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AN ANALOGUE OF PROBLEM 26 OF P. TURÁN

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Explicit formulas for Cotes numbers of the Gaussian Hermite quadrature formula based on the zeros of the nth Chebyshev polynomial of the second kind and their asymptotic behaviour as $n \to \infty$ are given. This provides an answer to an analogue of Problem 26 of Turán.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with explicit formulas for Cotes numbers of the Gaussian Hermite quadrature formula based on the zeros of the nth Chebyshev polynomial of the second kind and their asymptotic behaviour as $n \to \infty$.

Let $\alpha(x)$ be a nonnegative function on $[-1, 1]$ with infinitely many points of increase such that all moments of $\alpha(x)$ are finite and let $\mathbf{P}_N$ denote the set of polynomials of degree $\leqslant N$. According to Theorem 4 in [**3**], given integers $m \geqslant 1$ and $p \geqslant 0$, if $\omega_n(x) := \prod_{k=1}^{n} (x - x_{kn})$ with

$$1 = x_{0n} > x_{1n} > x_{2n} > \cdots > x_{nn} > x_{n+1,n} = -1, \quad n \geqslant 1 \tag{1.1}$$

satisfies

$$\int_{-1}^{1} (1 - x^2)^p |\omega_n(x)|^m \, d\alpha(x) = \min_{P = x^n + \cdots} \int_{-1}^{1} (1 - x^2)^p |P(x)|^m \, d\alpha(x), \tag{1.2}$$

then the quadrature formula with certain numbers $c_{ikm} := c_{ikmn}$ (called Cotes numbers of higher order)

$$\int_{-1}^{1} f(x)\sigma(x) d\alpha(x) = \sum_{k=0}^{n+1} \sum_{i=0}^{\mu_k} c_{ikm} f^{(i)}(x_k) \tag{1.3}$$

is exact for all $f \in \mathbf{P}_{mn+2p-1}$, where

$$\sigma(x) := \operatorname{sgn} \omega_n(x)^m \tag{1.4}$$

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and

$$\mu_k := \begin{cases} m-2, & 1 \leqslant k \leqslant n, \\ p-1, & \text{otherwise}. \end{cases} \tag{1.5}$$

As Turán pointed out in [**10**, p.46], particularly interesting is the case

$$p = 0, \quad d\alpha(x) = \frac{dx}{\sqrt{1-x^2}}.$$

By a theorem of Bernstein [**2**], in this case the nth Chebyshev polynomial of first kind $2^{1-n}T_n(x)$ is the solution of (1.2) for all values of $m \geqslant 1$. Meanwile, Turán raised the following problem for even m [**10**, p.47], in which c_{ikmn} stands for the Cotes number based on the zeros of $T_n(x)$:

PROBLEM 26. *Give an explicit formula for c_{ikmn} and determine its asymptotic behaviour as $n \to \infty$.*

In [**6, 7**] we got a solution of this problem for each (even and odd) m.

Another particular interesting case is

$$p = \left[\frac{m}{2}\right], \quad d\alpha(x) = d\alpha_m(x) := \left(1-x^2\right)^{[(m+1)/2]-(m+1)/2} dx. \tag{1.6}$$

By a well known result (see, say, [**8**]), in this case the nth Chebyshev polynomial of the second kind $2^{-n}U_n(x)$ is the solution of (1.2) for all values of $m \geqslant 1$. Then the quadrature formula with certain numbers c_{ikm}

$$\int_{-1}^{1} f(x)\sigma_m(x)d\alpha_m(x) = \sum_{k=0}^{n+1}\sum_{i=0}^{m_k} c_{ikm} f^{(i)}(x_k) \tag{1.7}$$

is exact for all $f \in \mathbf{P}_{mn+2[m/2]-1}$, where

$$\sigma_m(x) := \operatorname{sgn} U_n(x)^m \tag{1.8}$$

and

$$m_k := [n_k(m-2)], \quad n_k := \begin{cases} 1, & 1 \leqslant k \leqslant n, \\ \frac{1}{2}, & \text{otherwise}. \end{cases} \tag{1.9}$$

