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HOW SMALL CAN POLYNOMIALS BE IN AN INTERVAL OF GIVEN LENGTH?

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Abstract. In this paper, we provide two new extensions to a lemma of Bernik (1983). Applications are also discussed.

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1. Introduction. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be an integer polynomial and define $H(P) = H := \max_{0 \le j \le n}\{|a_j|\}$ to be the height of the polynomial $P(x)$. Define the classes of polynomials $\mathcal{P}_n$ and $\mathcal{P}_n(Q)$ by

$$\mathcal{P}_n := \left\{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n\right\},$$
$$\mathcal{P}_n(Q) := \left\{P(x) \in \mathbb{Z}[x] : \deg(P(x)) = n,\, H(P) \le Q\right\}.$$

We are interested in the following question: Given some $w \in \mathbb{R}$ how big is the set of $x \in \mathbb{R}$ for which $|P(x)| < H(P)^{-w}$ for infinitely many polynomials $P(x) \in \mathcal{P}_n$. Denote this set by $\mathcal{L}_n(w)$, i.e.

$$\mathcal{L}_n(w) := \left\{x \in \mathbb{R} : |P(x)| < H(P)^{-w} \text{ for i.m } P \in \mathcal{P}_n\right\},$$

where i.m stands for infinitely many. For $n > 1$, it follows from Dirichlet's pigeonhole principle that $\mathcal{L}_n(n) = \mathbb{R}$. In 1932, Mahler [**17**] conjectured that the Lebesgue measure of the set $\mathcal{L}_n(w)$, $\mu(\mathcal{L}_n(w))$, was zero when $w > n$, and indeed this was shown to be true by Sprindžuk [**20**] in 1964. Before Sprindžuk, there were some partial results. Mahler himself [**17**] proved the conjecture for $w > 4n$. This was improved by W. M. Schmidt [**19**] in 1961 to $w > 2n$. A further improvement was given by Volkmann [**21**] in 1962 who showed that the result was true for $w > \frac{4}{3}n$.

With regard to the Hausdorff dimension of the set, $\dim_{\mathcal{H}}(\mathcal{L}_n(w))$, the first result was proved by Jarník [**12**] in 1928 and then independently, using a different method to Jarník, by Besicovitch [**9**] in 1932. It is known as the Jarník–Besicovitch Theorem:

Theorem 1 (Jarník–Besicovitch Theorem (1932)). *Let* $w > 1$. *Then*

$$\dim_{\mathcal{H}}(\mathcal{L}_1(w)) = \frac{2}{w+1}.$$

This was extended for $n > 1$ by A. Baker and W. M. Schmidt [1] in 1970 who showed that

$$\frac{n+1}{w+1} \leq \dim_{\mathcal{H}}(\mathcal{L}_n(w)) < 2\frac{n+1}{w+1}.$$

In the same paper, it was conjectured that for $w > n$

$$\dim_{\mathcal{H}}(\mathcal{L}_n(w)) = \frac{n+1}{w+1}.$$

It had been shown previously by Kasch and Volkmann [13] that

$$\dim_{\mathcal{H}}(\mathcal{L}_2(w)) \leq \frac{3}{w+1}.$$

These two results prove that the conjecture is true for $n = 2$. In 1976, R. C. Baker [2] showed that when $w > 3$, one has $\dim_{\mathcal{H}}(\mathcal{L}_3(w)) \leq \frac{4}{w+1}$ and furthermore, when $n \geq 4$ if $w > (n^2 + n - 3)/3$ one has $\dim_{\mathcal{H}}(\mathcal{L}_n(w)) \leq \frac{n+1}{w+1}$. This result together with that of A. Baker and W. M. Schmidt proves the conjecture for $n = 3$ and for $n \geq 4$ if $w > (n^2 + n - 3)/3$. The conjecture was finally proved in 1983 by Bernik [6]. In his paper Bernik uses different methods to those of Baker and Schmidt which are based on the following lemma from the same paper.

LEMMA 1. *Let $\delta, \eta, \mu \in \mathbb{R}^+$ and let $Q_0(\delta, s)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials of degree $s > 1$ without common roots such that $\max(H(P), H(T)) = Q^\mu$, where $Q > Q_0(\delta, s)$. Assume that the interval $I \subset (-s, s) \subset \mathbb{R}$ with $|I| = Q^{-\eta}$. If there exists $\tau > 0$ such that for all $x \in I$*

$$\max\big(|P(x)|, |T(x)|\big) < Q^{-\tau},$$

then

$$\tau + \mu + 2\max\big(\tau + \mu - \eta, 0\big) < 2\mu s + \delta.$$

Lemma 1 can be thought of as a quantitative description of the fact that two relatively prime polynomials in $\mathbb{Z}[x]$ cannot both have very small absolute values (in terms of their degrees and heights) in an interval unless that interval is extremely short.

In [6], and for many results since, Lemma 1 was a key tool in disproving the existence of certain cases by obtaining contradictions. Generally speaking, Lemma 1 is useful when dealing with problems that are concerned with small first derivatives since Lemma 1 also shows that two polynomials $P(x), T(x) \in P_n(Q)$ cannot be simultaneously small at a point as well as having simultaneously small derivatives at that point. See [7], [8] and [10] for just some of the many examples of Lemma 1 being used.

In this paper, we improve on Lemma 1 and give an example of its use when considering the number of polynomials with bounded discriminants. We also provide an extension to the lemma and give reference to another application.

2. Main results. To begin, the necessary notation must first be introduced. It should be acknowledged that almost all of the following notation and definitions are due to Sprindžuk. Considering the roots of the polynomial P define $\alpha_1(P), \ldots, \alpha_{n_1}(P)$ to be the real roots and $\beta_1(P), \ldots, \beta_{\frac{n_2}{2}}(P)$ to be the non-real roots located in the upper-half plane. The set of non-real roots located in the lower-half plane will be denoted $\beta_{\frac{n_2}{2}+1}(P), \ldots, \beta_{n_2}(P)$. It is clear that each non-real root in the lower-half plane is just

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the complex conjugate of one of the non-real roots in the upper-half plane. With this in mind, the non-real roots in the lower-half plane are labelled so that $\overline{\beta}_i(P) = \beta_{\frac{n_2}{2}+i}(P)$ for $i = 1, \ldots, \frac{n_2}{2}$. Clearly $n_1 + n_2 = n$. Furthermore, define the roots of P in $\mathbb{Q}_p$ as $\gamma_1(P), \ldots, \gamma_{n_3}(P)$ with $n_3 \le n$. The roots of a second polynomial T are similarly spilt into the sets $\{\alpha_i(T)\}$, $\{\beta_j(T)\}$ and $\{\gamma_k(T)\}$, where $1 \le i \le m_1$, $1 \le j \le m_2$ with $m_1 + m_2 = n$, and $1 \le k \le m_3 \le n$.

