Can. J. Math., Vol. XXX, No. 1, 1978, pp. 1–21

INDEX FOUR SIMPLE GROUPS

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1. Introduction

1.1 An *index four simple group* is a finite simple group, G, with a self-centralizing Sylow p-subgroup whose normalizer in G has order $4p$. In this paper index four simple groups having a non-principal ordinary irreducible character of small degree in the principal p-block are studied.

In Section 2 several preliminary results primarily dealing with the values of the characters in $B_0(p)$ are obtained. In particular, inequalities relating the degrees of these characters are derived thus simplifying the task of solving the degree equation for $B_0(p)$. Also quite precise information regarding the values of characters of the group on involutions which normalize Sylow p-subgroups is obtained.

In Section 3 the index four simple groups with a non-principal irreducible character of degree $n \leqq 15$ in $B_0(p)$ are found. First given n, the degree equation for $B_0(p)$ is solved. Then the possible degree equations are studied using the character information form Section 2, class algebra coefficients and various other character theoretic techniques. Thus it is shown that the only such groups with $n \leqq 15$ are $O(5, 3)$, A_7, M_{11} and $Sz(8)$.

1.2 *Notation.* In general, upper case letters denote groups, and Sp is used to denote a Sylow p-subgroup. If A is a subgroup of G, then $N(A)$, $C(A)$, $|G : A|$, $|A|$ denote the normalizer of A in G, the centralizer of A in G, the index of A in G, and the order of A, respectively.

The notation x_n is used for a group element of order n. Then $C(x_n)$ denotes the centralizer of the element x_n in G. Lower case Greek letters denote characters and a character of degree m is denoted by χ_m.

The notation $a(x_i, x_j, x_k)$ denotes the class algebra coefficient which is the number of ways each element of the conjugacy class of x_k can be written as a product of an element of the class of x_i and an element of the class of x_j.

2. Preliminary results. In the sequel G is an index four simple group. Thus there is an odd prime p dividing $|G|$ to the first power only such that $|N(Sp) : Sp| = 4$ for the self-centralizing Sylow p-subgroup Sp.

Received February 16, 1976 and in revised form, June 27, 1977.

The research of the first author was partially supported by a grant from State University College at Oneonta, and that of the second author was partially supported by a grant from the Research Foundation of the State University of New York.

Brauer's work [**4**] yields the following information concerning $B_0(p)$, the principal p-block of G.

Let x_p be an element of order p and let x_q be a p-regular element. Then $B_0(p)$ contains the principal character, 1, 3 other non-exceptional characters χ_1, χ_2, χ_3, and $(p-1)/4$ exceptional characters $\chi^{(m)}$, $m = 1, 2, \ldots, (p-1)/4$. There are signs $\delta_i = \pm 1$, $i = 1, 2, 3$ and $\delta' = \pm 1$ such that $\chi_i(x_p) = \delta_i$, $\chi_i(1) \equiv \delta_i (\text{mod } p)$, $i = 1, 2, 3$, $\sum \chi^{(m)}(x_p) = \delta'$, $\chi^{(m)}(1) \equiv -4\delta' \pmod p$, $m = 1, 2, \ldots, (p-1)/4$ and

$$(2.1) \quad 1 + \delta_1\chi_1(x_q) + \delta_2\chi_2(x_q) + \delta_3\chi_3(x_q) + \delta'\chi^{(m)}(x_q) = 0.$$

If $x_q = 1$ in (2.1) we obtain the following degree equation for $B_0(p)$.

$$(2.2) \quad 1 + \delta_1\chi_1(1) + \delta_2\chi_2(1) + \delta_3\chi_3(1) + \delta'\chi^{(m)}(1) = 0.$$

We next list several results which are extremely important in using the degree equation for $B_0(p)$ to obtain information about the structures of various subgroups of G. The first two lemmas appear in the work [**6**] of Brauer and Tuan.

LEMMA 2.1. *Let G be a simple group of order $p\, q^b\, r$ where p and q are primes, $(pq, r) = 1$. Suppose the degree equation for $B_0(p)$ is $\sum \delta_i\chi_i(1) = 0$, and G has no elements of order pq. Then for any q-block, $B(q)$, $\sum \delta_i\chi_i(1) \equiv 0 \pmod{q^b}$, where the summation is taken over all characters in $B_0(p) \cap B(q)$.*

LEMMA 2.2. *If G is a simple group, χ is an irreducible character of G of degree p^s, $s > 0$, then χ cannot be in $B_0(p)$.*

It follows from the work [**6**] of Brauer and Tuan that there are three possible trees for $B_0(p)$. These trees are illustrated in Figure 2.1. Here $\chi_w, \chi_x, \chi_y, \chi_z$, denote respectively characters of degree w, x, y and z.

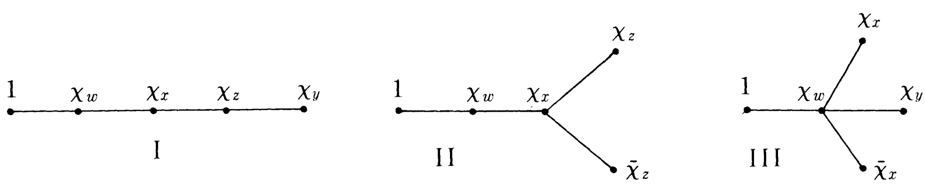

FIGURE 2.1

These trees determine the signs of the terms in the degree equation (2.2) for $B_0(p)$. Thus with the Type I straight line tree, (2.2) has the form

$$(2.3) \quad 1 + x + y = z + w,$$

with the Type II tree the equation is

$$(2.4) \quad 1 + x = w + 2z$$

and with the Type III tree, the equation is

$$(2.5) \quad 1 + 2x + y = w.$$

Note that we have not specified which vertex of the tree corresponds to the exceptional characters $\chi^{(m)}$.

The following lemma and its corollary give some important information concerning the characters in $B_0(p)$. The lemma gives information regarding constituents of products of characters in $B_0(p)$, and the corollary gives some useful inequalities relating the degrees of the characters in $B_0(p)$ when the tree for $B_0(p)$ is of Type I.

LEMMA 2.3. *Let G be an index four simple group. In the tree for* $B_0(p)$, (cf. Fig. 2.1), *let* χ_w *be the character adjacent to the principal character*, 1, *and let* χ *and* θ *be any two adjacent non-principal characters. Then*

1) *The character* $\chi\bar{\theta}$ *has* χ_w *as a constituent, and*

2) *If* χ *is not an endpoint of the tree, then* $\chi\bar{\chi}$ *has* χ_w *as a constituent.*

Proof. Since χ and θ are adjacent, they share a modular irreducible character. Thus $\chi\bar{\theta}$ has the modular identity character, $\hat{1}$, as a modular constituent. But $\hat{1}$ appears only as a modular constituent of the principal character, 1, and χ_w. Statement 1) now follows from the irreducibility of χ and θ.

When χ is not an endpoint of the tree, $\chi\bar{\chi}$ must have $\hat{1}$ as a modular constituent at least twice. Since $\chi\bar{\chi}$ has the principal character, 1, as a constituent exactly once, it follows that $\chi\bar{\chi}$ has χ_w as a constituent at least once.

COROLLARY 2.4. *Let G be an index four simple group with degree equation* (2.3) *for* $B_0(p)$. *Then*

1) $yz \geqq w$, $yw \geqq z$, $wz \geqq y$, $xz \geqq w$, $xw \geqq z$, $zw \geqq x$, $w^2 \geqq x$, $w^3 \geqq z$,

2) $z^2 \geqq w$, $x^2 \geqq w$, $z^3 \geqq x$, $z^3 \geqq y$, *and*

3) *if the endpoint character* χ_y *is not exceptional, then* $y^4 + y^3 - (y + 1) \geqq x$.

Proof. Statement 1) follows from statement 1) of Lemma 2.3. Statement 2) follows from statement 2) of Lemma 2.3 and statement 1) of the corollary. Note that if λ is a constituent of $\eta\gamma$, η, γ irreducibles, then η is a constituent of $\lambda\bar{\gamma}$.

If χ_y is not exceptional and x_p is any element of order p, then it follows from the relations above Eq. (2.1) that $\chi_y(x_p) = 1$. Since the character $\chi_y{}^2$ has at least two ordinary irreducible constituents, it is clear that the character $\chi_y{}^3$ has χ_y as a constituent at least twice. Then consideration of character values at x_p implies that $\chi_y{}^3$ has at least one of the characters χ_w, and χ_z as a constituent. Thus $y^3 \geqq w$ which implies that $y^4 \geqq z$, or $y^3 \geqq z$ which implies that $y^4 \geqq w$. Thus in any case $x \leqq y^4 + y^3 - (y + 1)$ and Corollary 2.4 is proved.

Remark. Lemma 2.3 and Corollary 2.4 are slight generalizations of results obtained by Brauer which appear in a preprint of [**3**].

Next we note the well-known and, for us, extremely useful fact that if χ is any irreducible character of a group, then the character χ^2 can be expressed as $\chi^2 = \theta + \phi$, where the characters θ and ϕ are respectively the alternating and symmetric parts of χ^2. Also if g is any element of the group, then

(2.6) $\theta(g) = \frac{1}{2}[\chi^2(g) - \chi(g^2)]$ and $\phi(g) = \frac{1}{2}[\chi^2(g) + \chi(g^2)]$.

The following lemma bounds the size of the prime divisors of G in terms of the degree of an irreducible character of G.

