ON THE NUMBER OF DISSIMILAR GRAPHS BETWEEN A GIVEN GRAPH-SUBGRAPH PAIR

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The purpose of this paper is to integrate the theorems on enumerating subgraphs and supergraphs in **(2)** and **(3)** respectively by generalizing to a result which includes both of these as special cases. In this process we again utilize the powerful enumeration method of Pólya **(4)**.

A *graph* may be defined as a set of p *points* together with a prescribed subset of the $\frac{1}{2}p(p-1)$ *lines* joining pairs of distinct points. Two points of a graph are *adjacent* if there is a line joining them. Two graphs are *isomorphic* if there is a one-to-one correspondence between their point sets which preserves adjacency. An *automorphism* of a graph G is an isomorphism of G with G. It is well known that the set $\Gamma(G)$ of all automorphisms of a graph G forms a group, called *the group of the graph.* The *line group* $\Gamma_1(G)$ of a graph G has been defined in **(2)** as the permutation group acting on the lines of G which is induced by the automorphisms of G. Two points of a graph are *similar* if there is an automorphism mapping one onto the other. Similarity of two lines or of two subgraphs is defined analogously.

The *complement* G' of a graph G is that graph whose point set coincides with that of G and in which two points are adjacent whenever they are not adjacent in G. Let K_p be the *complete graph* of p points, that is, the graph of p points in which every two distinct points are adjacent. Then K_p' is the totally disconnected graph with p points and no lines. Let K_{mn} be the graph of $m + n$ points $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ and all mn lines of the form a_ib_j.

A *spanning subgraph* (called a line-subgraph in **(2)**) of a graph G is a subgraph of G with the same point set. The main result of **(2)** is an enumeration formula for the number of dissimilar spanning subgraphs of a given graph G. If G has p points, this result may be described as the number of dissimilar graphs between G and K_p'. The number of dissimilar supergraphs of G was obtained in **(3)**. This is, of course, the number of dissimilar graphs between K_p and G. Even the generating function of **(1)** whose coefficients for a given value of p are the numbers of non-isomorphic graphs of p points and q lines may be regarded as enumerating the dissimilar graphs between K_p and K_p'. More recently a formula (to appear elsewhere) has been obtained which enumerates bicoloured graphs essentially as the dissimilar graphs between K_{mn} and K_p' for $m + n = p$.

We wish to derive here a generating function for the number of dissimilar

Received November 14, 1957. This work was supported by a grant from the National Science Foundation.

graphs between any given graph G and a given spanning subgraph H. Let q and q_1 be the number of lines of G and H respectively. We denote this function

$$F_{G,H}(X) = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r \tag{1}$$

where $r = q - q_1$ and a_k is the number of dissimilar graphs K between G and H having $k + q_1$ lines.

We now require Pólya's Theorem precisely in the one variable form in (**1**, p. 447). Since the definitions appear there we include only a statement of the theorem here.

Pólya's Theorem. *The configuration counting series $F(x)$ is obtained by substituting the figure counting series $f(x)$ into the cycle index $Z(\Gamma)$ of the configuration group Γ. Symbolically,*

$$F(x) = Z(\Gamma, f(x)). \tag{2}$$

This theorem reduces the problem of finding the configuration counting series to the determination of the figure counting series and the cycle index of the configuration group. The desired configuration counting series in the present context is the function $F_{G,H}(x)$ of equation (1).

Let L and L_1 be the line sets of G and H respectively. Let $G - H$ be the spanning subgraph of G whose line set is $L - L_1$. Then the figures are the pairs of points of G adjacent in $G - H$. In a configuration K, that is, a graph between G and H, the content of a given figure is 1 if the points of the pair are adjacent in K and is 0 otherwise. Hence the figure counting series is

$$f(x) = 1 + x. \tag{3}$$

The appropriate configuration group for this problem is the permutation group which acts on $L - L_1$ described as follows. Consider the subgroup of $\Gamma_1(G)$, the line group of G, which leaves the set L_1 invariant. If in this subgroup we cut down the object set from L to $L - L_1$, which can be done since L_1 is invariant, we obtain the required configuration group. We denote this group by $\Gamma_1(G - H|H)$ to indicate that it is the line group of $G - H$ subject to the auxiliary condition that H is left invariant. On substituting these observations into equation (2), we obtain the formula:

$$F_{G,H}(x) = Z(\Gamma_1(G - H|H), 1 + x). \tag{4}$$

Theorem. *The generating function for the number of dissimilar graphs between G and H is given by equation* (4).