For each real root $\alpha_i(P)$, the set $\mathcal{S}^1(\alpha_i(P))$ will be defined by

$$\mathcal{S}^1(\alpha_i(P)) = \left\{ x \in \mathbb{R} : |x - \alpha_i(P)| = \min_{l=1,\ldots,n_1} |x - \alpha_l(P)| \right\}.$$

In a similar fashion, analogues for $\mathbb{C}$ and $\mathbb{Q}_p$ are defined in the obvious way as follows:

$$\mathcal{S}^2(\beta_j(P)) = \left\{ z \in \mathbb{C}^+ : |z - \beta_j(P)| = \min_{l=1,\ldots,\frac{n_2}{2}} |z - \beta_l(P)| \right\},$$

$$\mathcal{S}^2(\overline{\beta}_j(P)) = \left\{ z \in \mathbb{C}^- : |z - \overline{\beta}_j(P)| = \min_{l=1,\ldots,\frac{n_2}{2}} \left|z - \overline{\beta}_l(P)\right| \right\},$$

$$\mathcal{S}^3(\gamma_k(P)) = \left\{ \omega \in \mathbb{Q}_p : |\omega - \gamma_k(P)|_p = \min_{l=1,\ldots,n_3} |\omega - \gamma_l(P)|_p \right\},$$

where $|.|_p$ denotes the p-adic norm, $\mathbb{C}^+ := \{z \in \mathbb{C} : Im(z) \ge 0\}$ and $\mathbb{C}^- := \{z \in \mathbb{C} : Im(z) \le 0\}$. Clearly, if $z \in \mathbb{C}^+$ and $z \in \mathcal{S}^2(\beta_j(P))$ for some $j \in 1, \ldots, \frac{n_2}{2}$, then $\overline{z} \in \mathcal{S}^2(\overline{\beta}_j(P))$.

In Lemma 2, z will be taken in a disk in $\mathbb{C}$ and differences of the form $|z - \beta_j|$ for some $z \in \mathbb{C}$ and some non-real root β_j will be estimated. If it can be assumed that $z \in \mathcal{S}^2(\beta_j(P))$ (i.e. z is such that $im(z) > 0$) and that $j \in \left\{1, \ldots, \frac{n_2}{2}\right\}$ then, as will be seen, estimating $|z - \beta_j|$ will be simplified greatly. This is the reason for considering the sets $\mathcal{S}^2(\beta_j(P))$ and $\mathcal{S}^2(\overline{\beta}_j(P))$ separately. Furthermore, by symmetry, estimating $|z - \beta_j|$ will give an estimate for $|\overline{z} - \overline{\beta}_j|$. Differences of the form $|z - \overline{\beta}_j|$ and $|\overline{z} - \beta_j|$ will also have to be considered but unfortunately nothing is known about these and so Lemma 6, which is stated in Section 3, will be used to estimate these.

The following notation will also be used:

$$\mathcal{S}\big(\alpha_i(P), \beta_j(P), \gamma_k(P)\big) = \mathcal{S}^1(\alpha_i(P)) \times \mathcal{S}^2(\beta_j(P)) \times \mathcal{S}^3(\gamma_k(P)).$$

The sets $\mathcal{S}^1(\alpha_i(T))$, $\mathcal{S}^2(\beta_j(T))$, $\mathcal{S}^2(\overline{\beta}_j(T))$ and $\mathcal{S}^3(\gamma_k(T))$ are defined similarly.

Suppose $(x, y, z) \in \mathcal{S}(\alpha_1(P), \beta_1(P), \gamma_1(P)) \cap \mathcal{S}(\alpha_1(T), \beta_1(T), \gamma_1(T))$. The other roots are then ordered according to their distance from $\alpha_1(J)$, $\beta_1(J)$ and $\gamma_1(J)$, where $J(x) = P(x)$ or $T(x)$, as follows:

$$|\alpha_1(J) - \alpha_2(J)| \ \le |\alpha_1(J) - \alpha_3(J)| \ \le \cdots \le |\alpha_1(J) - \alpha_{n_1}(J)|,$$

$$|\beta_1(J) - \beta_2(J)| \ \le |\beta_1(J) - \beta_3(J)| \ \le \cdots \le |\beta_1(J) - \beta_{\frac{n_2}{2}}(J)|,$$

$$|\gamma_1(J) - \gamma_2(J)|_p \le |\gamma_1(J) - \gamma_3(J)|_p \le \cdots \le |\gamma_1(J) - \gamma_{n_3}(J)|_p.$$

Note that the set of differences $|\beta_1(P) - \beta_i(P)|$ is only taken up as far as $i = \frac{n_2}{2}$ since $|\beta_1(P) - \beta_i(P)| = |\overline{\beta}_1(P) - \overline{\beta}_i(P)|$, and so only $i \le \frac{n_2}{2}$ need to be considered since any resulting calculations, as already discussed, will be the same for $\frac{n_2}{2} < i \le n_2$. This is a common technique, see, for example, [8].

Define the real numbers $\rho_i(J)$, $\lambda_i(J)$, $\sigma_i(J)$ such that

$$|\alpha_1(J) - \alpha_i(J)| = Q^{-\rho_i(J)}, \quad i = 2, \ldots, n_1,$$

$$|\beta_1(J) - \beta_i(J)| = Q^{-\lambda_i(J)}, \quad i = 2, \ldots, \frac{n_2}{2},$$

$$|\gamma_1(J) - \gamma_i(J)|_p = Q^{-\sigma_i(J)}, \quad i = 2, \ldots, n_3.$$

Furthermore, define

$$q_i(J) = \rho_{i+1}(J) + \cdots + \rho_{n_1}(J), \quad i = 1, \cdots, n_1 - 1,$$

$$r_i(J) = \lambda_{i+1}(J) + \cdots + \lambda_{\frac{n_2}{2}}(J), \quad i = 1, \cdots, \frac{n_2}{2} - 1,$$

$$s_i(J) = \sigma_{i+1}(J) + \cdots + \sigma_{n_3}(J), \quad i = 1, \cdots, n_3 - 1.$$

Let $I \subset \mathbb{R}$ be an interval, $C \subset \mathbb{C}$ be a disk and $K \subset \mathbb{Q}_p$ be a cylinder, and define the parallelepiped $\Omega = I \times C \times K \subset \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$. Fix $\delta_1 > 0$. Any complex number z lying in C with $|\mathrm{Im}(z)| < \delta_1$ will be excluded. As long as δ_1 is an arbitrary small number, this can be done without loss of generality. Later in the paper will appear inequalities of the form $|z - \beta| < Q^{-\nu}$. From this, with the condition $|\mathrm{Im}(z)| \geq \delta_1$, one obtains $|\mathrm{Im}(\beta)| \geq \frac{\delta_1}{2}$ i.e. $\beta \notin \mathbb{R}$. In particular, this implies that $|\beta_i - \overline{\beta}_j| > \delta_1$, and for any real root α_i, $|\alpha_i - \beta_j| = |\alpha_i - \overline{\beta}_j| > \delta_1$.

Let $\mu_P(A)$ be the Haar measure of a measurable set $A \subset \mathbb{Q}_p$. The first of the main results of this paper is the following extension of Lemma 1.