In the present paper, using a modification of the main idea of [**6**], we intend to answer to the same problem for this quadrature formula. It turns out that, in the

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present case the problem is more difficult and complicated. To state our results we need the notation:

$$l_k(x) := l_{kn}(x) := \frac{\omega_n(x)}{(x-x_k)\omega_n'(x_k)}, \quad k = 1, 2, \dots, n,$$

$$(1.10) \qquad \Delta_m(x) := \Delta_{mn}(x) := \left(1-x^2\right)^{[m/2]} U_n(x)^m,$$

$$(1.11) \qquad d_{km} := \Delta_{mn}^{([n_k m])}(x_k) = \begin{cases} m!\left(1-x_k^2\right)^{[m/2]} U_n'(x_k)^m, & 1 \leqslant k \leqslant n, \\ (-2)^{[m/2]}\left(\left[\frac{m}{2}\right]\right)! U_n(1)^m, & k = 0, \\ 2^{[m/2]}\left(\left[\frac{m}{2}\right]\right)! U_n(-1)^m, & k = n+1, \end{cases}$$

$$(1.12) \qquad L_{km}(x) := L_{kmn}(x) := \frac{([n_k m])!\Delta_{mn}(x)}{d_{km}(x-x_k)^{[n_k m]}}, \quad k = 0, 1, \dots, n+1,$$

$$(1.13) \quad b_{ikm} := b_{ikmn} := \frac{1}{i!}\left[L_{km}(x)^{-1}\right]_{x=x_k}^{(i)}, \quad i = 0, 1, \dots; \ k = 0, 1, \dots, n+1,$$

$$(1.14) \quad B_{ikm} := B_{ikmn} := \frac{1}{i!}\left\{\sum_{\nu\in\{0,n+1\}\setminus\{k\}} \left[2(x_\nu - x)L_{km}(x)\right]^{-1}\right\}_{x=x_k}^{(i)}, \quad i = 0, 1, \dots; \ k = 0, 1, \dots, n+1,$$

$$(1.15) \qquad s_m := \begin{cases} 2, & \text{if } m \text{ is odd}, \\ \pi, & \text{otherwise}. \end{cases}$$

The main results in this paper is

Theorem. *Let (1.1) be the zeros of* $\Omega_n(x) := \left(1-x^2\right)U_n(x)$ *and let* $m \geqslant 1$ *be an integer. Then for each* i, $0 \leqslant i \leqslant m_k$, *and for each* k, $0 \leqslant k \leqslant n+1$,

$$(1.16) \qquad \begin{cases} c_{m_k,k,m} = \dfrac{n_k s_m (m-2)!}{d_{k,m-2}[(m-2)!!]^2(n+1)}, & m \geqslant 2, \\ c_{m_k+1,k,m} = 0, \end{cases}$$

$$(1.17) \quad c_{ikm} = c_{i,k,m-2} + \frac{m_k! c_{m_k,k,m}}{i! n_k(m-2)}\Big\{(i + n_k(m-2) - m_k) b_{m_k-i,k,m-2} - \frac{1}{2}\left[1+(-1)^{m+1}\right] B_{m_k-i-1,k,m-2}\Big\}, \quad m \geqslant 3.$$

Moreover

$$(1.18) \qquad |c_{ikmn}| \leqslant \begin{cases} \dfrac{\left(1-x_{kn}^2\right)^{[(m-1)/2]-[(m-1-i)/2]}}{n^{m+1-2[(m-i)/2]}}, & 1 \leqslant k \leqslant n, \\ \dfrac{1}{n^{m-2[m/2]+2i+1}}, & \textit{otherwise}, \end{cases}$$

$$(1.19) \quad c_{ikmn} \sim \frac{\left(1-x_{kn}^2\right)^{i/2}}{n^{i+1}}, \quad m = \text{even}; \ i = 0, 2, 4, \dots, m-2; \ 1 \leqslant k \leqslant n.$$

REMARK. For sake of notation we still use the idex m in (1.16) and (1.17). In fact, they remain true if we replace m by any suitable integer r (say, $r \geqslant 3$ in (1.17)). Thus these two formulas provide an explicit expression for c_{ikm}.