LEMMA 2.5. *Let G be an index four simple group, let χ be a non-principal irreducible character of G, and let r be any prime dividing $|G|$. Then $\chi(1) \geqq r - 1$.*

Proof. Suppose not. Then there is a prime r dividing $|G|$ and a non-principal irreducible character χ of G such that $\chi(1) < r - 1$. Then the work [**9**] of Feit implies that G is isomorphic to $PSL(2, r)$ or $r - 1$ is a power of 2 and G is isomorphic to $PSL(2, r - 1)$. It is well known (cf. [**11**, Ch. 2]) that the self-centralizing cyclic Hall subgroups of $PSL(2, q)$ are of index 2 or $\frac{1}{2}(q - 1)$ in their normalizers. The latter could only occur when $q = r$ for a Sylow r-subgroup. But since G is an index four simple group we would have $r = 9$, contradicting the fact that r is a prime.

Our final preliminary results deal with involutions in an index four simple group. In the sequel ω denotes an involution which is in the normalizer of the Sylow p-subgroup of G. The following lemma is an immediate consequence of the work [**5**] of Brauer and Fowler.

LEMMA 2.6. *Let G be an index four simple group, let x_2 be any involution in G not conjugate to ω, and let x_p be an element of order p. Then*
1) $a(\omega, \omega, x_p) = p$,
2) $a(x_2, x_2, x_p) = 0$, *and*
3) $a(\omega, x_2, x_p) = 0$.

Our next lemma gives information regarding character values at the involution ω.

LEMMA 2.7. *Let G be an index four simple group, and let χ be a character of G such that $\chi(x_p) = c$ for all x_p in $Sp^\#$. Then*

$$|\chi(\omega)| \leqq \chi(1) - \left(\frac{p-1}{p}\right)(\chi(1) - c).$$

Proof. Let η be any one of the $(p - 1)/4$ characters of degree 4 of $N(Sp)$. Then

$$(\chi|_{N(Sp)}, \eta) = \frac{1}{p}(\chi(1) - c).$$

The result now follows since the other 4 characters of $N(Sp)$ are linear.

Remark. If χ is any irreducible character of G other than an exceptional character for the prime p, then the value of the constant c in Lemma **2.7** is **1**, -1 or 0. The lemma may be applied to the exceptional characters $\chi^{(m)}$, $m = 1, 2, \ldots, (p - 1)/4$, by letting $\chi = \sum \chi^{(m)}$. Then $c = \delta' = \pm 1$.

The following lemma on involution values is proved in Hall [**12**].

LEMMA 2.8. *Let G be a simple group, x_2 any involution in G, and χ a character of G. Then*

$$\chi(x_2) \equiv \chi(1) \pmod 4.$$

LEMMA 2.9. *Let G be a simple group, χ an irreducible character of degree pm, p a prime, $(p, m) = 1$, $m < p - 1$. Then if χ is rational on elements of order p, then χ is not in $B_0(p)$.*

Proof. Suppose $\chi \in B_0(p)$; let z_p be a central p-element and let K be the class of z_p. Then

$$\frac{|K|\chi(z_p)}{pm} \equiv |K| \pmod p.$$

Thus $\chi(z_p) \equiv pm \pmod{p^2}$. Since G is simple, $\chi(z_p) \not\equiv pm$. Hence since $|\chi(z_p)| < pm < p^2$, $\chi(z_p) = pm - p^2$. But then $(\chi|_{\langle z_p \rangle}, 1) = [(p - 1)(pm - p^2) + pm]/p < 0$, since $m < p - 1$. This contradiction proves Lemma 2.9.

3. Proof of the main theorem.

Next we apply the results of Section 2 to find all index four simple groups with a non-principal irreducible character of degree $n \leqq 15$ in $B_0(p)$.

3.1. *The case $n \neq 14$.*

LEMMA 3.1. *Let G be an index four simple group with a non-principal irreducible character of degree n in $B_0(p)$. Then*

1) *it is impossible that $n = 1, 2, 3, 4, 5, 7, 8, 10, 13$, or 15;*

2) *if $n = 6$, then G is isomorphic to $O(5, 3)$ or A_7.*

Proof. If $n = 1$, G is not simple. If $n = 2, 3, 5, 7, 8, 10$ or 15 there is no choice for $p \equiv 1 \pmod 4$ consistent with the relations above Equation (2.1). If $n = 4$, the work of Blichfeldt [**2**] shows that no group exists, completing statement 1). If $n = 13$, the relations above Equation (2.1) imply that $p = 17$, contradicting Lemma 2.5.

If $n = 6$, the work of Lindsey [**13**] shows that G is isomorphic to $O(5, 3)$ or A_7. The work of Brauer [**3**] shows that $O(5, 3)$ is an index four simple group with $p = 5$ and $B_0(5) = \{1, \chi_6, \chi_{81}, \chi_{24}, \chi_{64}\}$. It is an easy matter to verify that A_7 is an index four simple group with $p = 5$ and $B_0(5) = \{1, \chi_6, \chi_{14}, \chi_{14}', \chi_{21}\}$. This proves 2).

LEMMA 3.2. *There is no index four simple group with an irreducible character of degree 9 in $B_0(p)$.*

Proof. The relations above Equation (2.1) imply $p = 5$ or 13. Then Lemma 2.5 implies $p = 5$ and $|G| = 2^a\, 3^b\, 5\, 7^d$. We divide the work into two cases.

https://doi.org/10.4153/CJM-1978-001-x Published online by Cambridge University Press

Case I. (The irrational case) When χ_9 has irrational values, it has a distinct conjugate in $B_0(5)$. Obviously the tree for $B_0(5)$ cannot be of Type III. If the tree for $B_0(5)$ is of Type I, the degree equation in $1 + x + y = 9 + 9$. There is no solution with $x, y \equiv 1 \pmod 5$ and x, y dividing $|G|$ with $x \neq 1$, $y \neq 1$. If the tree for $B_0(5)$ is of Type II, the degree equation is $1 + x = w + 9 + 9$. In this case χ_9 is not real, so that $(\chi_9{}^2, 1) = 0$. The alternating part θ_{36} of $\chi_9{}^2$ is $+1$ on 5-elements, so θ_{36} involves the character of degree x. Thus $x \leqq 36$. Now the only solution to the degree equation is $1 + 21 = 4 + 9 + 9$. This contradicts Lemma 2.5.

Case II. (The rational case) Suppose $B_0(5)$ has a rational valued character of degree 9. The work [**15**] of Schur implies $|G| = 2^a\, 3^b\, 5\, 7^d$ with $a \leqq 16$, $b \leqq 5$, and $d \leqq 1$.

The tree for $B_0(5)$ cannot be of Type III. If the tree is of Type II, the degree equation is $1 + x = 9 + 2z$. By Lemma 2.3, $\chi_9{}^2$ involves χ_9. Since $\chi_9{}^2$ also involves 1 and $\chi_9{}^2$ is $+1$ on 5-elements, $\chi_9{}^2$ must involve the character of degree x which is $+1$ on 5-elements. Thus $x \leqq 45$, the degree of the symmetric part of $\chi_9{}^2$. Now the only solutions to the degree equation are $1 + 16 = 9 + 4 + 4$ and $1 + 36 = 9 + 14 + 14$. The first solution violates Lemma 2.5. For the second solution, $|G| = 2^a\, 3^b\, 5\, 7$. Now Lemma 2.1 applied to $B_0(5) \cap B_0(7)$ implies $1, \chi_9, \chi_{36} \in B_0(7)$. Now Brauer's work [**4**] implies that $|N(S_7): C(S_7)| = 2$, as $\chi_9(1) \equiv 2 \pmod 7$. But this is inconsistent with the degree equation for $B_0(7)$.

Now suppose that the tree for $B_0(5)$ is of Type I with degree equation $1 + x + y = z + w$. If $\chi_z = \chi_9$ (see Fig. 2.1), then Lemma 2.3 implies $(\chi_9{}^2, \chi_w) > 0$, so that $w < 45$. If $\chi_w = \chi_9$, then Lemma 2.3 implies $(\chi_9{}^2, \chi_x) > 0$, so that $x < 45$. Lemma 2.3 also implies $z \leqq xw < 405$. Now the only solutions to the degree equation, meeting these requirements and dividing the order of G are:

1) $1 + 6 + 6 = 9 + 4$	5) $1 + 16 + 56 = 9 + 64$
2) $1 + 6 + 16 = 9 + 14$	6) $1 + 36 + 36 = 9 + 64$
3) $1 + 16 + 16 = 9 + 24$	7) $1 + 36 + 56 = 9 + 84$
4) $1 + 6 + 56 = 9 + 54$	8) $1 + 16 + 216 = 9 + 224$.

We eliminate each solution in turn.

Solutions 1), 2) and 4) contradict Lemma 2.5. In solution 3), if 7 divides $|G|$, intersection of $B_0(5)$ with $B_0(7)$ yields a contradiction. So, in this case $|G| = 2^a\, 3^b\, 5$, and the work of Brauer [**3**] yields a contradiction.

In solution 5), intersection of $B_0(5)$ with $B_0(2)$ and Lemma 2.1 imply $a = 6$, so that $|G| = 2^6\, 3^b\, 5\, 7$. Now a count of Sylow 5-subgroups implies $b = 5$. But, as $\chi_9 \notin B_0(3)$, Lemma 2.1 implies $b < 5$, a contradiction.

In solution 6), application of Lemmas 2.1 and 2.2 to $B_0(5) \cap B_0(2)$ and $B_0(5) \cap B_0(3)$ yields $|G| = 2^6\, 3^2\, 5\, 7$ or $2^6\, 3^2\, 5$. Now a count of Sylow 5-subgroups yields a contradiction.

In solution 7), $|G| = 2^a\, 3^b\, 5\, 7$. Consideration of $B_0(5) \cap B_0(7)$ implies χ_9

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and $\chi_{36} \in B_0(7)$. Then the work of Brauer [**4**] implies $|N(S_7): C(S_7)| = 2$, which is inconsistent with the degree equation for $B_0(7)$.