LEMMA 2. *Let $\delta, \eta_r \in \mathbb{R}^+$ for $r = 1, 2, 3$ and let $Q_0(\delta, n)$ be a sufficiently large real number. Furthermore, let $P, T \in \mathcal{P}_n(Q)$ be polynomials without common roots such that* $\max(H(P), H(T)) = Q$, *where $Q > Q_0(\delta, n)$. Take $\Omega = I \times C \times K \subset \mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$ with $\mu(I) = Q^{-\eta_1}$, diam$(C) = Q^{-\eta_2}$, $\mu_p(K) = Q^{-\eta_3}$. If there exist $\tau_1, \tau_2, \tau_3 > 0$ such that for all* $(x, z, \omega) \in \Omega \cap \mathcal{S}(\alpha_1(P), \beta_1(P), \gamma_1(P)) \cap \mathcal{S}(\alpha_1(T), \beta_1(T), \gamma_1(T))$

$$\max\left(|P(x)|, |T(x)|\right) < Q^{-\tau_1},$$

$$\max\left(|P(z)|, |T(z)|\right) < Q^{-\tau_2},$$

$$\max\left(|P(\omega)|_p, |T(\omega)|_p\right) < Q^{-\tau_3},$$

and for $J(x) = P(x)$ or $T(x)$

$$\tau_1 + 1 \geq q_1(J) + \rho_2(J),$$

$$\tau_2 + 1 \geq r_1(J) + \lambda_2(J),$$

$$\tau_3 \geq s_1(J) + \sigma_2(J), \tag{1}$$

then

$$\tau_1 + 2\tau_2 + \tau_3 + 3 + 2\left(\sum_{i=1}^{n_1-1} q_i(J) + 2\sum_{j=1}^{\frac{n_2}{2}-1} r_j(J) + \sum_{k=1}^{n_3-1} s_k(J)\right) \leq 2n + \delta, \tag{2}$$

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and furthermore,

$$\tau_1+2\tau_2+\tau_3+3+2\left(\sum_{j=1}^{n_1-1}\max\left(\tau_1+1-j\eta_1,0\right)+2\sum_{j=1}^{\frac{n_2}{2}-1}\max\left(\tau_2+1-j\eta_2,0\right)\right.$$
$$\left.+\sum_{j=1}^{n_3-1}\max\left(\tau_3-j\eta_3,0\right)\right)<2n+\delta. \tag{3}$$

We will in fact show that (3) follows from (2). Although (2) is more powerful than (3), it is more difficult to use; see remark 1 in Section 5.1.

The second result we present is the first generalisation of Bernik's original lemma that removes the restriction on the size of the polynomials and allows for some of the values, τ_i, $i=1,2,3$, to be negative. In particular, this allows $|P(x)|$ to be very large. More notation is needed. Define the set Π_3 by

$$\Pi_3=I_1\times I_2\times I_3\subset\mathbb{R}^3,$$

with

$$\Pi_3\cap\left\{(x_1,x_2,x_3)\in\mathbb{R}^3:|x_i-x_j|<\epsilon_0,\ 1\le i<j\le 3\right\}=\emptyset.$$

So Π_3 is a three-dimensional box which does not intersect the thickened (by $2\epsilon_0$) planes $x_i=x_j$.

Let $P(x),T(x)\in\mathbb{Z}[x]$ be of degrees n_1 and n_2, respectively, with $3<n_1,n_2\le n$. Furthermore, let $\alpha_1,\alpha_2,\ldots,\alpha_{n_1}$ be the roots of $P(x)$ and let $\beta_1,\beta_2,\ldots,\beta_{n_2}$ be the roots of $T(x)$. Define the intervals

$$\nu_i^r(P):=I_r\cap\mathcal{S}^1(\alpha_i),\quad i=1,\ldots,n_1,\ r=1,2,3,$$
$$\nu_j^r(T):=I_r\cap\mathcal{S}^1(\beta_j),\quad j=1,\ldots,n_2,\ r=1,2,3. \tag{4}$$

Although it is possible that for some i and j, $\nu_i^r(P)=\emptyset$, $\nu_j^r(T)=\emptyset$, the following lemma guarantees that the sets are not empty for all i and j.

LEMMA 3. *There exist at least one pair i and j, such that for each $r=1,2,3$,*

$$|\nu_i^r(P)|\ge\frac{|I_r|}{n}\ \textit{and}\ |\nu_j^r(T)|\ge\frac{|I_r|}{n}.$$

Proof. Assume that for all $i=1,\ldots,n_1$,

$$|\nu_i^r(P)|<\frac{|I_r|}{n},$$

and note that since $\nu_i^r(P)=I_r\cap\mathcal{S}^1(\alpha_i)$,

$$\bigcup_{i=1}^{n_1}\nu_i^r(P)=I_r,$$

and so

$$\left|\bigcup_{i=1}^{n_1}\nu_i^r(P)\right|=|I_r|<\frac{|\bigcup_{i=1}^{n_1}I_r|}{n}<\sum_{i=1}^{n_1}\frac{|I_r|}{n}=\frac{n_1|I_r|}{n}<|I_r|,$$

which is a contradiction. The proof is the same for the set $\nu_j^r(T)$. $\square$

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We will denote one such pair of roots for which $\nu_i^r(P) \neq \emptyset$, $\nu_j^r(T) \neq \emptyset$ by α_1^r and β_1^r for each $r = 1, 2, 3$. Using this new notation, the second of the main results is now presented.

LEMMA 4. *Let $\delta, \mu, \eta_r \in \mathbb{R}^+$ for $r = 1, 2, 3$, and let $H_0(\delta, n)$ be a sufficiently large real number. Furthermore, let $P(x), T(x) \in \mathbb{Z}[x]$ be polynomials without common roots of degree n_1 and n_2, respectively, with $3 \leq n_1, n_2 \leq n$ such that $\max(H(P), H(T)) = H^\mu$, where $H > H_0(\delta, n)$. Assume that the intervals $I_r \subset \mathbb{R}$ with $|I_r| = H^{-\eta_r}$ for $r = 1, 2, 3$. If there exists $\tau_1 > 0$ and $\tau_2, \tau_3 \in \mathbb{R}$ such that for all $(x_1, x_2, x_3) \in \Pi_3 \cap S_P(\alpha_1^1, \alpha_1^2, \alpha_1^3) \cap S_T(\beta_1^1, \beta_1^2, \beta_1^3)$ with*

$$\alpha_1^r \neq \alpha_1^{r'} \text{ and } \beta_1^r \neq \beta_1^{r'}, \text{ for } 1 \leq r < r' \leq 3 \tag{5}$$

the inequality

$$\max\left(|P(x_r)|, |T(x_r)|\right) < H^{-\tau_r}, \quad 1 \leq r \leq 3,$$

holds, then

$$\sum_{r=1}^{3} \left(\tau_r + \mu + 2\max\left(\tau_r + \mu - \eta_r, 0\right)\right) < (n_1 + n_2)\mu + \delta.$$

It will become evident from the proof of Lemma 4 that there is nothing special about choosing to state the lemma for three values. In fact, it will be clear that the proof can be adapted for any k values with $2 \leq k < n$ provided that (5) holds.

If $\tau_1, \tau_2, \tau_3 > 0$, one does not need (5) since by definition of Π_3 for $x \in S(\alpha_1^r)$ and $y \in S(\alpha_1^{r'})$, $1 \leq r < r' \leq 3$, one has that $|x - y| > \epsilon_0$. Using this and Lemma 5, which is stated in Section 3, one can easily show that $|\alpha_1^r - \alpha_1^{r'}| > \frac{\epsilon_0}{2}$. Similarly, $|\beta_1^r - \beta_1^{r'}| > \frac{\epsilon_0}{2}$ for $1 \leq r < r' \leq 3$. Otherwise, if even one of the $\tau_i < 0$, then without (5) it would not be possible to ensure that there exist three distinct roots which are essential for the proof of Lemma 4 as will be seen.

3. Preliminary results.

In this section, several very useful lemmas are presented. The proof for the real inequalities below can be found in [**5**] and for the complex and p-adic inequalities in [**16**].

