there is a $\varepsilon_0 > 0$ such that $x_1, x_2 \in B_{\alpha-\varepsilon_0}$; as H_α^ε is concave on $B_{\alpha-\varepsilon}$ and $H_\alpha^\varepsilon(x_1) = H_\alpha^\varepsilon(x_2) = 1$, we have $H_\alpha^\varepsilon(x) = 1$ for $\varepsilon \in (0, \varepsilon_0]$. Thus $H_\alpha^\varepsilon(x) \to 1$ for $x \in I_\alpha$. Similarly we get $H_\alpha^\varepsilon(x) \to -1$ for $x \in \tilde{I}_\alpha$, as $\varepsilon \to 0+$.

Now let (x_0, x_1) be a component of J_α, $x_0 \in A_\alpha^c$, $x_1 \in B_\alpha^c$, so that $H_\alpha^\varepsilon(x_0) = 1$, $H_\alpha^\varepsilon(x_1) = -1$. Then for small $\varepsilon > 0$ there are $\delta(\varepsilon) > 0$, $\eta(\varepsilon) > 0$ such that H_α^ε is concave and decreasing on $[x_0, x_1 - \eta(\varepsilon)]$, convex and decreasing on $[x_1 + \delta(\varepsilon), x_1]$ and linear on $[x_0 + \delta(\varepsilon), x_1 - \eta(\varepsilon)]$, and

$$\lim_{\varepsilon \to 0+} \delta(\varepsilon) = \lim_{\varepsilon \to 0+} \eta(\varepsilon) = 0.$$

Hence $\lim_{\varepsilon\to 0+} H_\alpha^\varepsilon(x)$ exists for $x\in[x_0,x_1]$ and is linear and continuous. Thus

$$\lim_{\varepsilon\to 0+} H_\alpha^\varepsilon(x) = -\int_a^x h_\alpha(u)\,du \tag{12}$$

by (11) and the fact that the right-hand side is piecewise linear and continuous.

(k) By (10) and Fubini's theorem (note that $|h_\beta| \leqq (2/\min_i l_{i,\alpha})$ for $\beta\geqq\alpha$),

$$F_\alpha'(x) = \int_\alpha^\infty h_\beta(x)\,d\beta + \frac{F(b)-F(a)}{b-a} \quad \text{a.e.}$$

If F_α is not differentiable at x, F_α' denotes the right derivative which exists because of the concavity and convexity properties of F_α. As $v(F_\alpha')$ and $v(\int_\alpha^\infty h_\beta(.)\,d\beta)$ can be computed by only considering partitions of $[a,b]$ contained in a countable dense set, we get

$$v(F_\alpha') = v\left(\int_\alpha^\infty h_\beta(.)\,d\beta\right). \tag{13}$$

By (9), (13) and the assumption $\int_0^\infty s(\alpha)\,d\alpha < \infty$, it follows that

$$\sup_{\alpha>0} v(F_\alpha') < \infty. \tag{14}$$

By Helly's selection principle either there exists a function $f\colon[a,b]\to\mathbb{R}$ such that $F_{\alpha_j}'\to f$ pointwise for some sequence $\alpha_j\to 0+$ or there is a $x_0\in[a,b]$ such that $|F_{\alpha_j}'(x_0)|\to\infty$ for some sequence $\alpha_j\to 0+$. In the second case we have without restriction of generality $F_{\alpha_j}'(x)\to\infty$ for all $x\in[a,b]$ (use (14)) so that $F_{\alpha_j}(x)=\int_a^x F_{\alpha_j}'(u)\,du\to\infty$ for $x\in(a,b]$. Thus this possibility is excluded. In the first case however,

$$F(x)-F(a) = \lim_{j\to\infty} F_{\alpha_j}(x) - F(a) = \lim_{j\to\infty}\int_a^x F_{\alpha_j}'(t)\,dt = \int_a^x f(t)\,dt. \tag{15}$$

(l) We shall now prove

$$v(f) = 4\int_0^\infty s(\alpha)\,d\alpha. \tag{16}$$

Firstly, by (k) and (9),

$$v(f) \leqq \liminf_{j\to\infty} v(F_{\alpha_j}') = 4\int_0^\infty s(\alpha)\,d\alpha. \tag{17}$$

Next we show that

$$v(F_\alpha') \leqq \sum |D^+F(x) - D^-F(x')|, \tag{18}$$

where D^+ and D^- denote right and left derivative and the sum is taken over all components $[x', x]$ of $I_\alpha \cup \tilde{I}_\alpha$. Note that because of $v(f) < \infty$,

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1} \int_x^{x+\varepsilon} f(t)\,dt \quad \text{and} \quad \lim_{\varepsilon \to 0+} \varepsilon^{-1} \int_{x-\varepsilon}^{x} f(t)\,dt$$

exist for all $x \in (a, b)$, are continuous from the right resp. left and are both equal to $F'(x)$ almost everywhere (see [**2**]). Especially, the right-hand sum in (18) is well-defined.