In solution 8), $\chi_{16} \notin B_0(2)$ by Lemma 2.2, and then Lemma 2.1 yields a contradiction to $a \geqq 5$. This completes the proof of Lemma 3.2.

LEMMA 3.3. *If G is an index four simple group with an irreducible character of degree* 11 *in $B_0(p)$, then G is isomorphic to the Mathieu group M_{11}.*

Proof. The relations above Equation (2.1) imply $p = 5$. By Lemma 2.5 we have $|G| = 2^a\, 3^b\, 5\, 7^d\, 11^e$.

Case I. (The irrational case) When $\chi_{11} \in B_0(5)$ has irrational values, it has a distinct conjugate in $B_0(5)$. Obviously the tree for $B_0(5)$ cannot be of Type II. If it is of Type I, the degree equation in $1 + 11 + 11 = z + w$. The only solution with $z, w \equiv -1 \pmod 5$ and dividing $|G|$ is $1 + 11 + 11 = 14 + 9$; but this would contradict Lemma 2.5.

If the tree for $B_0(5)$ is of Type III, then the degree equation is $1 + 11 + 11 + y = w$. Now

$$\chi_{11}{}^2\bar{\chi}_{11} = h1 + k\chi_{11} + m\bar{\chi}_{11} + \eta,$$

where $k = (\chi_{11}{}^2, \chi_{11}{}^2) \geqq 2$. Checking $\chi_{11}{}^2\, \bar{\chi}_{11}$ on a 5-element, we see that it must involve χ_w (which is the only character in $B_0(5)$ which is -1 on 5-elements). So $w \leqq 11^3 = 1331$. It is easy to check that the only possible solutions to the degree equation are:

1) $1 + 11 + 11 + 21 = 44$ and

2) $1 + 11 + 11 + 121 = 144$.

In solution 2), intersection with $B_0(11)$ shows $e = 1$, but this contradicts the existence of a character of degree 121.

In solution 1), Lemma 2.9 implies $\chi_{21} \notin B_0(7)$. Then Lemma 2.1 applied to $B_0(5) \cap B_0(7)$ yields a contradiction.

Case II. (The rational case) Suppose $B_0(5)$ has a rational valued character of degree 11. Then a theorem of Feit [**8**] implies G has a subgroup of index 11 or 12. Therefore, as G is simple, G is isomorphic to a subgroup of A_{12}. In A_{12} an S_{11}-subgroup is self-centralizing and index 5 in its normalizer. So, by Burnside's Theorem the same is true in G. Now a count of S_5's and S_{11}'s yields $|G| = 2^4\, 3^2\, 5\, 11$ or $|G| = 2^7\, 3^2\, 5\, 7\, 11$. By the work of Parrot [**14**], G is isomorphic to M_{11} or M_{22}. But M_{22} has no irreducible character of degree 11. Thus G is isomorphic to M_{11}. It is easy to verify that M_{11} is an index four simple group with $p = 5$ and $B_0(5) = \{1, \chi_{11}, \chi_{16}, \bar{\chi}_{16}, \bar{\chi}_{44}\}$. This completes the proof of Lemma 3.3.

LEMMA 3.4. *There is no index four simple group with a character of degree* 12 *in $B_0(p)$.*

Proof. The relations above Equation (2.1) imply $p = 13$. Now $N(S_{13})$ is a Frobenius group of order 52 which has three characters η_1, η_2, η_3 of degree 4. By taking inner products we find that $\chi_{12}|_{N(S_{13})} = \eta_1 + \eta_2 + \eta_3$. It is easy to see that, in the matrix representation affording η_i, the matrix for an element of order 4 is similar to diag $\{\sqrt{-1}, -\sqrt{-1}, 1, -1\}$. Thus, the matrix for an element of order four has determinant -1. Since χ_{12} on $N(S_{13})$ is a direct sum of 3 such representations, we get that the matrix for an element of order 4 in the representation affording χ_{12} must also have determinant -1. This implies G is not simple, proving the lemma.

3.2 *The case* $n = 14$. Here the relations above Equation (2.1) imply that $p = 5$ or 13. The next three lemmas provide useful information about a character of degree 14 and its values.

LEMMA 3.5. *Let G be an index four simple group with an irreducible character,* χ_{14}, *of degree* 14 *in* $B_0(p)$. *Let* x_4 *and* ω *denote elements of order* 4 *and* 2, *respectively, in* $N(S)p$. *Then*

1) $\chi_{14}(x_4) = 0$ *or* $\pm 2i$.
2) *If* $\chi_{14}(x_4) = \pm 2i$, *then* $\chi_{14}(\omega) = -2$.
3) *If* $\chi_{14}(x_4) = 0$, *then* $\chi_{14}(\omega) = 2$.

Proof. Suppose first that $p = 13$. Then $N = N(S_{13})$ has three characters η_1, η_2, η_3 of degree 4 and four linear characters $\psi_0 = 1_N, \psi_1, \psi_2, \psi_3$ where $\psi_1(x_4) = -1, \psi_2(x_4) = i$, and $\psi_3 = \bar{\psi}_2$. Note that

$$\chi_{14}|_N = \eta_1 + \eta_2 + \eta_3 + \phi_1 + \phi_2$$

where ϕ_1 and ϕ_2 are linear characters. Now, in the matrix representation affording η_i, the matrix for x_4 is similar to diag $\{1, -1, i, -i\}$ and has determinant -1. But in the matrix representation affording χ_{14}, the matrix for x_4 must have determinant 1 as G is simple. As a consequence, we must have $\phi_1 + \phi_2 = \psi_0 + \psi_1$, $2\psi_2$, or $2\psi_3$. These possibilities yield $\chi_{14}(x_4) = 0, 2i, -2i$ and $\chi_{14}(\omega) = 2, -2, -2$, respectively.

When $p = 5$, $\chi_{14}|_N = 3\eta + \phi_1 + \phi_2$ where η is the unique character of degree 4 for $N = N(S_5)$ and ϕ_1, ϕ_2 are linear. The same argument as above yields the desired result.

LEMMA 3.6. *Let G be an index four simple group with an irreducible character* $\chi_{14} \in B_0(p)$. *Suppose that* $\chi_{14}(\omega) = -2$ *for an involution* $\omega \in N(Sp)$. *Then if Y is a subgroup of odd order in G,* $\chi_{14}|_Y$ *is rational-valued.*

Proof. If x_4 is an element of order four in $N(Sp)$, Lemma 3.5 implies $\chi_{14}(x_4) = \pm 2i$. By switching to $\bar{\chi}_{14}$, if necessary, we may assume $\chi_{14}(x_4) = 2i$.

Now let α be a primitive $|G|/2^a$th root of unity and β a primitive 2^ath root of unity, $a \geqq 3$. Put $K = Q(\alpha)$ and $L = K(\beta)$. There is an automorphism, σ, of L which fixes K and takes $\beta \rightarrow \beta^{-1}$. Since $i = \beta^n$ with $n = 2^{a-2}$, we have $i^\sigma = -i$. Then χ_{14}^σ is an irreducible character in $B_0(p)$ with $\chi_{14}^\sigma(x_4) = -2i$.

Consideration of the trees in Figure 2.1 clearly shows that $\chi_{14}^{\sigma} = \bar{\chi}_{14}$. Thus, since σ fixes K, χ_{14} and $\bar{\chi}_{14}$ agree on elements of odd order. If χ_{14} was not rational on elements of odd order, it would have an algebraic conjugate distinct from $\bar{\chi}_{14}$ in $B_0(p)$. But then Equations (2.4) and (2.5) imply G has a character of degree 41 or 43, so that 41 or 43 divides $|G|$, in violation of Lemma 2.5. This completes the proof of the lemma.

LEMMA 3.7. *Let G be a simple group whose order is divisible by exactly 7 to the first power. If G has a rational-valued character χ_{14} of degree 14, then*

1) *if x_3 is an element of order 3 in $C(S_7)$, then $\chi_{14}(x_3) = -7$, and*
2) *$|C(S_7)|$ divides 21.*

Proof. By Schur [**15**], $|G| = 2^a\, 3^b\, 5^c\, 7\, 11^e\, 13^f$. Let X be a cyclic subgroup of $C(S_7)$ of order $7q$, where q is a prime dividing $|G|$. Now

$$(\chi_{14}|_X, 1) = (1/7q)(14 + (q - 1)\chi_{14}(x_q))$$

because χ_{14} is 0 on 7-singular elements. In all cases we must have $\chi_{14}(x_q) \equiv 0 \pmod 7$. However, $\chi_{14}(x_q) \equiv 14 \pmod q$ yields a contradiction unless $q = 2$ or 3. If $q = 2$, $\chi_{14}(x_2) = 0$, in violation of Lemma 2.8. If $q = 3$, $\chi_{14}(x_3) = -7$. Thus, it follows that $C(S_7)$ is a $\{3, 7\}$-group.

Since $\chi_{14}(x_3) = -7$ for any element of order 3 in $C(S_7)$, $C(S_7)$ cannot have an elementary Abelian subgroup of order 9. Furthermore, if Y is a cyclic subgroup of $C(S_7)$ of order 9, consideration of $(\chi_{14}|_Y, 1)$ yields a contradiction to the fact that $\chi_{14}(x_9) \equiv 2 \pmod 3$. We thus conclude that $|C(S_7)|$ divides 21.

Now let $p = 5$. In our present case Lemma 2.5 implies

(3.1) $\quad |G| = 2^a\, 3^b\, 5\, 7^d\, 11^e\, 13^f.$

We begin with the irrational case. Here, Lemmas 2.7 and 2.8 imply $\chi_{14}(\omega) = \pm 2$, where ω is an involution in $N(S_5)$.