LEMMA 5. *Let $P \in P_n(Q)$ and let u represent x or z and θ represent α or β. Then for $u \in \mathcal{S}^1(\theta_1)$ or $u \in \mathcal{S}^2(\theta_1)$ and $w \in \mathcal{S}^3(\gamma_1)$, the inequalities*

$$|u - \theta_1| \leq n\frac{|P(u)|}{|P'(u)|} \qquad \text{for} \quad P'(u) \neq 0,$$

$$|w - \gamma_1|_p \leq n\frac{|P(w)|_p}{|P'(w)|_p} \qquad \text{for} \quad P'(w) \neq 0,$$

$$|u - \theta_1| \leq 2^{n-1}\frac{|P(u)|}{|P'(\theta_1)|} \qquad \text{for} \quad P'(\theta_1) \neq 0,$$

$$|w - \gamma_1|_p \leq 2^{n-1}\frac{|P(w)|_p}{|P'(\gamma_1)|_p} \qquad \text{for} \quad P'(\gamma_1) \neq 0$$

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hold, together with

$$|u-\theta_1| \leq \min_{2\leq j\leq n}\left(2^{n-j}|P(u)||P'(\theta_1)|^{-1}\prod_{k=2}^{j}|\theta_1-\theta_k|\right)^{\frac{1}{j}} \quad \textit{for} \quad P'(\theta_1)\neq 0,$$

$$|w-\gamma_1|_p \leq \min_{2\leq j\leq n}\left(2^{n-j}|P(w)||P'(\gamma_1)|^{-1}\prod_{k=2}^{j}|\gamma_1-\gamma_k|_p\right)^{\frac{1}{j}} \quad \textit{for} \quad P'(\gamma_1)\neq 0.$$

The proof for the following lemma can be found in [11].

LEMMA 6. *Let $P(x)$ be a polynomial of degree n, with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and leading coefficient a_n. Then for any k-tuple of distinct roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$, $1\leq i_1 < i_2 < \ldots < i_k \leq n$, $k\leq n$,*

$$|\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}| < c(n)\frac{H(P)}{|a_n|},$$

where c(n) is a positive constant depending only on n.

4. Proof of main results. Under the assumptions of Lemma 2, it was shown in [6] that $q_j(P)\geq \tau_1+1-j\eta_1$. In [8], it was shown that $r_j(P)\geq\tau_2+1-j\eta_2$ and $s_j(P)\geq\tau_3-j\eta_3$. Thus under the assumptions of Lemma 2, the following system of inequalities can be taken to hold,

$$\begin{aligned} q_j(P) &\geq \tau_1+1-j\eta_1,\\ r_j(P) &\geq \tau_2+1-j\eta_2,\\ s_j(P) &\geq \tau_3-j\eta_3. \end{aligned} \tag{6}$$

It is clear that using (6), (3) follows immediately from (2). Thus proving Lemma 2 is now equivalent to proving (2) holds.

Proof of Lemma 2. All the following calculations are analogous to those carried out in [6] and [8]. To begin define $\mathcal{K}(\alpha_i,\beta_j)=|\alpha_i(P)-\beta_j(P)||\alpha_i(P)-\beta_j(T)||\alpha_i(T)-\beta_j(P)||\alpha_i(T)-\beta_j(T)|$ and note that since, by assumption, P and T have no common roots:

$$\begin{aligned} 1 &\leq |R(P,T)||R(P,T)|_p\\ &\leq |a_n|^n|b_n|^n\prod_{1\leq i\leq j\leq n_1}|\alpha_i(P)-\alpha_j(T)|\prod_{1\leq i\leq j\leq n_2}|\beta_i(P)-\beta_j(T)|\\ &\quad\times\prod_{1\leq i\leq j\leq n_3}|\gamma_i(P)-\gamma_j(T)|_p\times\prod_{\substack{1\leq i\leq n_1\\ 1\leq j\leq n_2}}\mathcal{K}(\alpha_i,\beta_j). \end{aligned}$$

Here the basic property of the p-adic norm that for any $a\in\mathbb{Z}$ one always has $1\leq|a||a|_p$ is being used.

Suppose that $|a_n|=Q^{\zeta_1}$, for $0\leq\zeta_1\leq 1$, and $|b_n|=Q^{\zeta_2}$, for $0\leq\zeta_2\leq 1$. Furthermore, recall that $\beta_{i+\frac{n_2}{2}}(P):=\overline{\beta}_i(P)$ and note

$$\prod_{1\le i\le j\le n_2}|\beta_i(P)-\beta_j(T)| = \prod_{1\le i\le j\le \frac{n_2}{2}}|\beta_i(P)-\beta_j(T)|\prod_{1\le i\le j\le \frac{n_2}{2}}|\overline{\beta}_i(P)-\overline{\beta}_j(T)|$$
$$\times\prod_{1\le i\le \frac{n_2}{2},1\le j\le\frac{n_2}{2}}\left|\beta_i(P)-\overline{\beta}_j(T)\right|\prod_{1\le i\le \frac{n_2}{2},1\le j\le\frac{n_2}{2}}|\overline{\beta}_i(P)-\beta_j(T)|$$
$$=\left(\prod_{1\le i\le j\le \frac{n_2}{2}}|\beta_i(P)-\beta_j(T)|\prod_{1\le i\le \frac{n_2}{2},1\le j\le\frac{n_2}{2}}|\beta_i(P)-\overline{\beta}_j(T)|\right)^2,$$

since it is clear that $|\beta_i(P)-\beta_j(T)|=|\overline{\beta}_i(P)-\overline{\beta}_j(T)|$ and $|\beta_i(P)-\overline{\beta}_j(T)|=|\overline{\beta}_i(P)-\beta_j(T)|$. Nothing is known about the distances $|\beta_i(P)-\overline{\beta}_j(T)|$ and in fact these could be very large. Similarly, nothing is known about the distances $|\alpha_i-\beta_j|$ which again could be very large. Using Lemma 6, however, the differences $|\beta_i(P)-\overline{\beta}_j(T)|$ and $|\alpha_i-\beta_j|$ can be bounded. This can be done since for each i,j, the triangle inequality gives that $|\beta_i(P)-\overline{\beta}_j(T)|\le 2\max\{|\beta_i(P)|,|\overline{\beta}_j(T)|\}$, and so for some $0\le f_1,\dots,f_{\frac{n_2}{2}},g_1,\dots,g_{\frac{n_2}{2}}\le\frac{n_2}{2}$

$$\prod_{1\le i\le j\le \frac{n_2}{2}}\left|\beta_i(P)-\overline{\beta}_j(T)\right|\le 2^{\frac{n_2^2}{4}}|\beta_1(P)|^{f_1}\dots|\beta_{\frac{n_2}{2}}(P)|^{f_{\frac{n_2}{2}}}\left|\overline{\beta}_1(T)\right|^{g_1}\dots\left|\overline{\beta}_{\frac{n_2}{2}}(T)\right|^{g_{\frac{n_2}{2}}}$$
$$<2^{\frac{n_2^2}{4}}\left(\frac{H(P)}{|a_n|}\right)^{\frac{n_2}{2}}\left(\frac{H(T)}{|b_n|}\right)^{\frac{n_2}{2}}\le 2^{\frac{n_2^2}{4}}Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)}$$
$$<c_1(n)Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)},$$

for some constant $c_1(n)>0$.