For purposes of clarity, we illustrate this theorem using two examples other than those in (**2**) and (**3**).

Example 1. Let $G = K_{33}$, and let $H = C_6$, a cycle of length 6. Then $\Gamma_1(G - H|H) = S_3$, the symmetric group of degree 3. Since it is well known

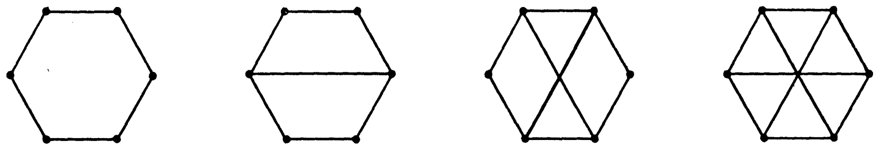

FIGURE 1

that $Z(S_n, 1 + x) = 1 + x + x^2 + \ldots + x^n$, it follows that $F_{G,H}(x) = 1 + x + x^2 + x^3$. This counting polynomial is illustrated in Figure 1, in which the first graph is C_6 and the last is K_{33}.

Example 2. Let $G = Q_3$, the graph of the three dimensional cube, and let $H = C_8$, any Hamiltonian circuit of Q_3. (A Hamiltonian or complete circuit of a graph is one containing all its points. All complete circuits of Q_3 are similar.) Then one readily sees that $\Gamma_1(Q_3 - C_8|C_8)$ is the direct product of two copies of the permutation group S_2, denoted $S_2 \times S_2$ or more briefly S_2^2. Since $Z(S_2) = \frac{1}{2}(f_1^2 + f_2)$ and $Z(S \times T) = Z(S) \cdot Z(T)$ for any permutation groups S and T, we have $Z(S_2^2) = \frac{1}{4}(f_1^4 + 2f_1^2f_2 + f_2^2)$.

Hence

$$Z(\Gamma_1(Q_3 - C_8|C_8),\ 1 + x) = \tfrac{1}{4}[(1 + x)^4 + 2(1 + x)^2(1 + x^2) + (1 + x^2)^2]$$

so that

$$F_{Q_3, C_8}(x) = 1 + 2x + 3x^2 + 2x^3 + x^4.$$

This generating function enumerates the graphs of Figure 2, in which C_8 appears first and Q_3 last.

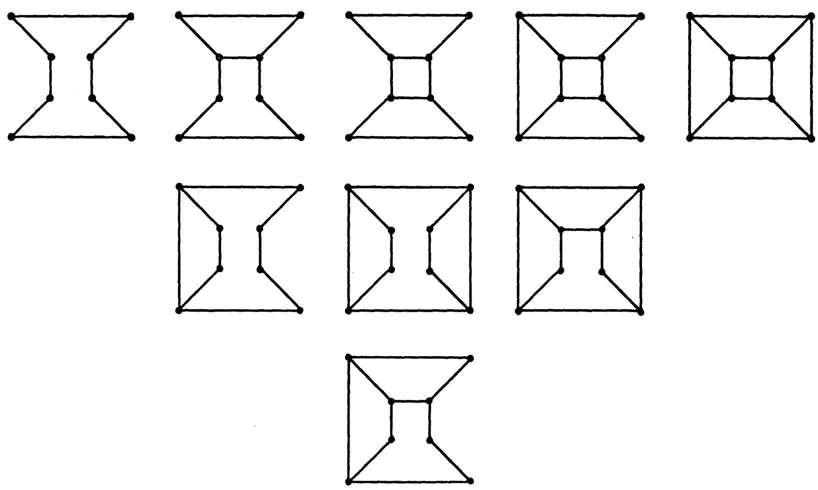

FIGURE 2

These considerations leave open the subtle problem of enumerating the dissimilar Hamiltonian circuits in an n-cube.

https://doi.org/10.4153/CJM-1958-051-3 Published online by Cambridge University Press

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and
The Institute for Advanced Study

https://doi.org/10.4153/CJM-1958-051-3 Published online by Cambridge University Press