LEMMA 3.8. *Let G be an index four simple group with an irreducible character χ_{14} of degree 14 in $B_0(5)$. If χ_{14} is irrational, then the degree equation for $B_0(5)$ is $1 + 91 = 64 + 14 + 14$ and the tree for $B_0(5)$ has Type II.*

Proof. As χ_{14} is irrational it has a distinct conjugate in $B_0(5)$, so that $B_0(5)$ has at least two characters of degree 14. Obviously the tree for $B_0(5)$ is not of Type III. If it is of Type I, the degree equation is $1 + x + y = 14 + 14$. The only possible solutions are $1 + 6 + 21 = 14 + 14$ and $1 + 11 + 16 = 14 + 14$. The first solution implies G is isomorphic to A_7 by Lemma 3.1. But in A_7 the two characters of degree 14 are rational. The second solution contradicts Lemma 3.3.

If the tree for $B_0(5)$ is of Type II, then the degree equation is $1 + x = w + 14 + 14$. In this case χ_{14} is not real, so χ_{14}^2 does not involve 1. Now the alternating part, θ_{91}, of χ_{14}^2 is $+1$ on 5-elements, so it must involve the character of degree x. Thus $x \leqq 91$ and the only possible solutions to the degree equation are (1) $1 + 36 = 9 + 14 + 14$, (2) $1 + 81 = 54 + 14 + 14$ and (3) $1 + 91 = 64 + 14 + 14$.

The first solution contradicts Lemma 3.2. In the second solution Lemmas 2.1 and 2.2 yield $b = 4$ with $\chi_{54} \in B_0(3)$. This is impossible. This contradiction completes the proof of Lemma 3.8.

LEMMA 3.9. *Let G be an index four simple group with an irrational irreducible character of degree* 14 *in* $B_0(5)$. *Then G is isomorphic to the Suzuki group* $Sz(8)$.

Proof. The previous Lemma 3.8 implies that the degree equation for $B_0(5)$ is $1 + 91 = 64 + 14 + 14$, where the two characters of degree 14 are complex conjugates. In Equation (3.1), consideration of $B_0(5) \cap B_0(2)$ yields $a = 6$ by Lemma 2.1.

Let ω be an involution in $N(S_5)$. As χ_{64} is defect zero for 2, $\chi_{64}(\omega) = 0$. Now as we saw in the proof of Lemma 3.6, χ_{91} must be the alternating part of χ_{14}^2. So $\chi_{91}(\omega) = \frac{1}{2}(\chi_{14}^2(\omega) - 14)$ and $\chi_{91}(\omega) = \chi_{14}(\omega) + \bar{\chi}_{14}(\omega) - 1$. This information implies $\chi_{14}(\omega) = -2$ and $\chi_{91}(\omega) = -5$. Computation of the coefficient $a(\omega, \omega, x_5)$ yields $|C(\omega)|^2 = 2^{12} 3^b 7^{d-1} 11^e 13^{f-1}$.

In the present case Lemma 3.6 implies χ_{14} is rational on elements of odd order. In particular, Schur [**15**] implies $|G| = 2^6 3^b 5\, 7^d\, 11^e\, 13^f$ with $d \leqq 2$, $e \leqq 1$, and $f \leqq 1$. From the form of $|C(\omega)|^2$ given above it is clear that $d = 1$, $e = 0$, and $f = 1$.

If z_3 is an element of order 3 in $Z(S_3)$, $\chi_{14}(z_3) \equiv 2 \pmod 3$. Consideration of the coefficient $a(z_3, z_3, x_5)$ shows that at most 3^2 divides $|G|$. At this point we know (from Equation 3.1) that $|G| = 2^6\, 3^b\, 5\, 7\, 13$, and a count of Sylow 5-subgroups shows $b \equiv 0 \pmod 4$. Thus we conclude that $b = 0$.

Next suppose x_2 represents a class of involutions other than ω's. Put $\chi_{14}(x_2) = s$ and $\chi_{91}(x_2) = t$. Since $\chi_{64}(x_2) = 0$, Equation (2.1) implies $t = 2s - 1$. On the other hand χ_{91} is the alternating part of χ_{14}^2, so that $t = \frac{1}{2}(s^2 - 14)$. The only values of s satisfying both equations are $s = -2$ and 6. Now Lemma 2.6 implies $a(\omega, x_2, x_5) = 0$, which gives the relation $5t = 26s + 91$. This is a contradiction. So G has one class of involutions.

Now $|G| = 2^6\, 5\, 7\, 13$, G has 1 class of involutions, and $|C(\omega)| = 2^6$. By Suzuki's classification theorem in [**17**], we see that G is isomorphic to $Sz(8)$. It is an easy matter to verify that $Sz(8)$ is an index four simple group with $p = 5$ and $B_0(5) = \{1, \chi_{91}, \chi_{64}, \chi_{14}, \bar{\chi}_{14}\}$. This proves Lemma 3.9.

We next consider the rational case with $n = 14$ and $p = 5$. Throughout this case, Lemma 3.5 implies $\chi_{14}(\omega) = 2$, where ω is an involution in $N(S_5)$. The work of Schur [**15**] implies

(3.2) $|G| = 2^a\, 3^b\, 5\, 7^d\, 11^e\, 13^f$ with $a \leqq 25$, $b \leqq 9$, $d \leqq 2$, $e \leqq 1$ and $f \leqq 1$.

LEMMA 3.10. *Let G be an index four simple group with a rational valued irreducible character of degree* 14 *in* $B_0(5)$. *Then the possible solutions for the degree equation of* $B_0(5)$ (*in which all degrees are at least* 14) *are*:

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1) $1 + 91 = 39 + 39 + 14$	14) $1 + 26 + 176 = 189 + 14$
2) $1 + 16 + 21 = 24 + 14$	15) $1 + 36 + 81 = 104 + 14$
3) $1 + 16 + 36 = 39 + 14$	16) $1 + 56 + 56 = 99 + 14$
4) $1 + 16 + 81 = 84 + 14$	17) $1 + 66 + 91 = 144 + 14$
5) $1 + 16 + 96 = 99 + 14$	18) $1 + 66 + 486 = 539 + 14$
6) $1 + 21 + 36 = 44 + 14$	19) $1 + 81 + 156 = 224 + 14$
7) $1 + 21 + 56 = 64 + 14$	20) $1 + 81 + 196 = 264 + 14$
8) $1 + 21 + 91 = 99 + 14$	21) $1 + 81 + 256 = 324 + 14$
9) $1 + 21 + 96 = 104 + 14$	22) $1 + 91 + 156 = 234 + 14$
10) $1 + 21 + 216 = 224 + 14$	23) $1 + 91 + 216 = 294 + 14$
11) $1 + 26 + 26 = 39 + 14$	24) $1 + 91 + 286 = 364 + 14$
12) $1 + 26 + 36 = 49 + 14$	25) $1 + 91 + 546 = 624 + 14$
13) $1 + 26 + 91 = 104 + 14$	26) $1 + 91 + 1056 = 1134 + 14$.

Proof. If the tree for $B_0(5)$ is of Type III, the degree equation for $B_0(5)$ is $1 + 2x + y = 14$ and there is no solution. If the tree is of Type II, the degree equation is $1 + x = 14 + 2x$. Since χ_{14} and χ_x are adjacent in the tree, Lemma 2.3 implies $\chi_{14}\chi_x$ involves χ_{14} and consequently χ_{14}^2 involves χ_x. Therefore $x < 105$. This leads to solution 1).

If the tree is of Type I, then the degree equation is $1 + x + y = z + w$, where z or w is 14. If $z = 14$, Lemma 2.3 implies χ_w is involved in χ_{14}^2, so that $w < 105$. This yields solutions 2) – 9), 11) – 13), 15) and 16).

If $w = 14$, Lemma 2.3 implies χ_x is involved in χ_{14}^2. So $x < 105$ and $x \equiv 1 \pmod 5$. If $x = 96$, then χ_{96} must be involved in ϕ_{105}, the symmetric part of χ_{14}^2, so that $\phi_{105} = \chi_{96} + \phi_9$ where ϕ_9 has value -1 on 5-elements. However, this would imply ϕ_9 involves χ_z. This is ridiculous because $1 + 96 + y = 14 + z$. Consequently, we must have $x \leqq 91$.

Now Lemma 2.3 implies $\chi_{14}\chi_x$ involves χ_{14}, χ_x and χ_z, so that $z \leqq 14x - x - 14 = 13x - 14$. With this information one finds that the possible solutions are 2) – 26). This finishes the proof of Lemma 3.10.

LEMMA 3.11. *Under the hypotheses of Lemma* 3.10, *solutions* 2) – 6), 8) – 9), 11) – 23), *and* 26) *can be eliminated.*

Proof. Recall that Equation (3.2) is in effect. Solutions 3), 9), 12), 14), 15), 19), 23) and 26) can be eliminated by consideration of $B_0(5) \cap B_0(13)$ using Lemma 2.1, the work of Brauer [**4**], and the work of Stanton [**16**]. As a typical example, look at solution 19). Here Lemma 2.1 implies $\chi_{14} \in B_0(13)$. Then Stanton [**16**] gives that $C(S_{13}) = S_{13}$. It then follows that all irreducible characters not in $B_0(13)$ are defect zero for 13. Therefore χ_{81} and $\chi_{224} \in B_0(13)$ and both must be exceptional by Brauer [**4**]. But this is ridiculous.

Solutions 5), 6), 8), 16), 18), 20) can be eliminated by consideration of $B_0(5) \cap B_0(11)$ as above. For example consider solution 20). Here Brauer's work [**4**] implies $\chi_{14}, \chi_{81}, \chi_{196} \notin B_0(11)$, contradicting Lemma 2.1.