The same argument can be made for the differences $|\alpha_i-\beta_j|$, so that $\prod|\alpha_i-\beta_j|<c_2(n)Q^{\frac{n}{2}(1-\zeta_1)+\frac{n}{2}(1-\zeta_2)}$, for some constant $c_2(n)>0$. Thus, for Q sufficiently large, $Q^\delta>c_1(n)c_2(n)$ and $R(P,T)$ can be rewritten as

$$\begin{aligned}1&\le|R(P,T)||R(P,T)|_p\\&\le(a_n)^n(b_n)^nQ^{n(1-\zeta_1)+n(1-\zeta_2)+\delta}\prod_{1\le i\le j\le n_1}|\alpha_i(P)-\alpha_j(T)|\prod_{1\le i\le j\le\frac{n_2}{2}}|\beta_i(P)-\beta_j(T)|^2\\&\quad\times\prod_{1\le i\le j\le n_3}|\gamma_i(P)-\gamma_j(T)|_p\\&\le Q^{2n+\delta}\prod_{1\le i\le j\le n_1}|\alpha_i(P)-\alpha_j(T)|\prod_{1\le i\le j\le\frac{n_2}{2}}|\beta_i(P)-\beta_j(T)|^2\prod_{1\le i\le j\le n_3}|\gamma_i(P)-\gamma_j(T)|_p.\end{aligned}\tag{7}$$

Now the proof revolves around bounding each of the products of (7). It is assumed without loss of generality that

$$\begin{aligned}q_1(T)&\le q_1(P),\\r_1(T)&\le r_1(P),\\s_1(T)&\le s_1(P).\end{aligned}\tag{8}$$

This assumption can be made since the real, complex and p-adic roots can now be considered separately.

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First consider the differences of the real roots and recall the identity, for $P \in P_n(Q)$, $|P'(\alpha_i)| = \prod_{\substack{j=1,\dots,n,\\ j\neq i}} |a_n||\alpha_i - \alpha_j|$. Then for $x \in \mathcal{S}^1(\alpha_1(P))$ by Lemma 5,

$$|x - \alpha_1(P)| \leq 2^{n-1}\frac{|P(x)|}{|P'(\alpha_1)|} \ll Q^{-\tau_1 - 1 + q_1(P)}.$$

Similarly for $y \in \mathcal{S}^1(\alpha_1(T))$,

$$|y - \alpha_1(T)| \ll Q^{-\tau_1 - 1 + q_1(T)}.$$

So for $x \in \mathcal{S}^1(\alpha_1(P))$ and $y \in \mathcal{S}^1(\alpha_1(T))$, using (6) and (8),

$$\begin{aligned}|\alpha_1(P) - \alpha_1(T)| &\leq |\alpha_1(P) - x| + |x - y| + |y - \alpha_1(T)|\\ &\ll Q^{-\tau_1-1+q_1(P)} + Q^{-\eta_1} + Q^{-\tau_1-1+q_1(T)} \ll Q^{-\tau_1-1+q_1(P)}.\end{aligned}$$

This gives

$$\begin{aligned}\prod_{2\leq j\leq n_1} |\alpha_1(P) - \alpha_j(T)| &\leq \prod_{2\leq j\leq n_1}\Big(|\alpha_1(P) - \alpha_1(T)| + |\alpha_1(T) - \alpha_j(T)|\Big)\\ &\ll \prod_{2\leq j\leq n_1}\Big(Q^{-\tau_1-1+q_1(P)} + Q^{-\rho_j(T)}\Big).\end{aligned}$$

Recall that (1) gives $\tau_1 + 1 - q_1(P) \geq \rho_2(P)$. Using this it is seen that

$$\prod_{2\leq j\leq n_1} |\alpha_1(P) - \alpha_j(T)| \ll \prod_{2\leq j\leq n_1}\Big(Q^{-\rho_2(P)} + Q^{-\rho_j(T)}\Big) \ll \prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P), -\rho_j(T))}. \tag{9}$$

Similarly

$$\prod_{2\leq i\leq n_1} |\alpha_i(P) - \alpha_1(T)| \ll \prod_{2\leq i\leq n_1}\Big(Q^{-\rho_i(P)} + Q^{-\rho_2(P)}\Big) \ll \prod_{2\leq i\leq n_1} Q^{-\rho_i(P)} = Q^{-q_1(P)}. \tag{10}$$

Combining (9) and (10) gives

$$\prod_{2\leq j\leq n_1} |\alpha_1(P) - \alpha_j(T)| \prod_{2\leq i\leq n_1} |\alpha_i(P) - \alpha_1(T)| \ll \prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P), -\rho_j(T))} Q^{-q_1(P)}.$$

If there exists $\phi \in \mathbb{Z}$ with $2 \leq \phi \leq n_1$, such that

$$-\rho_2(P) < -\rho_j(T) \ \ \forall\ j \in [2, \phi] \quad \text{and} \quad -\rho_2(P) \geq -\rho_j(T) \ \ \forall\ j \in (\phi, n_1],$$

then

$$\begin{aligned}\prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P), -\rho_j(T))} &= Q^{-\rho_2(T) - \dots - \rho_\phi(T) - (n_1-\phi)\rho_2(P)}\\ &\leq Q^{-\rho_2(T) - \dots - \rho_\phi(T) - \rho_{\phi+1}(P) - \dots - \rho_{n_1}(P)} \leq Q^{-q_1(P)}.\end{aligned}$$

If, on the other hand, no such ϕ exists, i.e. $-\rho_2(P) \geq -\rho_j(T)$ for all $j \in [2, n_1]$, then

$$\prod_{2\leq j\leq n_1} Q^{\max(-\rho_2(P), -\rho_j(T))} = Q^{-(n_1-1)\rho_2(P)} \leq Q^{-\rho_2(P) - \rho_3(P) - \dots - \rho_{n_1}(P)} = Q^{-q_1(P)}.$$

In either case

$$\prod_{2\leq j\leq n_1}|\alpha_1(P)-\alpha_j(T)|\prod_{2\leq i\leq n_1}|\alpha_i(P)-\alpha_1(T)|\ll Q^{-2q_1(P)}. \tag{11}$$

Now consider

$$\begin{aligned}\prod_{3\leq j\leq n_1}|\alpha_2(P)-\alpha_j(T)| &\leq \prod_{3\leq j\leq n_1}\big(|\alpha_2(P)-\alpha_1(P)|+|\alpha_1(P)-\alpha_1(T)|+|\alpha_1(T)-\alpha_j(T)|\big)\\ &\ll \prod_{3\leq j\leq n_1} Q^{\max(-\rho_2(P),-\rho_j(T))}.\end{aligned}$$

Similarly

$$\prod_{3\leq i\leq n_1}|\alpha_i(P)-\alpha_2(T)|\ll \prod_{3\leq i\leq n_1} Q^{\max(-\rho_i(P),\rho_2(T))}.$$

Arguing in an identical fashion to (11), it is clear that

$$\prod_{3\leq i\leq n_1}|\alpha_i(P)-\alpha_2(T)|\prod_{3\leq j\leq n_1}|\alpha_2(P)-\alpha_j(T)|\ll Q^{-2q_2(P)}.$$

More generally, using the same approach

$$\prod_{k\leq i\leq n_1}|\alpha_i(P)-\alpha_{k-1}(T)|\prod_{k\leq j\leq n_1}|\alpha_{k-1}(P)-\alpha_j(T)|\ll Q^{-2q_{k-1}(P)}.$$

The final case that needs considering is when $i=j\geq 2$. Note

$$\begin{aligned}|\alpha_i(P)-\alpha_i(T)| &\leq |\alpha_i(P)-\alpha_1(P)|+|\alpha_1(P)-\alpha_1(T)|+|\alpha_1(T)-\alpha_i(T)|\\ &\ll Q^{-\rho_i(P)}+Q^{-\rho_2(P)}+Q^{-\rho_i(T)}\ll Q^{\max(-\rho_i(P),-\rho_i(T))}.\end{aligned} \tag{12}$$