2. LEMMAS

To prove our theorem we need several lemmas.

LEMMA 1. *Let*

$$a_{ik} := a_{ikmn} := \sum_{\substack{\nu=0 \\ \nu\neq k}}^{n+1} \frac{[n_\nu m]}{(x_\nu - x_k)^i}, \quad i = 1, 2, \ldots; \quad 0 \leqslant k \leqslant n+1. \tag{2.1}$$

Then

$$b_{ikmn} = \frac{1}{i}\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu,k,m}, \quad i = 1, 2, \ldots; \quad 0 \leqslant k \leqslant n+1. \tag{2.2}$$

PROOF: Let k, $0 \leqslant k \leqslant n+1$, be fixed. It is easy to check that

$$\frac{L'_{km}(x)}{L_{km}(x)} = -\sum_{\substack{\nu=0 \\ \nu\neq k}}^{n+1} \frac{[n_\nu m]}{x\nu - x}. \tag{2.3}$$

Hence

$$a_{ik} = \frac{1}{(i-1)!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}\right]^{(i-1)}_{x=x_k}. \tag{2.4}$$

Using (1.13) and applying the Newton-Leibniz rule we get

$$\begin{aligned}\frac{1}{i}\sum_{\nu=1}^{i} a_{\nu k} b_{i-\nu,k,m} &= \frac{1}{i}\sum_{\nu=1}^{i}\frac{1}{(\nu-1)!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}\right]^{(\nu-1)}_{x=x_k}\frac{1}{(i-\nu)!}[L_{km}(x)^{-1}]^{(i-\nu)}_{x=x_k} \\ &= \frac{1}{i!}\left[-\frac{L'_{km}(x)}{L_{km}(x)}L_{km}(x)^{-1}\right]^{(i-1)}_{x=x_k} = \frac{1}{i!}[L_{km}(x)^{-1}]^{(i)}_{x=x_k} = b_{ikm}.\end{aligned}$$ ∎

LEMMA 2. *Let (1.1) be the zeros of $\Omega_n(x)$ and let $m \geqslant 0$ be an integer. Then for each k, $0 \leqslant k \leqslant n+1$,*

$$\int_{-1}^{1}\frac{\left|(1-x^2)^{1/2}U_n(x)\right|^m \Omega_n(x)}{\Omega'_n(x_k)(x-x_k)}\frac{dx}{\sqrt{1-x^2}} = \frac{n_k s_m(m!)}{(m!!)^2(n+1)}. \tag{2.5}$$

PROOF: We distinguish two cases.

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CASE 1. $1 \leqslant k \leqslant n$.

We know [5, Volume 3, Chapter 5, Section 4]

$$\int_{-1}^{1} l_k(x)\left(1-x^2\right)^{1/2} dx = \frac{\pi\left(1-x_k^2\right)}{n+1}. \tag{2.6}$$

Expand $|\sin\theta|^m$ as

$$|\sin\theta|^m = \sum_{\nu=0}^{\infty} \beta_\nu \cos\nu\theta. \tag{2.7}$$

Here we can use [4, Formula 3.621-3 and 4, p.369] to calculate the required first two coefficients:

$$\beta_0 = \frac{1}{\pi}\int_0^{\pi} |\sin\theta|^m\, d\theta = \frac{s_m(m!)}{\pi(m!!)^2},\quad \beta_1 = 0. \tag{2.8}$$

Replacing θ by $(n+1)\theta$ in (2.7) yields

$$|\sin(n+1)\theta|^m = \sum_{\nu=0}^{\infty} \beta_\nu \cos\nu(n+1)\theta. \tag{2.9}$$

By making the substitution $\theta = \arccos x$, we obtain

$$\left|\left(1-x^2\right)^{1/2} U_n(x)\right|^m = \sum_{\nu=0}^{\infty} \beta_\nu T_{\nu(n+1)}(x). \tag{2.10}$$