Solutions 2), 4), 11), 13), 17), 21), and 22) can be eliminated by the following argument. If 7^2 divides $|G|$, then Lemma 2.9 implies $\chi_{14} \notin B_0(7)$. But

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in each solution cited, this is inconsistent with Lemma 2.1 so 7^2 does not divide $|G|$. Now consideration of $B_0(5) \cap B_0(7)$ leads to a contradiction as above. For example, in solution 13), Lemma 2.1 implies $B_0(5) \cap B_0(7) = \{1, \chi_{26}, \chi_{104}\}$. Then by Brauer [**4**], $|N(S_7): C(S_7)| = 2$ and the degree equation for $B_0(7)$ would be $1 + 26 = 27$, a contradiction.

LEMMA 3.12. *Let G be an index four simple group with $p = 5$. Then the degree equation for $B_0(5)$ is not* $1 + 91 = 39 + 39 + 14$.

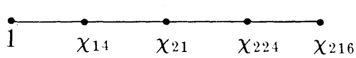
Proof. We saw in the proof of Lemma 3.10 that in this case the two characters of degree 39 are complex conjugates and χ_{91} is involved in χ_{14}^2. Obviously $B_0(5) \cap B_0(13) = \{1, \chi_{14}\}$, so that $C(S_{13}) = S_{13}$ by Stanton [**16**]. If $d = 2$ in Equation (3.2) then Lemma 2.1 implies $\chi_{14} \in B_0(7)$, contradicting Lemma 2.9. So $d = 1$ and $B_0(5) \cap B_0(7) = \{1, \chi_{39}, \bar{\chi}_{39}\}$. Brauer [**4**], then implies $|N(S_7): C(S_7)| = 3$ and that the degree equation for $B_0(7)$ is $1 + 39 = x + y$, where $x, y \equiv -1 \pmod 7$ and $x, y \equiv 0 \pmod 5$. The only possible solution to the degree equation is $1 + 39 = 20 + 20$. Now Lemma 2.1 implies $B_0(7) \cap B_0(13) = \{1, \chi_{20}, \chi_{20}'\}$ so that $|N(S_{13}): C(S_{13})| = 6$. If 11 divides $|G|$, then $B_0(5) \cap B_0(11) = \{1, \chi_{39}, \bar{\chi}_{39}\}$. This would imply that χ_{39} and $\bar{\chi}_{39}$ were exceptionals for $p = 11$. However, since they are already exceptionals for $p = 7$, they are integer valued on 11-elements. This contradiction implies 11 does not divide $|G|$. Thus, by Equation (3.2), $|G| = 2^a\, 3^b\, 5\, 7\, 13$. A count of S_5's and S_{13}'s shows $|G| = 2^9\, 3^3\, 5\, 7\, 13,\ 2^{21}\, 3^3\, 5\, 7\, 13,\ 2^5\, 3^7\, 5\, 7\, 13$, or $2^{17}\, 3^7\, 5\, 7\, 13$.

Since χ_{91} is involved in χ_{14}^2, either $\chi_{91} = \theta_{91}$ or $\phi_{105} = \chi_{91} + \chi_{14}$, where θ_{91} is the alternating part of χ_{14}^2 and ϕ_{105} is the symmetric part. First suppose $\chi_{91} = \theta_{91}$. Since $\chi_{14}(\omega) = 2$, we have $\chi_{91}(\omega) = -5$ and Equation (2.1) implies $\chi_{39}(\omega) = -3$. This contradicts Lemma 2.8. On the other hand, if $\phi_{105} = \chi_{91} + \chi_{14}$, then $\chi_{14}(\omega) = 2$ implies $\chi_{91}(\omega) = 7$ so that $\chi_{39}(\omega) = 3$. Now consideration of $a(\omega, \omega, x_5)$ shows b is even, a contradiction.

LEMMA 3.13. *Let G be an index four simple group with $p = 5$. Then the degree equation for $B_0(5)$ is not* $1 + 21 + 216 = 14 + 224$.

Proof. In Equation (3.2) if $d = 2$, Lemma 2.1 implies $\chi_{14} \in B_0(7)$, contradicting Lemma 2.9. Thus $d = 1$. Consideration of $B_0(5) \cap B_0(11)$ and $B_0(5) \cap B_0(13)$ shows that $e = f = 0$. Consequently $|G| = 2^a\, 3^b\, 5\, 7$. A count of S_5's now gives $a \equiv b + 1 \pmod 4$.

The tree for $B_0(5)$ is of Type I. From Corollary 2.4 it is apparent that the tree is



Then Lemma 2.3 implies

(3.3) $\chi_{14}\chi_{21} = \chi_{14} + \chi_{21} + \chi_{224} + \theta_{35}$,

where θ_{35} is a character of degree 35.

Let ω be an involution in $N(S_5)$. By Lemmas 2.7 and 2.8, $\chi_{21}(\omega) = -3$, 1 or 5, and $\theta_{35}(\omega) = -5, -1, 3$ or 7. Using Equation (3.3) above and Equation (2.1), consideration of the coefficient $a(\omega, \omega, x_5)$ yields a contradiction in all but one case. In this case we get the following information:

$$\chi_{14}(\omega) = 2, \quad \chi_{21}(\omega) = 1, \quad \chi_{216}(\omega) = 0, \quad \chi_{224}(\omega) = 0 \quad \text{and} \quad |C(\omega)|^2 = 2^{a+4}\, 3^{b-1}.$$

Here ω is not a central involution as $a \geqq 5$. So let x_2 be a central involution. Then Lemma 2.6 implies $a(\omega, x_2, x_5)$ and $a(x_2, x_2, x_5)$ are both zero. Let $r = \chi_{14}(x_2)$, $s = \chi_{21}(x_2)$ $t = \chi_{216}(x_2)$, and $u = \chi_{224}(x_2)$. Since x_2 is a central involution, $t \equiv 0 \pmod 8$ and $u \equiv 0 \pmod{32}$. Then we get $s = 3r - 21$ and

$$(3.4) \quad 2^5\, 3^3\, 7 + 2^5\, 3^2\, s^2 + 2^2\, 7\, t^2 - 2^4\, 3^3\, r^2 - 3^3\, u^2 = 0.$$

We also know from Equation (2.1) that $1 + s + t = r + u$. One can easily check that there are no integral solutions with r even, $|r| < 14$ and $|s| < 21$. This completes the proof of Lemma 3.13.

LEMMA 3.14. *Let G be an index four simple group with $p = 5$. Then the degree equation for $B_0(5)$ is not* $1 + 21 + 56 = 14 + 64$.

Proof. In Equation (3.2), consideration of $B_0(5) \cap B_0(2)$ gives $a = 6$, by Lemma 2.1. Furthermore, we must have $d = 1$, for if $d = 2$ Lemma 2.1 implies $\chi_{14} \in B_0(7)$ contradicting Lemma 2.9. If 11 divides $|G|$, $\chi_{14} \notin B_0(11)$, so that $B_0(5) \cap B_0(11) = \{1, \chi_{21}\}$ or $\{1, \chi_{56}, \chi_{64}\}$. In the first case, Stanton [**16**] gives $C(S_{11}) = S_{11}$. But then any irreducible character not in $B_0(11)$ is defect zero for 11, a contradiction. In the second case, since $\chi_{64} \in B_0(11)$, we must have $|N(S_{11}): C(S_{11})| = 2$. Consequently, the degree equation for $B_0(11)$ would be $1 + 64 = 65$, which is absurd. Thus 11 does not divide $|G|$. Similarly, consideration of $B_0(5) \cap B_0(13)$ shows 13 does not divide $|G|$. At this point, Equation (3.2) implies $|G| = 2^6\, 3^b\, 5\, 7$; a count of S_5's yields $b = 1, 5$ or 9.

Now let ω be an involution in $N(S_5)$. Since χ_{64} is defect zero for 2, $\chi_{64}(\omega) = 0$. It follows from Lemmas 2.7 and 2.8 that $\chi_{21}(\omega) = -3$, 1 or 5. Then consideration of the coefficient $a(\omega, \omega, x_5)$ yields a contradiction in all but one case: $\chi_{14}(\omega) = 2$, $\chi_{21}(\omega) = 1$, $\chi_{56}(\omega) = 0$, $\chi_{64}(\omega) = 0$ and $|C(\omega)|^2 = 2^{10}\, 3^{b-1}$.

Now Lemma 3.7 implies $|C(S_7)|$ divides 21 and if $x_3 \in C(S_7)$ then $\chi_{14}(x_3) = -7$. Note first that $|N(S_7): C(S_7)| \neq 2$ because of the resulting degree equation for $B_0(7)$. Then a count of S_7's yields $|G| = 2^6\, 3^5\, 5\, 7$ with $|N(S_7)| = 126$ and $|C(S_7)| = 21$ or $|G| = 2^6\, 3^9\, 5\, 7$ with $|N(S_7)| = 63$ and $|C(S_7)| = 21$. Consideration of $(\chi_{C(S_7)}, 1)$, for each $\chi \in B_0(5)$, and the coefficient $a(x_3, x_3, x_5)$ show $\chi_{21}(x_3) = 0$, $\chi_{56}(x_3) = -28$, and $\chi_{64}(x_3) = -20$, where $x_3 \in C(S_7)$.

