ON THE LATENT ROOTS OF A DOUBLY STOCHASTIC MATRIX

Dedicated to the memory of Hanna Neumann

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Under certain general conditions an $n \times n$ stochastic matrix $P = (p_{ij})$, where

$$0 \leqq p_{ij} \leqq 1\,, \quad \sum_j p_{ij} = 1 \quad (i,j = 1,2,\cdots,n),$$

is known to possess the "ergodic" property. This means that there exists a finite matrix E such that

(1) $$\lim_{k \to \infty} P^k = E\,.$$

Clearly one of the latent roots of P is equal to unity. By the Frobenius-Perron Theorem ([1], p. 64), if the p_{ij} are strictly positive, all the other latent roots are of modulus less than unity. Denoting the latent roots by $\alpha_1, \alpha_2, \cdots, \alpha_n$ we may therefore assume that

$$\alpha_1 = 1\,, \; 1 > |\alpha_2| \geqq |\alpha_3| \geqq \cdots \geqq |\alpha_n|\,.$$

In the simplest case, in which all the latent roots are distinct, the matrix P possesses a spectral resolution

$$P = E_1 + \alpha_2 E_2 + \alpha_3 E_3 + \cdots + \alpha_n E_n\,,$$

in which $E_1, E_2, \cdots, E_n$ are mutually orthogonal idempotents. Hence, for any positive integer k, we have that

$$P^k = E_1 + \alpha_2^k E_2 + \alpha_3^k E_3 + \cdots + \alpha_n^k E_n\,.$$

It follows that $P^k \to E_1$, so that E_1 is identical with the matrix E mentioned in (1). The speed with which the limit E_1 is attained, depends on the magnitude of $|\alpha_2|$. *Following a verbal suggestion of J. F. C. Kingman it would be desirable* to obtain an upper bound for $|\alpha_2|$, strictly less than unity, so that the speed of convergence can be estimated effectively.

It is the purpose of this note to obtain such a bound for the class of positive doubly stochastic matrices, that is, matrices whose elements satisfy the conditions

$$0 < p_{ij} \leqq 1\,,\ 1 = \sum_j p_{ij} = \sum_i p_{\cdot j}, \quad (i,j = 1, 2, \cdots, n).$$

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We begin by considering a positive doubly stochastic matrix $Q = (q_{ij})$ which happens to be symmetric. We adopt the convention that all our vectors are column vectors, and we use the dash to denote transposition. Thus if e is a latent vector corresponding to the unit latent root, we may put

(2) $$e' = (1, 1, \cdots, 1).$$

Let α be a latent root of Q other than unity, and let t be a corresponding latent vector, thus

(3) $$Qt = \alpha t,$$

where

$$t' = (t_1, t_2, \cdots, t_n).$$

Since $Q' = Q$, α and t are real. Moreover, e and t are orthogonal, that is

(4) $$t_1 + t_2 + \cdots + t_n = 0.$$

If the rows and columns of Q are permuted in the same manner, the latent roots are unchanged. Also, the vector t may be multiplied by an arbitrary non-zero scalar. We may therefore assume that

(5) $$t_1 = 1\,,\ 1 \geqq t_2 \geqq t_3 \geqq \cdots \geqq t_n.$$

On writing out the first component of (3) in full we obtain that

(6) $$\alpha = \sum_{j=1}^{n} q_{1j} t_j = 1 - \sum_{j=1}^{n} q_{1j}(1 - t_j).$$

Now, by hypothesis, Q is a (strictly) positive matrix. Hence there exists a number μ such that

(7) $$q_{ij} \geqq \mu > 0$$

for all i,j. Clearly

$$\mu \leqq \frac{1}{n},$$

because the row (column) sums of Q are equal to unity.

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Since $1 - t_j \geqq 0$, we infer from (6) that

$$\alpha \leqq 1 - \mu \sum_j (1 - t_j),$$

which, by (4), reduces to

$$\alpha \leqq 1 - n\mu. \tag{8}$$

This estimate is best possible, because the latent roots of

$$Q = \begin{bmatrix} 1-(n-1)\mu & \mu & \cdots & \mu \\ \mu & 1-(n-1)\mu & \cdots & \mu \\ \cdots & \cdots & & \cdots \\ \mu & \mu & \cdots & 1-(n-1)\mu \end{bmatrix}$$

are 1 and $1 - n\mu$ (repeated $n-1$ times).

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We shall now drop the assumption that the matrix is symmetric. If P is any doubly stochastic matrix,

$$e'P = e', \quad Pe = e. \tag{9}$$

The matrix

$$Q = P'P$$

is then both doubly stochastic and symmetric. In order to apply the result of the preceding section, we assume that

$$\mu = \min_{i,j} \sum_{k=1}^{n} p_{ki} p_{kj} > 0. \tag{10}$$

Let ρ be a latent root of P, other than unity, with latent vector u of unit length in the complex metric. Thus we have that

$$Pu = \rho u, \ \bar{u}'u = 1.$$

Using (9) we find that

$$e'Pu = e'u = \rho e'u,$$

whence

$$e'u = 0.$$

Let the latent roots of Q be

$$1 = \alpha_1 > \alpha_2 \geqq \alpha_3 \geqq \cdots \geqq \alpha_n.$$

It is known ([2], p. 28) that α_2 can be characterized as the maximum of the Hermitian form

$$\phi(z) = \bar{z}'Qz$$

subject to the constraints

$$\bar{z}'z = 1, \; e'z = 0. \tag{11}$$

Since the vector u satisfies (11), it follows that

$$\bar{u}'Qu = \bar{u}'P'Pu = |\rho|^2 \leqq \alpha_2.$$

On applying (8) to α_2 we conclude that

$$|\rho| \leqq (1 - n\mu)^{\frac{1}{2}}.$$

This is the estimate we wished to establish. It holds for every doubly stochastic matrix which satisfies (10), whether or not there are multiple roots.

NOTE (ADDED IN PROOF): My attention hat been drawn to the article "Bounds for Eigenvalues of Doubly Stochastic Matrices" by Miroslav Fiedler (*Linear Algebras and its Applications*, 5, 297–310 (1972)), in which more elaborate results have been obtained by somewhat different methods.

References

[1] F. R. Gantmacher, *Applications of the theory of matrices* (Interscience, New York, 1959).
[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 1, (Interscience, New York, 1953).

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