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A NOTE ON ANNIHILATOR BANACH ALGEBRAS

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Let A be a semisimple Banach algebra with $\|\cdot\|$, which is a dense subalgebra of a semisimple Banach algebra B with $|\cdot|$ such that $\|\cdot\|$ majorises $|\cdot|$ on A. The purpose of this paper is to investigate the annihilator property between the algebras A and B.

1. INTRODUCTION

Let A be a simisimple Banach algebra with norm $\|\cdot\|$, which is a dense subalgebra of a semisimple Banach algebra B with norm $|\cdot|$ such that $\|\cdot\|$ majorises $|\cdot|$ on A.

We show that A is an annihilator algebra if and only if B is an annihilator algebra and A and B have the same socle which is dense in A. This improves greatly a result by Tomiuk and Yood [**7**, Theorem 4.5, p. 246].

2. NOTATION AND PRELIMINARIES

Definitions not explicitly given are taken from Rickart [**6**].

Let A be a Banach algebra. For any subset E of A, let $\mathrm{cl}_A(E)$ denote the closure of E in A and $\mathrm{l}_A(E)$ (respectively $\mathrm{r}_A(E)$) the left (respectively right) annihilator of E in A. Then A is called a *modular annihilator algebra* if, for every maximal modular left ideal M and for every modular maximal right ideal N we have $\mathrm{r}_A(M) = (0)$ if and only if $M = A$ and $\mathrm{l}_A(N) = (0)$ if and only if $N = A$ (see [**8**] and [**12**]). Also, A is called an *annihilator algebra* if $\mathrm{l}_A(A) = \mathrm{r}_A(A) = (0)$ and if for every proper closed right ideal I and every proper closed left ideal J, $\mathrm{l}_A(I) \neq (0)$ and $\mathrm{r}_A(J) \neq (0)$. If, in addition, $\mathrm{r}_A(\mathrm{l}_A(I)) = I$ and $\mathrm{l}_A(\mathrm{r}_A(J)) = J$, then A is called a *dual algebra.*

In this paper, all algebras and linear spaces under consideration are over the field C of complex numbers, and the norms on A and B will be denoted by $\|\cdot\|$ and $|\cdot|$ respectively.

The following result is well-known.

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Lemma 2.1. *Let A be a semisimple Banach algebra which is a dense two sided ideal of a semisimple Banach algebra B. Then:*

(1) there exists a constant K such that $K\|\cdot\| \geqslant |\cdot|$;

(2) there exists a constant M such that

$$\|ab\| \leqslant M\|a\|\,|b| \text{ and } \|ba\| \leqslant M\|a\|\,|b|,$$

for all a in A and b in B;

(3) A and B have the same socle S.

Proof:

(1) This is [**2**, Proposition 2.2, p. 3].

(2) This is [**2**, Theorem 2.3, p.3]. See also [**5**, Lemma 4, p. 18].

(3) Let e be a minimal indempotent of B. Since $eBe = eAe = Ce$, $e \in A$ and so $eA = eB$ and $Ae = Be$. It follows that A and B have the same socle S. ∎

3. The main result

In this section, A will be a semisimple Banach algebra which is a dense subalgebra of a semisimple Banach algebra B such that $\|\cdot\|$ majorises $|\cdot|$ on A.

Lemma 3.1. *Let A be an annihilator algebra and e a minimal idempotent in A. Then the following statements are true:*

(1) the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on Ae and eA; also $AE = Be$ and $eA = eB$;

(2) A and B have the same socle S, which is a dense two-sided ideal of both A and B.

Proof: By [**6**, Corollary (2.8.16), p. 100], the socle S of A is dense in A.

1. Let $I = Ae$ and $K = \mathrm{cl}_A(AeA)$. Then K is a topologically simple annihilator algebra. Also, K can be considered as an operator algebra on I and K contains all continuous linear operators with finite rank (see [**6**, p. 101]). Let E be a proper closed subspace of I. We claim that E is not a dense subspace of Be. Suppose otherwise and let f be a non-zero bounded linear functional on I and

$$R = \{T \in K : T(I) \subseteq E\}.$$

If $u \notin E$, then $u \otimes f \notin R$. Hence R is a proper closed right ideal of K. Therefore $\mathrm{l}_K(R) \neq (0)$ and so there exists a minimal idempotent p in K such that $pR = (0)$. Let $y \in E$. Then $y \otimes f \in R$. Therefore, for any x in I, we have

$$(p(y \otimes f))(x) = f(x)py = 0.$$

Since f and x are arbitary, we have $py = 0$ and so $pE = (0)$. If E is dense in Be, then $pBe = (0)$. Hence $pI = (0)$ and so $p = 0$, which is impossible. Therefore E is not a dense subspace of Be.