Let us consider an arbitrary component $[x_1, x_2]$ of I_α. Then $F_\alpha(x_1) = F(x_1) + \alpha$, $F_\alpha(x_2) = F(x_2) + \alpha$, and F_α is convex on $[x_1 - \varepsilon,\ x_2 + \varepsilon]$ for $\varepsilon > 0$ so small that $(x_1 - \varepsilon, x_2 + \varepsilon) \subset I_\alpha \cup J_\alpha$. Thus the total variation of F'_α in $[x_1 - \varepsilon,\ x_2 + \varepsilon]$ is equal to $F'_\alpha(x_2 + \varepsilon) - F'_\alpha(x_1 - \varepsilon)$. On the other hand, by definition of I_α and ε it is clear that for all $\delta \in (0, \varepsilon)$

$$F_\alpha(x_2 + \delta) < F(x_2 + \delta) + \alpha, \qquad F_\alpha(x_1 - \delta) < F(x_1 - \delta) + \alpha.$$

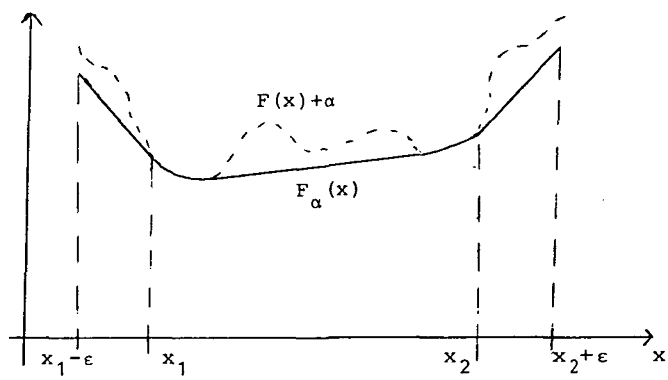

FIGURE 2

Hence,

$$\begin{aligned} D^+F(x_2) &= \lim_{\delta \to 0+} \delta^{-1} \int_{x_2}^{x_2+\delta} f(t)\,dt \\ &= \lim_{\delta \to 0+} \delta^{-1}[F(x_2 + \delta) + \alpha - (F(x_2) + \alpha)] \\ &\geqq \lim_{\delta \to 0+} \delta^{-1}[F_\alpha(x_2 + \delta) - F_\alpha(x_2)] \\ &= F'_\alpha(x_2 + \varepsilon) \end{aligned} \tag{19}$$

(the last equation follows, because F_α is linear on a set containing $[x_2, x_2 + \varepsilon]$). Similarly it is seen that $D^-F(x_1) \leqq F'_\alpha(x_1 - \varepsilon)$. Therefore the total variation of F'_α in $[x_1 - \varepsilon, x_2 + \varepsilon]$

is at most as large as $|D^+F(x_2)-D^-F(x_1)|$. An analogous argument applies to the components of $\tilde{I}_\alpha$. $v(F'_\alpha)$ is the sum of the above estimated total variations.

This yields (18).

Since $D^+F(x)$ $(D^-F(x))$ is continuous from the right (left) and almost everywhere equal to $f(x)$, we obtain from (18) that

$$v(F'_\alpha) \leqq v(f) \quad \text{for all } \alpha > 0. \tag{19}$$

(17) and (19) together imply (16).

(m) Finally suppose that there is a f: $[a,b] \to \mathbb{R}$ with bounded variation for which $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$. As in (1) it is shown that $v(F'_\alpha) \leqq v(f)$ for all $\alpha > 0$. As in (k) it is then proved that there is a $\tilde{f}$ of bounded variation coinciding with f almost everywhere such that $\tilde{f}(x) = \lim_{j\to\infty} F'_{\alpha_j}(x)$ for all $x \in [a,b]$. Note that also

$$\int_\alpha^\infty s(\beta)\,d\beta = v(F'_\alpha) \leqq v(\tilde{f}) \quad \text{for all } \alpha > 0. \tag{20}$$

Thus

$$\int_0^\infty s(\beta)\,d\beta < \infty, \tag{21}$$

and the first part of the theorem applies.

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