Now consideration of the coefficient $a(\omega, x_3, x_5)$ shows $|C(x_3)|$ divides $2^2\, 3^{1/2(b+1)}\, 7$. By the orthogonality relations, we have

$$|C(x_3)| \geqq \sum \chi(x_3)\overline{\chi(x_3)} = 1234,$$

where the sum is over all $\chi \in B_0(5)$. Therefore $b = 9$, $|G| = 2^6\, 3^9\, 5\, 7$, and

$|N(S_7)| = 63$. We also have $|C(x_3)| = 2^2\,3^4\,7, 3^5\,7, 2\,3^5\,7$ or $2^2\,3^5\,7$. Now $C(x_3)$ is solvable in all these cases (cf. Wales [**19**]), so $X(x_3)$ has a subgroup K of order $3^4\,7$ or $3^5\,7$. But $|N_K(S_7)| = 21$ or 63, which contradicts Sylow's Theorem. This final contradiction completes the proof of Lemma 3.14.

LEMMA 3.15. *Let G be an index four simple group with $p = 5$. Then the degree equation for $B_0(5)$ is not* $1 + 91 + 286 = 14 + 364$.

Proof. If 7^2 divides $|G|$, Lemma 2.1 implies $\chi_{14} \in B_0(7)$, contradicting Lemma 2.9. Consequently, from Equation (3.2) we have $|G| = 2^a\,3^b\,5\,7\,11\,13$. A count of S_5's yields $a \equiv b + 2 \pmod 4$. Let $m = |N(S_{11}): C(S_{11})|$. If $m = 2$, then the degree equation for $B_0(11)$ would be $1 + 364 = 365$; but 365 doesn't divide $|G|$. Also, since $C(S_5) = S_5$, $m \not\equiv 10$, so that $m = 5$ by Burnside's Theorem. Now we see that $C(S_{11})$ has a fixed-piont-free automorphism of order 5. Hence, by Thompson's thesis [**18**], $C(S_{11})$ is nilpotent. It follows that $|C(S_{11})| = 2^h\,3^k\,11$, where $h, k \equiv 0 \pmod 4$. Now a count of S_{11}'s shows that a is odd, and consequently b is odd too.

In Lemma 3.10, the present degree equation arose with χ_{91} involved in $\chi_{14}{}^2$. Consequently, either $\theta_{91} = \chi_{91}$ is the alternating part of $\chi_{14}{}^2$ or the symmetric part is $\phi_{105} = \chi_{91} + \chi_{14}$. This makes $\chi_{91}(\omega) = -5$, or 7 since $\chi_{14}(\omega) = 2$. Now Lemmas 2.7 and 2.8 imply that $\chi_{286}(\omega) = u$ with $|u| \leqq 58$ and $u \equiv 2 \pmod 4$. Using Equation (2.1), consideration of the coefficient $a(\omega, \omega, x_5)$ leads to a contradiction of the information found above. This completes the proof.

LEMMA 3.16. *Let G be an index four simple group with $p = 5$. Then the degree equation for $B_0(5)$ is not* $1 + 91 + 546 = 14 + 624$.

Proof. Application of Lemmas 2.1 and 2.9 and consideration of block intersections, gives $|G| = 2^a\,3^b\,5\,7\,13$ from Equation (3.2). Furthermore, since $C(S_{13}) = S_{13}$, a count of S_{13}'s yields $a \equiv b + 2 \pmod 4$.

As in Lemma 3.15, χ_{91} is involved in $\chi_{14}{}^2$, so that either $\theta_{91} = \chi_{91}$ or $\phi_{105} = \chi_{91} + \chi_{14}$. Again $\chi_{14}(\omega) = 2$ implies $\chi_{91}(\omega) = -5$ or 7. Now Lemmas 2.7 and 2.8 imply that $\chi_{546}(\omega) = u$ with $|u| \leqq 110$ and $u \equiv 2 \pmod 4$. Consideration of the coefficient $a(\omega, \omega, x_5)$ yields a contradiction in all but the following cases:

	$\chi_{14}(\omega)$	$\chi_{91}(\omega)$	$\chi_{546}(\omega)$	$\chi_{624}(\omega)$	$\|C(\omega)\|^2$
1)	2	-5	6	0	$2^{a+5}3^{b+1}$
2)	2	-5	-90	-96	$2^{a+5}3^{b+1}$

If x_2 is an involution not conjugate to ω, Lemma 2.6 together with the coefficients $a(\omega, x_2, x_5)$ and $a(x_2, x_2, x_5)$ gives a contradiction. (Note that since $\theta_{91} = \chi_{91}$ or $\phi_{105} = \chi_{91} + \chi_{14}$, the value of χ_{14} determines the value of χ_{91} by Equation (2.6).) Thus, in both cases, G has one class of involutions and $a = 5$.

In both cases 1) and 2), Lemma 2.1 implies $\chi_{624} \in B_0(2)$, a contradiction since χ_{624} must be in a block of defect 1.

LEMMA 3.17. *If G is an index four simple group with an irreducible character of degree* 14 *in $B_0(5)$, then G is isomorphic to A_7 or $Sz(8)$.*

Proof. If the character of degree 14 is irrational, then G is isomorphic to $Sz(8)$ by Lemma 3.9. If the character of degree 14 is rational, Lemmas 3.10–3.16 imply that G must have a character of degree strictly less than 14 in $B_0(5)$. This degree must be 6, 9 or 11 by the relations above Equation (2.1). Now Lemmas 3.1–3.3 show that G is isomorphic to A_7.

Now let $p = 13$. If χ_{14} is irrational then it has at least one distinct conjugate which must also be in $B_0(13)$. It is then clear that the tree for $B_0(13)$ must be of Type III, and so the degree equation for $B_0(13)$ has the form

(3.5) $1 + 14 + 14 + x = y,$

where the primes dividing xy come from the set $\{2, 3, 5, 7, 11\}$.

LEMMA 3.18. *The solutions to Equation* (3.5) *are* $(x, y) = (35, 64), (27, 56), (48, 77)$, *and* $(196, 225)$.

Proof. Let ϕ_{105} and θ_{91} be respectively the symmetric and alternating constituents of the character χ_{14}^2. If ϕ_{105} is irreducible then $x = 105$. But then $y = 134$ which is impossible. Thus χ_{14}^2 has a norm of at least 3. Then the character $\chi_{14}^2\bar{\chi}_{14}$ contains χ_{14} at least 3 times as a constituent. Then an examination of character values at an element of order 13 implies that χ_y appears at least twice as a constituent of $\chi_{14}^2\bar{\chi}_{14}$. Thus $y \leqq \frac{1}{2}(14^3 - 42) = 1351$. Now a short calculation utilizing the relations above (2.1) yields the solutions to Equation (3.5) as listed.

LEMMA 3.19. *There are no index four simple groups possible in the cases* $(x, y) = (27, 56), (48, 77)$ *and* $(196, 225)$.

Proof. Let ω be an involution in $N(S_{13})$. Then Lemmas 2.7 and 2.8 imply that $\chi_{14}(\omega) = \pm 2$. When $(x, y) = (27, 56)$, Lemmas 2.7 and 2.8 give $\chi_{27}(\omega) = 3$ or -1. Then the class algebra coefficient $a(\omega, \omega, x_{13})$ and a count of Sylow 13-subgroups yield a contradiction in each possible case. Similarly when $(x, y) = (48, 77)$, Lemmas 2.7 and 2.8 give $\chi_{77}(\omega) = 1, 5$ or -3. Then $a(\omega, \omega, x_{13})$ and a Sylow 13-subgroup count give a contradiction.

When $(x, y) = (196, 225)$, consideration of $B_0(13) \cap B_0(7)$ implies that χ_{14} and $\bar{\chi}_{14}$ are in $B_0(7)$. Let x_7 be an element of order 7 in the center of a Sylow 7-subgroup. Let $\chi_{14}(x_7) + \bar{\chi}_{14}(x_7) = n$. Clearly n is an integer. Moreover, $n/14 \equiv 2 \pmod 7$. Thus $n \equiv 28 \pmod{49}$. But this is impossible since $|\chi_{14}(x_7)| < 14$. This completes the proof of Lemma 3.19.

LEMMA 3.20. *If G is an index four simple group with degree equation* $1 + 14 + 14 + 35 = 64$ *for* $B_0(13)$, *then G is isomorphic to* $Sz(8)$.

Proof. Here Lemmas 2.7 and 2.8 imply that $\chi_{14}(\omega) = \pm 2$. Also Lemmas 2.1 and 2.2 applied to $B_0(2) \cap B_0(13)$ imply that $|G| = 2^6\, 3^b\, 5^c\, 7^d\, 11^e\, 13$. Let x_2 be any involution in G. Let $u = \chi_{14}(x_2)$ then it is easy to verify that $a(x_2, x_2, x_{13})$ is positive. Thus by Lemma 2.6, it follows that G has exactly one class of involutions, and in particular ω is a central involution. Now consideration of the

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coefficient $a(\omega, \omega, x_{13})$ yields $\chi_{14}(\omega) = -2$, and $|C(\omega)|^2 = 2^6\, 3^b\, 5^{c-1}\, 7^{d-1}\, 11^e$. Since $\chi_{14}(\omega) = -2$, Lemma 3.6 implies that χ_{14} is rational-valued on elements of odd order. Consequently, Schur [**15**] gives $c \leqq 3$, $d \leqq 2$, $e \leqq 1$. From the form of $|C(\omega)|^2$ it follows that $c = 1$ or 3, $d = 1$, $e = 0$, and b is even. A count of Sylow 13-subgroups now yields $c = 1$ and $b = 0$ or 6.