Let X_1 and X_2 be the normed spaces $(Ae, \|\cdot\|)$ and $(Ae, |\cdot|)$, respectively. The identity mapping from X_1 onto X_2 is denoted by U. Suppose that E is maximal closed subspace of X_1. Since $\mathrm{cl}_B(E)$ is a proper closed subspace of Be, E is not dense in X_2. Hence E is contained in a maximal closed subspace N of X_2. Since $\|\cdot\|$ majorises $|\cdot|$, N is also closed in $\|\cdot\|$. Therefore, by the maximality of E, $E = N$ and so E is a maximal closed subspace of X_2. Hence U maps maximal closed subspaces of X_1 to maximal closed subspaces of X_2. Similarly the inverse of U has the same property. Therefore by [**4**, Lemma B, p. 246], $\|\cdot\|$ and $|\cdot|$ are equivalent on Ae and so $Ae = Be$. Similarly we can show that $eA = eB$. This proves (1).

2. By (1), S is a dense two-sided ideal of B. Let e be a minimal idempotent of B. Then $Se = Be$ and so $Be \subset S \subset A$. Therefore $e \in A$ and e is a minimal idempotent of A. Now it is clear that S is also the socle of B. This proves (2). ∎

Remark. If A is an annihilator algebra and B is a dual algebra, Lemma 3.1 is contained in [**9**, Lemma 3.2, p. 82] and [**10**, Lemma 5.1, p. 442].

We now have the main result of this section.

THEOREM 3.2. *Let A be a semisimple Banach algebra which is a dense subalgebra of a semisimple Banach algebra B such that $\|\cdot\|$ majorises $|\cdot|$ on A. Then the following statements are equivalent:*

(1) A is an annihilator algebra;

(2) B is an annihilator algebra, A and B have the same socle S, which is dense in A.

PROOF:

(1) $\Longrightarrow$ (2) Suppose that A is an annihilator algebra. Then by Lemma 3.1, A and B have the same socle S, which is a dense two-sided ideal in both A and B. Let R be a proper closed right ideal of B. Since $\mathrm{cl}_A(R \cap A)$ is a proper closed right ideal of A, $\mathrm{l}_A(R \cap A) \neq (0)$. Therefore there exists a minimal idempotent $e \in \mathrm{l}_A(R \cap A)$. We show that $eR = (0)$. Suppose otherwise. Since $eR \subseteq eB$ and eR is a right ideal of B, we have $eR = eB = eA$ and so $eRe = eAe = Ce$. Since $Re \subseteq Be = Ae \subseteq A$, $Re \subseteq R \cap A$. Because $e \in \mathrm{l}_A(R \cap A)$, $eRE = (0)$, which is impossible. Therefore $eR = (0)$ and so $\mathrm{l}_B(R) \neq (0)$. Similarly we can show that $\mathrm{r}_B(J) \neq (0)$ for any proper closed left ideal J of B. Therefore B is an annihilator algebra and this proves (2).

(2) $\Longrightarrow$ (1) Let M be a proper closed right ideal of A. Since the socle S is dense in A, there exists a minimal idempotent e such that $e \notin M$. We claim that $e \notin \mathrm{cl}_B(M)$. Suppose otherwise and write $e = \lim_n x_n$ in $|\cdot|$, with $x_n \in M$. Since

$Be \subseteq S \subseteq A$, we have $Ae = Be$ and so, by the Closed Graph Theorem, the two norms $\|\cdot\|$ and $|\cdot|$ are equivelent on Ae. Since $x_n e \to e$ and $x_n e \in M$, it follows that $e \in M$; a contradiction. Therefore $e \notin \mathrm{cl}_B(M)$ and so $\mathrm{cl}_B(M)$ is a proper closed right ideal of B. Let p be a minimal idempotent in $\mathrm{l}_B(\mathrm{cl}_B(M))$. Since $p \in S \subset A$, it follows that $p \in \mathrm{l}_A(M)$. Similarly we can show that $\mathrm{r}_A(N) \neq (0)$ for any proper closed left ideal N of A. Therefore A is an annihilator algebra. This completes the proof of the Theorem. ∎

Remark 1. The condition "S is dense in A" cannot be omitted in (2) of Theorem 3.2. In fact, let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra. Suppose that A is a modular annihilator algebra which is not an annihilator algebra (see an example in [**11**, p. 1033]). Then B is a dual B^*-algebra. By Lemma 2.1, A and B have the same socle S.

Remark 2. Theorem 3.2 greatly improves [**7**, Theorem 4.5, p. 264].

COROLLARY 3.3. *Let A be a semisimple Banach algebra which is a dense two-sided ideal of a semisimple Banach algebra B. Then the following statements are equivelent:*

(1) *A is an annihilator algebra;*
(2) *B is an annihilator algebra and A^2 is dense in A.*

PROOF: By Lemma 2.1, A and B have the same socle S. If condition (1) or (2) is satisfied, then S is dense in B. Let x and y be elements of A. Since S is dense in B, there exists a sequence $\{x_n\}$ in S such that $x_n \to x$ in $|\cdot|$. It follows from Lemma 2.1 that $x_n y \to xy$ in $\|\cdot\|$. Therefore $A^2 \subseteq \mathrm{cl}_A(S)$. If A^2 is dense in A, then $\mathrm{cl}_A(S) = A$. Now the Corollary follows immediately from Theorem 3.2. ∎

Remark. As seen before, the condition "A^2 is dense in A" cannot be omitted in (2) of Corollary 3.3.

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