Then

$$\begin{aligned}
&\int_{-1}^{1} \frac{\left|\left(1-x^2\right)^{1/2}U_n(x)\right|^m \Omega_n(x)}{\Omega_n'(x_k)(x-x_k)} \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{1}{1-x_k^2}\int_{-1}^{1} \left|\left(1-x^2\right)^{1/2}U_n(x)\right|^m l_k(x)\left(1-x^2\right)\frac{dx}{\sqrt{1-x^2}} \\
&= \frac{1}{1-x_k^2}\int_{-1}^{1} \left[\sum_{\nu=0}^{\infty}\beta_\nu T_{\nu(n+1)}(x)\right] l_k(x)\left(1-x^2\right)\frac{dx}{\sqrt{1-x^2}} \\
&= \frac{\beta_0}{1-x_k^2}\int_{-1}^{1} l_k(x)\left(1-x^2\right)^{1/2}dx = \frac{s_m(m!)}{(m!!)^2(n+1)}.
\end{aligned}$$

CASE 2. $k = 0,\ n+1$.

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We give the proof for the case $k = 0$ only, the proof for the case $k = n+1$ being similar. Since [**1**, Formula 22.3.12, p.776]

$$(2.11)\qquad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta} = \sum_{\nu=0}^{n} \cos(n-2\nu)\theta,$$

we have

$$(2.12)\qquad \int_0^{\pi} U_n(\cos\theta)(1+\cos\theta)d\theta = \pi.$$

This means

$$(2.13)\qquad \int_{-1}^{1} U_n(x)(1+x)\frac{dx}{\sqrt{1-x^2}} = \pi.$$

By (2.8) and (2.10) we conclude

$$\begin{aligned}
&\int_{-1}^{1} \frac{\left|(1-x^2)^{1/2}U_n(x)\right|^m \Omega_n(x)}{\Omega_n'(1)(x-1)}\frac{dx}{\sqrt{1-x^2}} \\
&= \frac{1}{2(n+1)}\int_{-1}^{1}\left|(1-x^2)^{1/2}U_n(x)\right|^m U_n(x)(1+x)\frac{dx}{\sqrt{1-x^2}} \\
&= \frac{1}{2(n+1)}\int_{-1}^{1}\left[\sum_{\nu=0}^{\infty}\beta_\nu T_{\nu(n+1)}(x)\right] U_n(x)(1+x)\frac{dx}{\sqrt{1-x^2}} \\
&= \frac{\beta_0}{2(n-1)}\int_{-1}^{1} U_n(x)(1+x)\frac{dx}{\sqrt{1-x^2}} = \frac{s_m(m!)}{2(m!!)^2(n+1)}.
\end{aligned}$$

□

Lemma 3. *Let (1.1) be the zeros of $\Omega_n(x)$ and let $m \geqslant 1$ be an integer. Then for each i, $i < [n_k m]$,*

$$(2.14)\qquad |b_{ikmn}| \leqslant \begin{cases} c\dfrac{n^{2[i/2]}}{(1-x_{kn}^2)^{[(i+1)/2]}}, & 1 \leqslant k \leqslant n, \\ cn^{2i}, & \textit{otherwise}, \end{cases}$$

$$(2.15)\qquad |B_{ikmn}| \leqslant \begin{cases} c\dfrac{n^{2[i/2]}}{(1-x_{kn}^2)^{[(i+3)/2]}}, & 1 \leqslant k \leqslant n, \\ cn^{2i}, & \textit{otherwise}. \end{cases}$$

Proof: We distinguish two cases.

CASE 1. $1 \leqslant k \leqslant n$.