So finally

$$\begin{aligned}\prod_{1\leq i\leq j\leq n_1}|\alpha_i(P)-\alpha_j(T)| &= \prod_{\substack{1\leq i\leq j\leq n_1\\ i\neq j}}|\alpha_i(P)-\alpha_j(T)||\alpha_1(P)-\alpha_1(T)|\cdots|\alpha_{n_1}(P)-\alpha_{n_1}(T)|\\ &\ll |\alpha_1(P)-\alpha_1(T)|\cdots|\alpha_{n_1}(P)-\alpha_{n_1}(T)|Q^{-2\sum_{i=1}^{n_1-1}q_i(P)}\\ &\ll Q^{-\tau_1-1+q_1(P)}\prod_{i=2}^{n_1}Q^{\max(-\rho_i(P),-\rho_i(T))}Q^{-2\sum_{i=1}^{n_1-1}q_i(P)}\\ &\leq Q^{-\tau_1-1+q_1(P)}Q^{-q_1(P)}Q^{-2\sum_{i=1}^{n_1-1}q_i(P)}\\ &= Q^{-\left(\tau_1+1+2\sum_{i=1}^{n_1-1}q_i(P)\right)}.\end{aligned} \tag{13}$$

Identical calculations are carried out when considering the differences between the complex roots. Using Lemma 5 along with equations (6) and (8) gives

$$|\beta_1(P)-\beta_1(T)|\ll Q^{-\tau_2-1+r_1(P)}.$$

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Using this and (1), it can be shown that

$$\prod_{2\le k\le i\le \frac{n_2}{2}} |\beta_i(P)-\beta_{k-1}(T)| \prod_{2\le k\le j\le \frac{n_2}{2}} |\beta_{k-1}(P)-\beta_l(T)| \ll Q^{-2r_{k-1}(P)}.$$

Furthermore, by the same method used to obtain (12), it can be shown that

$$|\beta_i(P)-\beta_i(T)| \ll Q^{\max(-\lambda_i(P),-\lambda_i(T))}.$$

So finally

$$\prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P)-\beta_j(T)| \ll Q^{-\left(\tau_2+1+2\left(r_1(P)+r_2(P)+\cdots+r_{\frac{n_2}{2}-1}(P)\right)\right)}.$$

Thus

$$\prod_{1\le i\le j\le \frac{n_2}{2}} |\beta_i(P)-\beta_j(T)|^2 \le Q^{-2\left(\tau_2+1+2\left(r_1(P)+\cdots+r_{\frac{n_2}{2}-1}(P)\right)\right)}. \tag{14}$$

Finally, the p-adic case is considered. Again almost identical calculations are carried out. Using Lemma 5 along with equations (6) and (8) gives

$$|\gamma_1(P)-\gamma_1(T)|_p \ll Q^{-\tau_3+s_1(P)}.$$

Using this and (1), it can be shown that

$$\prod_{2\le k\le i\le n_3} |\gamma_i(P)-\gamma_{k-1}(T)|_p \prod_{2\le k\le j\le n_3} |\gamma_{k-1}(P)-\gamma_l(T)|_p \ll Q^{-2s_{k-1}(P)}.$$

Furthermore, it can be shown that

$$|\gamma_i(P)-\gamma_i(T)|_p \ll Q^{\max(-\sigma_i(P),-\sigma_i(T))}.$$

So finally

$$\prod_{1\le i\le j\le n_3} |\gamma_i(P)-\gamma_j(T)|_p \ll Q^{-\left(\tau_3+2\left(s_1(P)+s_2(P)+\cdots+s_{n_3}(P)\right)\right)}. \tag{15}$$

Using (13), (14) and (15) in (7) gives

$$\begin{aligned} 1 &\le |R(P,T)||R(P,T)|_p \\ &\le Q^{2n+\delta} Q^{-\left(\tau_1+1+2(q_1(P)+\cdots+q_{n_1-1}(P))+2\left(\tau_2+1+2\left(r_1(P)+\cdots+r_{\frac{n_2}{2}-1}(P)\right)\right)+\tau_3+2\left(s_1(P)+\cdots+s_{n_3-1}(P)\right)\right)}, \end{aligned}$$

so that

$$\tau_1+2\tau_2+\tau_3+3+2\left(\sum_{i=1}^{n_1-1} q_i(P)+2\sum_{j=1}^{\frac{n_2}{2}-1} r_j(P)+\sum_{k=1}^{n_3-1} s_k(P)\right) \le 2n+\delta,$$

as required. □

The proof of Lemma 4 below requires only the real calculations from above. Furthermore, in Lemma 4, no assumptions such as (1) are made. In particular, as was shown in obtaining equations (34) and (35) in [6], when only dealing with real intervals, I_r,

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inequalities of the form of (1) can be shown to always hold under the assumptions of the lemma. See also [**18**] for more details.

With regard to the complex inequality of (1), it is almost certain that, under the assumptions of Lemma 2, this holds always also since the argument should follow the real argument made in [**6**]. With regard to the p-adic inequality of (1), it is not clear whether this will always hold, under the assumptions of Lemma 2, but it is not difficult to show that there are infinitely many cases for which it does. As an example of a polynomial for which Lemma 2 can be applied to consider $P(z)=2z^4+z^3-2z^2+2z-1$ with $I=\left[\frac{1}{4},\frac{1}{2}\right]$, $C=\left\{z\in\mathbb{C}:|z-\frac{1+i}{4}|\le\frac{1}{4}\right\}$ and $K\subseteq\left\{z\in\mathbb{Q}_{19}:|P(z)|_{19}<\frac{1}{19}\right\}\cup\{4\}$. One can easily check that in this case $\rho_2=-\log_2(\sqrt{5})$, $\lambda_2=-\log_2\left(\frac{\sqrt{7}}{2}\right)$, $\sigma_2=0$, $\tau_1=-\log_2\left(\frac{77}{128}\right)$, $\tau_2=-\log_2\left(\frac{61}{100}\right)$ and $\tau_3=1$. Thus (1) can be seen to hold.

Proof of Lemma 4. Let $\alpha_1,\dots,\alpha_{n_1}$ be the roots of the polynomial P and $\beta_1,\dots,\beta_{n_2}$ be the roots of the polynomial T, where n_1 and n_2 are the degrees of the polynomials P and T, with $n_1,n_2\le n$. Now defining $\nu_i^r(P)$ and $\nu_j^r(T)$ as in (4) and using Lemma 3, it will again be taken that one such pair of roots for which $\nu_i^r(P)\ne\emptyset$, $\nu_j^r(T)\ne\emptyset$ is denoted by α_1^r and β_1^r for each $r=1,2,3$. In particular, to ensure $\nu_i^r(P),\nu_j^r(T)\ne\emptyset$, from this point only the intervals

$$\nu^r(P):=I_r\cap\mathcal{S}(\alpha_1^r),\qquad r=1,2,3,\tag{16}$$

$$\nu^r(T):=I_r\cap\mathcal{S}(\beta_1^r),\qquad r=1,2,3\tag{17}$$

will be considered.

Throughout the proof, it will be necessary to consider differences of the form $|x_r-\alpha_i^r|$ for some $x_r\in\nu^r(P)$. From this point on, $x_r\in\nu^r(P)$ will be chosen so that $|x_r-\alpha_i^r|>\frac{1}{4}|\nu^r(P)|$. Similarly, when dealing with the roots of $T(x)$ choose $x_r\in\nu^r(T)$ such that $|x_r-\beta_i^r|>\frac{1}{4}|\nu^r(T)|$.