To show $b = 0$, we proceed as follows. The symmetric part ϕ_{105} of χ_{14}^2 has $\phi_{105}(x_{13}) = 1$ and ϕ_{105} does not involve 1_G as χ_{14} is not real. If ϕ_{105} involves χ_{14} or $\bar{\chi}_{14}$, then $\phi_{105} = \phi_{14} + \phi_{91}$. However $\phi_{14}(\omega) = -2$ implies $\phi_{91}(\omega) = 11$, which contradicts Lemma 2.7. Consequently, we must have $\phi_{105} = \sum_{i=1}^{3} \chi_{35}{}^{(i)}$. Let $z_3 \in Z(S_3)$ be an element of order 3. As $\chi_{14}(z_3)$ is rational, $\chi_{14}(z_3) \equiv 2 \pmod 3$. We have $\chi_{35}(z_3) = (1/3)\phi_{105}(z_3) = (1/6)(\chi_{14}{}^2(z_3) + \chi_{14}(z_3{}^2))$. With this information consideration of $a(z_3, z_3, x_{13})$ shows that $b \leqq 4$. Thus $b = 0$.

Now $|G| = 2^6\; 5\; 7\; 13$, G has one class of involutions, and $|C(\omega)| = 2^6$. So the classification [**17**] of Suzuki proves that G is isomorphic to $Sz(8)$. It is an easy matter to verify that $Sz(8)$ is an index four simple group with degree equation $1 + 14 + 14 + 35 = 64$ for $B_0(13)$. This completes the proof of Lemma 3.20.

Now we consider the case that χ_{14} is a rational character. Here it follows from Lemma 2.5 and Schur [**15**] that

(3.6) $\quad |G| = 2^a\, 3^b\, 5^c\, 7^d\, 11^e\, 13, \quad$ where $a \leqq 25$, $b \leqq 9$, $c \leqq 3$, $d \leqq 2$, and $e \leqq 1$.

Also Lemma 3.5 implies that $\chi_{14}(\omega) = 2$. Now consideration of Figure 2.1, the relations above Equation (2.1) and Lemma 2.5 yield that the tree for $B_0(13)$ cannot be of Type II.

LEMMA 3.21. *There are no index four simple groups having a rational character of degree* 14 *in* $B_0(13)$ *when the tree for* $B_0(13)$ *is of Type III.*

Proof. Suppose not, then Equation (2.5) implies that the degree equation for $B_0(13)$ has the form $15 + 2x = w$. It follows from the relations above Equation (2.1) that w is the degree of the 3 exceptional characters in $B_0(13)$. Since χ_{14} is rational, it must be the case that $\chi_{14}{}^2$ has a norm of at least 3. Then $\chi_{14}{}^3$ must have χ_{14} as a constituent at least 3 times. Thus $\chi_{14}{}^3$ has the sum of the 3 exceptional characters as a constituent at least twice. Thus $6w \leqq 14^3 - 42$, whence $w < 451$. It is now an easy matter to verify that the only possible degree equations for $B_0(13)$ consistent with the relations above Eq. (2.1) are $1 + 14 + 66 + 66 = 147$ and $1 + 14 + 105 + 105 = 225$. In the former case, Lemma 2.1 and Brauer's work [**4**] applied with 11 as the prime yield a contradiction.

In the latter case, Lemmas 2.7 and 2.8 imply that $|\chi_{105}(\omega)| \leqq 9$. Let x_2 be a central involution. Then the coefficients $a(\omega, \omega, x_{13})$, $a(\omega, x_2, x_{13})$, $a(x_2, x_2, x_{13})$ and a count of Sylow 13-subgroups yield a contradiction. This completes the proof of Lemma 3.21.

Thus we see that the tree for $B_0(13)$ must be of Type I and the degree equation for $B_0(13)$ has the form

(3.7) $15 + x = w + z.$

Here Corollary 2.4 implies that $x \leqq 14^4 + 14^3 - 15 = 41145$. Also if θ_{91} is the alternating part of $\chi_{14}{}^2$, then the character $\chi_{14}\,\theta_{91}$ has χ_{14} as a constituent. Thus $\chi_{14}\,\theta_{91}$ has χ_w or χ_z as a constituent; thus min $(w, z) \leqq 1260$. Using these bounds and the relations above Equation (2.1) it is a straightforward but tedious matter to verify that the only possible solutions to Equation (3.7) are

	x	w	z		x	w	z
1)	35	25	25	34)	924	264	675
2)	40	25	30	35)	945	64	896
3)	66	25	56	36)	945	168	792
4)	100	25	90	37)	945	480	480
5)	105	30	90	38)	1080	420	675
6)	105	56	64	39)	1080	220	875
7)	126	64	77	40)	1470	77	1408
8)	165	90	90	41)	1470	675	810
9)	196	64	147	42)	1470	225	1260
10)	243	90	168	43)	1470	693	792
11)	300	90	225	44)	1782	147	1650
12)	300	147	168	45)	1920	675	1260
13)	352	147	220	46)	2250	25	2240
14)	360	25	350	47)	2310	675	1650
15)	378	168	225	48)	2640	675	1980
16)	490	25	480	49)	2640	30	2625
17)	490	64	441	50)	2640	225	2430
18)	495	90	420	51)	2640	576	2079
19)	495	30	480	52)	2700	90	2625
20)	495	160	350	53)	2835	225	2625
21)	560	25	550	54)	2835	420	2430
22)	560	225	350	55)	2835	1200	1650
23)	594	168	441	56)	3402	792	2625
24)	729	264	480	57)	3675	90	3600
25)	729	168	576	58)	3675	1260	2430
26)	729	324	420	59)	4200	90	4125
27)	750	90	675	60)	5760	875	4900
28)	750	324	441	61)	6600	675	5940
29)	768	90	693	62)	6930	225	6720
30)	768	108	675	63)	8100	30	8085
31)	880	220	675	64)	9408	675	8748
32)	924	64	875	65)	11025	480	10560
33)	924	147	792	66)	12000	675	11340

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67) 12000	108	11907	70) 23760	675	23100
68) 12936	576	12375	71) 32340	675	31680
69) 13365	420	12960	72) 36960	225	36750.

We have here excluded solutions to Equation (3.7) with x, w, or $z < 14$ since they have been considered in previous cases when $n < 14$.

In the next several lemmas we show that none of the solutions to Equation (3.7) gives an index four simple group.

LEMMA 3.22. *There are no index four simple groups with degree equation* (3.7) *for* $B_0(13)$ *for the solutions* 1) – 5), 7) – 14), 16) – 35), 37) – 53), *and* 55) – 72).

Proof. Here Equation (3.6) and consideration of $B_0(13) \cap B_0(11)$ eliminates solutions 7), 8), 18) – 20), 24), 29), 31) – 34), 40), 43), 44), 47) – 51), 56) – 59), 61) – 63), 65) and 68) – 72).

Now Lemma 2.9 and consideration of $B_0(13) \cap B_0(7)$ eliminates solutions 2) – 5), 9), 10), 12) – 14), 16), 17), 21), 23), 25) – 28), 30), 37) – 39), 41), 42), 45), 52), 55), 57), 58), 60), 64), 66) and 67).

Solutions 35) and 46) are eliminated by Lemmas 2.1 and 2.2 applied to $B_0(13) \cap B_0(2)$ and $B_0(13) \cap B_0(5)$, respectively.

In the case of solution 1), if $c = 2$, then χ_{35} cannot be in $B_0(5)$, whence Lemma 2.1 yields a contradiction. Similar arguments eliminate solutions 11), 22) and 53). This completes the proof of Lemma 3.22.

In the remaining cases, class algebra coefficients involving ω and other involutions are crucial to the arguments.

LEMMA 3.23. *There are no index four simple groups with degree equation* (3.7) *for* $B_0(13)$ *for the solutions* 6) *and* 54).

Proof. With solution 6), Lemmas 2.1 and 2.2 applied to $B_0(2) \cap B_0(13)$ yield $a = 6$. Now Lemmas 2.7 and 2.8 imply that $\chi_{56}(\omega) = 0, 4$ or -4 for any exceptional character χ_{56}. Then if x_2 is a central involution, Lemma 2.8 and the coefficients $a(\omega, \omega, x_{13})$, $a(\omega, x_2, x_{13})$ and $a(x_2, x_2, x_{13})$ yield a contradiction in each case.

In the case of solution 54), Lemma 2.3 implies that $\chi_{14}\chi_{420} = \chi_{2835} + \theta_{3045}$, where χ_{420} is any one of the exceptional characters in $B_0(13)$, and θ_{3045} is the sum of the remaining irreducible constituents of $\chi_{14}\chi_{420}$. Now θ_{3045} cannot have χ_{14} as a constituent since χ_{14}^2 would then have χ_{420} as a constituent which is absurd. But since θ_{3045} is not defect zero for 13, it must contain some constituent from $B_0(13)$. Now consideration of character values at an element of order 13 leads to a contradiction, completing the proof of Lemma 3.23.

LEMMA 3.24. *There are no index four simple groups with degree equation* (3.7) *for* $B_0(13)$ *with solution* 15).

Proof. Here Equation (3.6), Lemma 2.9 and Lemma 2.1 applied to $B_0(13) \cap$

$B_0(7)$ and $B_0(13) \cap B_0(11)$ imply that $|G| = 2^a 3^b 5^c 7\ 13$ where $c = 2$ or 3. A count of S_{13} subgroups yields $a + 4b \equiv 9 \pmod{12}$ when $c = 2$ and $a + 4b \equiv 0 \pmod{12}$ when $c = 3$.

Now if $s = \chi_{168}(\omega)$ and $t = \chi_{225}^{(i)}(\omega)$, then Lemmas 2.7 and 2.8 give $|s| \leqq 12$ and $s \equiv 0 \pmod 4$, while $|t| \leqq 17$ and $t \equiv 1 \pmod 4$. Then Equation (2.1) and consideration of the coefficient $a(\omega, \omega, x_{13})$ yield a contradiction unless we have one of the following sets of values.