We have [11, Lemma 3.11]

$$\left|\left[l_k(x)^{-m}\right]^{(j)}_{x=x_k}\right| \leqslant c_1 \frac{n^{2[j/2]}}{(1-x_k^2)^{[(j+1)/2]}}, \quad j < m. \tag{2.16}$$

Meanwhile it is easy to see that

$$\left|\left[\left(\frac{1-x^2}{1-x_k^2}\right)^{-[m/2]}\right]^{(j)}_{x=x_k}\right| \leqslant c_2\left(1-x_k^2\right)^{-j}, \quad j < m. \tag{2.17}$$

Then by the Newton-Leibniz rule

$$|b_{ikm}| = \left|\frac{1}{i!}\left[\left(\frac{1-x^2}{1-x_k^2}\right)^{-[m/2]} l_k(x)^{-m}\right]^{(i)}_{x=x_k}\right| \leqslant c\frac{n^{2[i/2]}}{(1-x_k^2)^{[(i+1)/2]}}.$$

Here we used the relation

$$\left(1-x_k^2\right)^{-1/2} \leqslant c_3 n.$$

On the other hand, since in this case

$$B_{ikm} = \frac{1}{2i!}\left[\left(\frac{1}{1-x} - \frac{1}{1+x}\right)L_{km}(x)^{-1}\right]^{(i)}_{x=x_k},$$

we have in a similar way

$$|B_{ikm}| \leqslant c\frac{n^{2[i/2]}}{(1-x_k^2)^{[(i+3)/2]}}.$$

CASE 2. $k = 0,\ n+1$.

We discuss the case $k = 0$, the case $k = n+1$ leading to entirely analogous, symmetric considerations. In this case we obtain

$$|b_{i,0,m}| = \left|\frac{1}{i!}\left[\left(\frac{1+x}{2}\right)^{-[m/2]}\left(\frac{U_n(x)}{U_n(1)}\right)^{-m}\right]^{(i)}_{x=1}\right| \leqslant cn^{2i}.$$

Meanwhile

$$|B_{i,0,m}| = \left|\frac{1}{2i!}[(1+x)^{-1}L_{0,m}(x)^{-1}]^{(i)}_{x=1}\right| \leqslant cn^{2i}. \qquad \square$$

Lemma 4. *Let (1.1) be the zeros of $\Omega_n(x)$ and let $m \geqslant 1$ be an integer. Then (1.16) is true for each k, $0 \leqslant l \leqslant n+1$.*

Proof: That $c_{0,k,1} = 0$, $0 \leqslant k \leqslant n+1$, follows from (1.7). Now let $m \geqslant 2$ and let k, $0 \leqslant k \leqslant n+1$, be fixed. It is easy to see that

$$f(x) = \frac{\Delta_{m-2}(x)\Omega_n(x)}{d_{k,m-2}\Omega_n'(x_k)(x-x_k)} \tag{2.18}$$

satisfies the conditions:

$$f^{(i)}(x_j) = \delta_{i,m_k}\delta_{jk}, \quad i = 0, 1, \ldots, m_k; \quad j = 0, 1, \ldots, n+1. \tag{2.19}$$

In fact, it is sufficient to show

$$f^{(m_k)}(x_k) = 1. \tag{2.20}$$

By the Newton-Leibniz rule and (1.11) this is indeed the case:

$$f^{(m_k)}(x_k) = \left[\frac{\Delta_{m-2}(x)\Omega_n(x)}{d_{k,m-2}\Omega_n'(x_k)(x-x_k)}\right]^{(m_k)}_{x=x_k} = \frac{1}{d_{k,m-2}}\Delta_{m-2}^{(m_k)}(x_k) = 1.$$

Substituting f into (1.7) and using (2.5) gives

$$\begin{aligned} c_{m_k,k,m} &= \int_{-1}^{1} f(x)\sigma_m(x)d\alpha_m(x) \\ &= \frac{1}{d_{k,m-2}}\int_{-1}^{1} \frac{\left|(1-x^2)^{1/2}U_n(x)\right|^{m-2}\Omega_n(x)}{\Omega_n'(x_k)(x-x_k)}\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{n_k s_m (m-2)!}{d_{k,m-2}[(m-2)!!]^2(n+1)}. \end{aligned}$$

□

3. Proof of Theorem

The theorem for $m \leqslant 2$ or $i = m_k$ is given by Lemma 4. Now assume that $m \geqslant 3$ and that both i and k, $0 \leqslant i \leqslant m_k - 1$, $0 \leqslant k \leqslant n+1$, are fixed.