Choose $\epsilon_0>0$ so that for $1\le r<r'\le 3$, the following inequality holds:

$$\min\left(\left|\alpha_1^r-\alpha_1^{r'}\right|,\left|\beta_1^r-\beta_1^{r'}\right|\right)>\epsilon_0.$$

It is clear that such an ϵ_0 exists by (5). The roots of the polynomials P and T are then ordered in one of three ways depending on their distances from α_1^r and β_1^r as follows. Define $a_r,b_r\in\mathbb{Z}$ such that for $r=1,2,3$,

$$\left|\alpha_1^r-\alpha_2^r\right|\le\cdots\le\left|\alpha_1^r-\alpha_{a_r}^r\right|\le\frac{\epsilon_0}{2}\le\left|\alpha_1^r-\alpha_{a_r+1}^r\right|\le\cdots\le\left|\alpha_1^r-\alpha_{n_1}^r\right|,$$

$$\left|\beta_1^r-\beta_2^r\right|\le\cdots\le\left|\beta_1^r-\beta_{b_r}^r\right|\le\frac{\epsilon_0}{2}\le\left|\beta_1^r-\beta_{b_r+1}^r\right|\le\cdots\le\left|\beta_1^r-\beta_{n_2}^r\right|.$$

Define the real numbers $\rho_i^r,\lambda_j^r\in\mathbb{R}$ such that

$$\left|\alpha_1^r-\alpha_i^r\right|=H^{-\rho_i^r},\quad i=2,\dots,n_1,$$

$$\left|\beta_1^r-\beta_j^r\right|=H^{-\lambda_j^r},\quad j=2,\dots,n_2.$$

Furthermore, define

$$l_i^r=\rho_i^r+\cdots+\rho_{a_r}^r,\quad i=2,\dots,a_r,$$

$$\tilde{l}_i^r=\rho_i^r+\cdots+\rho_{n_1}^r,\quad i=a_r+1,\dots,n_1,$$

$$m_j^r=\lambda_j^r+\cdots+\lambda_{b_r}^r,\quad j=2,\dots,b_r,$$

$$\tilde{m}_j^r=\lambda_j^r+\cdots+\lambda_{n_2}^r,\quad j=b_r+1,\dots,n_2.$$

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For the polynomial $P(x) = a_{n_1}x^{n_1} + \cdots + a_1x + a_0$ suppose that $|a_{n_1}| = H^{\gamma_1}$, $0 \leq \gamma_1 \leq \mu$, and for the polynomial $T(x) = b_{n_2}x^{n_2} + \cdots + b_1x + b_0$ suppose that $|b_{n_2}| = H^{\gamma_2}$, $0 \leq \gamma_2 \leq \mu$. Then following the method used by Bernik in [6] in the natural way, one obtains (18) and (19) below which are similar to equations (25) and (26) of [6].
For $x_r \in \nu^r(P)$,

$$\left|x_r - \alpha_1^r\right| \ll \min_{1\leq j\leq a_r} H^{-\frac{\tau_r+\mu-l_j^r}{j}}, \tag{18}$$

and for $x_r \in \nu^r(T)$,

$$\left|x_r - \beta_1^r\right| \ll \min_{1\leq j\leq b_r} H^{-\frac{\tau_r+\mu-m_j^r}{j}}. \tag{19}$$

Let the minimum on the right-hand side of (18) be achieved at $j = j_r^\alpha$ and the minimum on the right-hand side of (19) be achieved at $j = j_r^\beta$.
From the definition of j_r^α, one obtains for any i, $1 \leq i \leq a_r$

$$H^{-\frac{\tau_r+\mu-l_{j_r^\alpha}^r}{j_r^\alpha}} \leq H^{-\frac{\tau_r+\mu-l_i^r}{i}}.$$

This gives the inequality

$$i\left(\tau_r + \mu - l_{j_r^\alpha}^r\right) \geq j_r^\alpha\left(\tau_r + \mu - l_i^r\right). \tag{20}$$

By the way, the interval $\nu^r(P)$ was defined for $r = 1, 2, 3$, Lemma 3 gives that $|\nu^r(P)| > c(n)H^{-\eta_r}$. Furthermore, recall that $x_r \in \nu_1^r(P)$ was chosen so that $|x_r - \alpha_1^r| \gg |\nu^r(P)|$. Thus by (18),

$$H^{-\frac{\tau_r+\mu-l_j^r}{j}} \geq H^{-\eta_r}.$$

Rearranging gives

$$\eta_r \geq \frac{\tau_r + \mu - l_j^r}{j}, \quad j = 1, \ldots, a_r. \tag{21}$$

For the polynomial T, the following analogous inequality to (21) can be obtained:

$$\eta_r \geq \frac{\tau_r + \mu - m_j^r}{j}, \quad j = 1, \ldots, b_r. \tag{22}$$

Using (20) and assuming without loss of generality that

$$\frac{\tau_r + \mu - m_{j_r^\beta}^r}{j_r^\beta} \geq \frac{\tau_r + \mu - l_{j_r^\alpha}^r}{j_r^\alpha}, \tag{23}$$

then again following an adapted method to that in obtaining equations (40) and (44) in [6] one finds that

$$\left|\alpha_1^r - \beta_1^r\right| \ll H^{-\frac{\tau_r+\mu-l_{j_r^\alpha}^r}{j_r^\alpha}}, \tag{24}$$

and

$$\prod_{1\leq i\leq a_r} \prod_{1\leq j\leq j_r^\beta} \left|\alpha_i^r - \beta_j^r\right| \ll H^{-j_r^\beta(\tau_r+\mu)}. \tag{25}$$

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Since the polynomials P and T have no common roots $|R(P, T)| \geq 1$ and so using (25), one obtains

$$\begin{aligned} 1 &\leq |R(P, T)| \\ &\leq H^{\gamma_1 n_2+\gamma_2 n_1} \prod_{\substack{1\leq i\leq a_1\\1\leq j\leq j_1^\beta}} \left|\alpha_i^1-\beta_j^1\right| \prod_{\substack{1\leq i\leq a_2\\1\leq j\leq j_2^\beta}} \left|\alpha_i^2-\beta_j^2\right| \prod_{\substack{1\leq i\leq a_3\\1\leq j\leq j_3^\beta}} \left|\alpha_i^3-\beta_j^3\right| \times \prod_{\mathcal{R}} \left|\alpha_i^r-\beta_j^r\right|, \end{aligned} \tag{26}$$

where the set $\mathcal{R}$ is defined by

$$\mathcal{R} := \Big\{(i, j) \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} : \left|\alpha_1^{r'}-\alpha_i^{r''}\right| > \frac{\epsilon_0}{2} \text{ and } \left|\beta_1^{r'}-\beta_j^{r''}\right| > \frac{\epsilon_0}{2} \\ \text{for each } r' = 1, 2, 3 \text{ and } r'' \in \{1, 2, 3\}\Big\}.$$

It is possible that $\mathcal{R} = \emptyset$ in which case recall that by definition $\gamma_1, \gamma_2 \leq \mu$ and so

$$\gamma_1 n_2 + \gamma_2 n_1 \leq \mu(n_1 + n_2).$$

If, however, $\mathcal{R} \neq \emptyset$, then using Lemma 6 the product

$$\prod_{\mathcal{R}} \left|\alpha_i^r-\beta_j^r\right|$$

is bounded by $H^{n_2(\mu-\gamma_1)+n_1(\mu-\gamma_2)}$. In either case, inequality (26) can be rewritten as

$$1 \ll H^{(n_1+n_2)\mu - j_1^\beta(\tau_1+\mu) - j_2^\beta(\tau_2+\mu) - j_3^\beta(\tau_3+\mu)}.$$

Rearranging gives that

$$j_1^\beta(\tau_1+\mu) + j_2^\beta(\tau_2+\mu) + j_3^\beta(\tau_3+\mu) \leq (n_1+n_2)\mu.$$

In the case of $j_1^\beta, j_2^\beta, j_3^\beta \geq 3$, one obtains the inequality

$$\begin{aligned} (n_1+n_2)\mu &\geq 3(\tau_1+\mu) + 3(\tau_2+\mu) + 3(\tau_3+\mu) \\ &\geq \sum_{r=1}^{3} (\tau_r + \mu + 2\max(\tau_r+\mu-\eta_r, 0)), \end{aligned}$$

which clearly proves Lemma 4. Thus, only the cases in which at least one of j_1^β, j_2^β or j_3^β is less than 3 needs to be considered. To do this two arguments which depend on whether $j_r^\beta = 2$ or $j_r^\beta = 1$ will be used.