	$\chi_{14}(\omega)$	$\chi_{378}(\omega)$	$\chi_{168}(\omega)$	$\chi_{225}^{(i)}(\omega)$	$\|C(\omega)\|^2$
1)	2	-6	-8	5	$2^{a+1}3^{b-2}5^c$
2)	2	6	4	5	$2^{a+2}3^{b-2}5^{c+1}$
3)	2	-18	0	-15	$2^{a+1}3^b 5^c$
4)	2	-30	-12	-15	$2^{a+2}3^{b-1}5^{c+1}$

It is apparent that ω is not a central involution in any of these cases. Let x_2 be a central involution. It follows from Lemma 2.1 that if $a \geqq 6$, then χ_{14} and χ_{378} are in $B_0(2)$. Even when $a < 6$, χ_{14} and χ_{378} must be in the same 2-block. Therefore $\chi_{14}(x_2) \equiv \chi_{378}(x_2) \pmod 4$. Now $\chi_{168}(x_2) \equiv 0 \pmod 8$, so Equation (2.1) implies that $\chi_{225}^{(i)}(x_2) \equiv 1 \pmod 4$, for $i = 1, 2$ and 3. Then Lemma 2.6 and the coefficients $a(\omega, x_2, x_{13})$ and $a(x_2, x_2, x_{13})$ yield a contradiction in cases 1) and 2) above.

To eliminate the cases 3) and 4), we concentrate on the prime 7. Now Lemma 2.1 implies $\chi_{225}^{(i)} \in B_0(7)$. The work of Brauer [**4**] then shows $|N(S_7): C(S_7)| = 6$ and the tree for $B_0(7)$ is a straight line. The degree equation for $B_0(7)$ is $1 + 3(225) = x + y + z$, where $x, y, z \equiv 13 \pmod{91}$. The only possible such equation is $1 + 3(225) = 104 + 104 + 468$.

Now by Lemma 3.7, and a count of S_7 and S_{13} subgroups in cases 3) and 4) we obtain $C(S_7) = S_7$. Consideration of the character χ_{14}^2 on 13-elements implies that the symmetric part, ϕ_{105}, of χ_{14}^2 has 1 as a constituent. Let $\phi_{105} = 1 + \phi_{104}$. Since $\phi_{104}(x_7) = -1$, ϕ_{104} must involve an irreducible from $B_0(7)$. Thus ϕ_{104} is irreducible. Now Brauer [**4**] implies that we have the following equation.

$$(3.8) \quad 1 + 3\chi_{225}^{(i)}(\omega) = \phi_{104}(\omega) + \chi_{104}(\omega) + \chi_{468}(\omega).$$

Since $\phi_{104}(\omega) = 8$ and $\chi_{225}^{(i)}(\omega) = -15$ in cases 3) and 4), Equation (3.8) simplifies to $\chi_{104}(\omega) + \chi_{468}(\omega) = -52$. This contradicts the fact, given by Lemmas 2.7 and 2.8 that $|\chi_{104}(\omega)| \leqq 8$ and $|\chi_{468}(\omega)| \leqq 36$. This completes the proof of Lemma 3.24.

Our final lemma deals with solution 36) of Equation (3.7).

LEMMA 3.25. *There are no index four simple groups with degree equation* (3.7) *for* $B_0(13)$ *with solution* 36).

Proof. Here Equation (3.6), Lemmas 2.1 and 2.9 and consideration of

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$B_0(11) \cap B_0(13)$ and $B_0(13) \cap B_0(7)$ yield

(3.9) $|G| = 2^a\, 3^b\, 5^c\, 7\; 11\; 13, \quad a \leqq 25, b \leqq 9, c \leqq 3.$

If $c = 1$ in Equation (3.9) the coefficient $a(x_{11}, x_{11}, x_{13})$ implies that G has no elements of order 55. Then consideration of $B_0(5) \cap B_0(11)$ implies that $|N(S_5): C(S_5)| \neq 2$. Thus $|N(S_5): C(S_5)| = 4$ and the trees for $B_0(5)$ are those illustrated in Figure 2.1. It is easy to verify that the tree for $B_0(5)$ must be of Type I. Now Lemma 2.3 implies that χ_{14}^2 involves the character, η, next to the principal character in the tree. Now $\eta \neq \chi_{14}$ so η must be a character not in $B_0(13)$ such that $\eta(1) \equiv 4 \pmod 5$ and $\eta(1) \leqq 196$. It is now clear that the degree equation for $B_0(5)$ must be $1 + 26 + 26 = 39 + 14$ or $1 + 26 + 91 = 14 + 104$. In each case Lemma 2.1 and consideration of $B_0(5) \cap B_0(11)$ yield a contradiction. Thus $c = 2$ or 3 in Equation (3.9).

Now set $s = \chi_{168}(\omega)$ and $t = \chi_{792}(\omega)$. Then Lemmas 2.7 and 2.8 give $|s| \leqq 12$ and $s \equiv 0 \pmod 4$, while $|t| \leqq 60$ and $t \equiv 0 \pmod 4$. Lemma 3.7 implies that ω does not centralize any 7-element. Now Equation (2.1), a count of S_{13} subgroups, and consideration of the coefficient $a(\omega, \omega, x_{13})$ yield a contradiction unless we have one of the following sets of values.

	$\chi_{14}(\omega)$	$\chi_{945}^{(i)}(\omega)$	$\chi_{168}(\omega)$	$\chi_{792}(\omega)$	$\lvert C(\omega)\rvert^2$
1)	2	5	0	8	$2^{a+9}3^{b-3}5^{c+1}$
2)	2	5	8	0	$2^{a+4}3^{b-3}5^{c}11^2$
3)	2	-39	8	-44	$2^{a+1}3^{b-2}5^{c-1}11^2$
4)	2	45	12	36	$2^{a+2}3^{b+2}5^{c}$
5)	2	45	0	48	$2^{a+3}3^{b}\,5^{c+1}$

Now let x_2 be any central involution. It follows from Lemma 2.1 that $\chi_{14} \in B_0(2)$, whence $\chi_{14}(x_2) \equiv 2 \pmod 4$. Also if $a \geqq 7$, then χ_{168} and χ_{792} are also in $B_0(2)$; thus $\chi_{168}(x_2) \equiv 8 \pmod{16}$ and $\chi_{792}(x_2) \equiv 8 \pmod{16}$. Even if χ_{168} and χ_{792} are not both in $B_0(2)$ they must be in the same 2-block, B, by Lemma 2.1. If $B \neq B_0(2)$, then $\chi_{168}(x_2) \equiv \chi_{792}(x_2) \pmod{16}$. These facts yield that ω cannot be a central involution in any of these cases. Next Lemma 2.6 and consideration of the coefficients $a(\omega, x_2, x_{13})$ and $a(x_2, x_2, x_{13})$ give a contradiction in cases 1) – 4) above.

In case 5) if $a \geqq 7$, then the coefficients $a(\omega, x_2, x_{13})$ and $a(x_2, x_2, x_{13})$ imply that $\chi_{14}(x_2) = 6$, $\chi_{945}^{(i)}(x_2) = -375$, $\chi_{168}(x_2) = -104$, and $\chi_{792}(x_2) = -264$. If y_2 is any other involution, then the coefficients $a(\omega, y_2, x_{13})$, $a(x_2, y_2, x_{13})$ and $a(y_2, y_2, x_{13})$ imply that $\chi_{14}(y_2) = 10$, $\chi_{945}^{(i)}(y_2) = 285$, $\chi_{168}(y_2) = 32$, and $\chi_{792}(y_2) = 264$. Now by Glauberman's Z^*-Theorem (cf. [**10**]), there is an involution x_2' in the same S_2 subgroup with x_2 such that x_2 and x_2' are conjugate. Let $y_2 = x_2 x_2'$, then restriction of χ_{168} and χ_{14} to $\langle x_2, y_2\rangle$ yields that $\chi_{168}(y_2) \geqq 40$. This contradiction eliminates case 5) when $a \geqq 7$.

In case 5) when $a < 7$, the coefficients $a(\omega, x_2, x_{13})$ and $a(x_2, x_2, x_{13})$ imply that $\chi_{14}(x_2) = -2$, $\chi_{945}^{(i)}(x_2) = -15$, $\chi_{168}(x_2) = -16$, and $\chi_{792}(x_2) = 0$. Here a count of S_{13} subgroups yields $|G| = 2^5\, 3^6\, 5^3\, 7\; 11\; 13$. Now by Lemma 3.7

$|C(S_7)|/21$ and if x_3 is a 3-element in $C(S_7)$, then $\chi_{14}(x_3) = -7$. Next set $u = \chi_{168}(x_3)$ and $v = \chi_{792}(x_3)$, whence $\chi_{945}(x_3) = u + v + 6$. Since the coefficient $a(x_2, x_3, x_{13})$ is non-negative, we find that $5u - v \geqq -120$. Similarly the coefficients $a(x_{11}, x_3, x_{13})$, $a(\omega, x_3, x_{13})$ and $a(x_3, x_{13}, x_{13})$ yield the inequalities $143u + 8v \leqq -3828$, $11u - 3v \geqq -66$, and $429u + 299v + 15444 \geqq 0$, respectively. Here x_{11} is an 11-element. This system of inequalities has no solution. This final contradiction completes the proof of Lemma 3.25.

We now restate our main theorem which follows immediately from Lemmas 3.1 – 3.3, 3.4, and 3.17 – 3.25.

THEOREM 3.26. *Let G be a finite simple group with a self-centralizing Sylow p-subgroup whose normalizer has order 4p. If there is a non-principal irreducible character in $B_0(p)$ of degree $n \leqq 15$, then G is isomorphic to one of the groups $O(5, 3)$, A_7, M_{11} and $Sz(8)$.*

Acknowledgement. The authors would like to express their gratitude to the referee for helpful suggestions.

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