By the same arguments as that in [**9**, Lemma 1] we conclude

$$f(x) = \frac{1}{i!}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}(x-x_k)^{i+j}L_{k,m-2}(x) \tag{3.1}$$

satisfies the conditions:

$$f^{\mu}(x_\nu) = \delta_{i\mu}\delta_{k\nu}, \quad \mu = 0, 1, \ldots, m_k - 1; \quad \nu = 0, 1, \ldots, n+1. \tag{3.2}$$

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Applying (1.7) to f twice yields

$$c_{i,k,m-2} = \int_{-1}^{1} f(x)\sigma_{m-2}(x)d\alpha_{m-2}(x) \tag{3.3}$$
$$= \int_{-1}^{1} f(x)\sigma_m(x)d\alpha_m(x) = c_{ikm} + \sum_{\nu=0}^{n+1} c_{m_\nu,\nu,m} f^{(m_\nu)}(x_\nu).$$

We distinguish two cases.

CASE 1. $\nu = k$.

From (1.11) and (1.12) it follows that $L_{km}(x_k) = 1$. Using (3.1) and (2.5) and applying the Newton-Leibniz rule twice, we obtain

$$\begin{aligned} f^{(m_k)}(x_k) &= \frac{1}{i!}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}\left[(x-x_k)^{i+j}L_{k,m-2}(x)\right]_{x=x_k}^{(m_k)} \\ &= \frac{(m_k)!}{i!}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}\frac{L_{k,m-2}^{(m_k-i-j)}(x_k)}{(m_k-i-j)!} \\ &= \binom{m_k}{i}\sum_{j=0}^{m_k-1-i}\binom{m_k-i}{j}\left[L_{k,m-2}(x)^{-1}\right]_{x=x_k}^{(j)}\frac{L_{k,m-2}^{(m_k-i-j)}(x_k)}{(m_k-i-j)!} \\ &= \binom{m_k}{i}\left\{[1]_{x=x_k}^{(m_k-i)} - (m_k-i)!b_{m_k-i,k,m-2}\right\} \\ &= -\frac{(m_k)!b_{m_k-i,k,m-2}}{i!}. \end{aligned}$$

CASE 2. $\nu \neq k$.

By using (3.1) and applying the Newton-Leibniz rule again we have

$$\begin{aligned} \sum_{\nu\neq k} c_{m_\nu,\nu,m} f^{(m_\nu)}(x_\nu) &= \frac{1}{i!}\sum_{\nu\neq k} c_{m_\nu,\nu,m}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}\left[(x-x_k)^{i+j}L_{k,m-2}(x)\right]_{x=x_\nu}^{(m_\nu)} \\ &= \frac{1}{i!}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}\sum_{\nu\neq k} c_{m_\nu,\nu,m}\left[(x-x_k)^{i+j}L_{k,m-2}(x)\right]_{x=x_\nu}^{(m_\nu)} \\ &= \frac{1}{i!}\sum_{j=0}^{m_k-1-i} b_{j,k,m-2}\sum_{\nu\neq k} c_{m_\nu,\nu,m}(x_\nu-x_k)^{i+j}L_{k,m-2}^{(m_\nu)}(x_\nu). \end{aligned}$$

According to (1.11) and (1.12) we see that

$$L_{k,m-2}^{(m_\nu)}(x_\nu) = \frac{d_{\nu,m-2}m_k!}{d_{k,m-2}(x_\nu-x_k)^{m_k}}, \quad \nu\neq k. \tag{3.4}$$

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Substituting (1.16) and (3.4) into the previous formula and applying Lemma 1, we get

$$\begin{aligned}
&\sum_{\nu\neq k} c_{m_\nu,\nu,m} f^{(m_\nu)}(x_\nu) \\
&= \frac{s_m(m-2)!m_k!}{i![(m-2)!!]^2(n+1)d_{k,m-2}} \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \sum_{\nu\neq k} \frac{n_\nu}{(x_\nu-x_k)^{m_k-i-j}} \\
&= \frac{m_k! c_{m_k,k,m}}{i! n_k(m-2)} \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \sum_{\nu\neq k} \frac{n_\nu(m-2)}{(x_\nu-x_k)^{m_k-i-j}} \\
&= \frac{m_k! c_{m_k,k,m}}{i! n_k(m-2)} \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \sum_{\nu\neq k} \frac{m_\nu+[n_\nu(m-2)-m_\nu]}{(x_\nu-x_k)^{m_k-i-j}} \\
&= \frac{m_k! c_{m_k,k,m}}{i! n_k(m-2)} \left\{ (m_k-i) b_{m_k-i,k,m-2} + \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \sum_{\nu\neq k} \frac{n_\nu(m-2)-m_\nu}{(x_\nu-x_k)^{m_k-i-j}} \right\}.
\end{aligned}$$

It remains to compute

$$S := \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \sum_{\nu\neq k} \frac{n_\nu(m-2)-m_\nu}{(x_\nu-x_k)^{m_k-i-j}}.$$

It is particularly simple for even m; in this case we have $m_\nu = n_\nu(m-2)$ and hence $S = 0$.

Determination of S for odd m in the same way first leads to a formula of different structure:

$$\begin{aligned}
\sum_{\nu\neq k} \frac{n_\nu(m-2)-m_\nu}{(x_\nu-x_k)^{m_k-i-j}} &= \sum_{\nu\in\{0,n+1\}\setminus\{k\}} \frac{1}{2(x_\nu-x_k)^{m_k-i-j}} \\
&= \frac{1}{(m_k-i-j-1)!} \sum_{\nu\in\{0,n+1\}\setminus\{k\}} [(2(x_\nu-x))^{-1}]^{(m_k-i-j-1)}_{x=x_k}.
\end{aligned}$$

Using the notation (1.14) we can get a simple formula for S:

$$\begin{aligned}
S &= \sum_{j=0}^{m_k-1-i} b_{j,k,m-2} \frac{1}{(m_k-i-j-1)!} \sum_{\nu\in\{0,n+1\}\setminus\{k\}} [(2(x_\nu-x))^{-1}]^{(m_k-i-j-1)}_{x=x_k} \\
&= \frac{1}{(m_k-i-1)!} \left\{ \sum_{\nu\in\{0,n+1\}\setminus\{k\}} [2(x_\nu-x)L_{k,m-2}(x)]^{-1} \right\}^{(m_k-i-1)}_{x=x_k} \\
&= B_{m_k-i-1,k,m-2}.
\end{aligned}$$

By substituting the values of S we obtain a simple formula:

$$\sum_{\nu\neq k} c_{m_\nu,\nu,m} f^{(m_\nu)}(x_\nu) \tag{3.5}$$
$$= \frac{m_k! c_{m_k,k,m}}{i! n_k(m-2)} \{(m_k - i) b_{m_k-i,k,m-2} + \frac{1}{2}[1+(-1)^{m+1}] B_{m_k-i-1,k,m-2}\}.$$

At last, from (3.3) we get the recurrence relation (1.17) with respect to the index m. Finally, (1.18) follows from (1.16), (1.17), (2.14), (2.15), and the relation

$$U_n'(x_k) = \frac{(-1)^{k+1}(n+1)}{1-x_k^2}.$$

To prove (1.19) by the same arguments as those in [**9**, Lemma 3] we claim $b_{ikm} > 0$ for even i. Thus if both i and m are even then for $1 \leqslant k \leqslant n$ by (1.16) and (1.17)

$$c_{ikm} \geqslant c_{i,k,m-2} \geqslant c_{i,k,i+2} = \frac{\pi(1-x_k^2)^{i/2}}{(i!!)^2(n+1)^{i+1}}.$$

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