First consider when $j_1^\beta = 2$ and $j_2^\beta, j_3^\beta \geq 3$. If

$$\lambda_3^1 \geq \frac{\tau_1+\mu-m^1_{j_1^\beta}}{j_1^\beta}, \tag{27}$$

then carrying out calculations in a similar manner to those in achieving (9) and (10) in the proof of Lemma 2 or those carried out in [**6**] and [**8**] one finds

$$\prod_{1\leq i\leq a_1} \left|\alpha_i^1-\beta_3^1\right| \ll H^{-(\tau_1+\mu)}. \tag{28}$$

Using this and (25), one obtains

$$\begin{aligned}1 &\leq |R(P, T)| \\ &\ll H^{(n_1+n_2)\mu-2(\tau_1+\mu)-(\tau_1+\mu)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)},\end{aligned}$$

which gives the lemma for $j_2^\beta, j_3^\beta \geq 3$. If (27) does not hold, then

$$\lambda_3^1 < \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}.$$

Using this and (24), one obtains that for any $j \geq 3$,

$$\left|\alpha_1^1 - \beta_j^1\right| \leq \left|\alpha_1^1 - \beta_1^1\right| + \left|\beta_1^1 - \beta_j^1\right| \ll H^{-\frac{\tau_1+\mu-l^1_{j_1^\alpha}}{j_1^\alpha}} + H^{-\lambda_j^1} \ll H^{-\lambda_j^1}.$$

Thus

$$\prod_{3\leq j\leq b_1} \left|\alpha_1^1 - \beta_j^1\right| \ll H^{-m_2^1}. \tag{29}$$

This together with (25) leads to

$$\prod_{1\leq i\leq n_1} \prod_{1\leq j\leq n_2} \left|\alpha_i^r - \beta_j^r\right| \ll H^{-2(\tau_1+\mu)-m_2^1-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+(n_1+n_2)\mu}.$$

From (22), $m_2^r \geq \tau_r + \mu - 2\eta$; therefore, the exponent above can be replaced by

$$-(\tau_1+\mu) - 2(\tau_1+\mu-\eta) - j_2^\beta(\tau_2+\mu) - j_3^\beta(\tau_3+\mu) + (n_1+n_2)\mu,$$

which also leads to the proof of Lemma 4.

Now consider the case when $j_1^\beta = 1$ and $j_2^\beta, j_3^\beta \geq 3$. Assume

$$\lambda_2^1 > \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta} \quad \text{and} \quad \lambda_3^1 > \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}.$$

Then as in the case of inequality (28), one obtains

$$\prod_{1\leq i\leq a_1} \prod_{2\leq j\leq 3} \left|\alpha_i^1 - \beta_j^1\right| \ll H^{-2(\tau_1+\mu)}.$$

This together with (25) implies the result. Next assume

$$\lambda_2^1 > \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta} \quad \text{and} \quad \lambda_3^1 \leq \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}.$$

The first inequality leads to the inequality

$$\prod_{1\leq i\leq a_1} \left|\alpha_i^1 - \beta_2^1\right| \ll H^{-(\tau_1+\mu)}$$

and the second to the inequality (29). Together these inequalities imply the result.

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Finally assume

$$\lambda_2^1 \leq \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta} \text{ and } \lambda_3^1 \leq \frac{\tau_1 + \mu - m^1_{j_1^\beta}}{j_1^\beta}.$$

Then one obtains

$$\prod_{2\leq j\leq b_1} \left|\alpha_1^1 - \beta_j^1\right| \ll H^{-m_1^1}. \tag{30}$$

Suppose furthermore

$$\rho_2^1 < \lambda_2^1,$$

then when $2 \leq i \leq a_1$, one gets

$$\left|\alpha_i^1 - \beta_2^1\right| \ll H^{-\rho_i^1},$$

leading to

$$\prod_{2\leq i\leq a_1} \left|\alpha_i^1 - \beta_2^1\right| \ll H^{-l_1^1}. \tag{31}$$

If on the other hand

$$\rho_2^1 \geq \lambda_2^1,$$

then for $2 \leq j \leq b_1$, one obtains

$$\left|\alpha_2^1 - \beta_j^1\right| \ll H^{-\lambda_j^1},$$

so that

$$\prod_{2\leq j\leq b_1} \left|\alpha_2^1 - \beta_j^1\right| \ll H^{-m_1^1}. \tag{32}$$

Using (25), (30), (31) and (32) leads to

$$1 \leq |R(P, T)| \ll H^{-(\tau_1+\mu)-m_1^1-\min(l_1^1,m_1^1)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+\mu(n_1+n_2)}. \tag{33}$$

From the definition of $v_i^r(P)$ and (23), one can show that

$$\min\left(l_1^1, m_1^1\right) \geq \tau_1 + \mu - \eta_1.$$

Thus inequality (33) can be rewritten as

$$\prod_{1\leq i\leq n_1} \prod_{1\leq j\leq n_2} \left|\alpha_i^r - \beta_j^r\right| \ll H^{-\tau_1-\mu-2\max(\tau_1+\mu-\eta_1,0)-j_2^\beta(\tau_2+\mu)-j_3^\beta(\tau_3+\mu)+\mu(n_1+n_2)},$$

which clearly proves Lemma 4 for $j_2^\beta, j_3^\beta \geq 3$. The arguments above will hold for $j_r^\beta \leq 2$ for any $r = 1, 2, 3$ and so all remaining cases are combinations of these arguments. □

It should now be evident that the proof of Lemma 4 can easily be adapted for any k variables, with $2 \leq k \leq n$, provided

$$\alpha_1^r \neq \alpha_1^{r'} \text{ and } \beta_1^r \neq \beta_1^{r'}, \text{ for } 1 \leq r < r' \leq k.$$

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When dealing with k variables, all arguments will be identical to those made in the proof for $k = 3$ with the two exceptions. Firstly, the indices will now be of the form $j_4^\beta + \cdots + j_k^\beta$. Secondly, increasing the number of variables will of course increase the number of cases to be considered. However, one can again begin by considering the cases in which $j_1^\beta < 3$ while $j_r^\beta \geq 3$ for $r = 2, 3, \ldots, k$, and then simply work down through all other cases (such as $j_1^\beta, j_2^\beta < 3$ while $j_r^\beta \geq 3$ for $r = 3, \ldots, k$) in an identical fashion to above. The arguments will not change; however, the number of different products required to be bounded will certainly become larger depending on how many of the $j_r^\beta < 3$.

5. Examples. For an application of Lemma 4, see [7]. In this section, an example of how Lemma 2 can be used will be discussed. It will also be shown why (2) is stronger than (3); see remark 1 in Section 5.1

Let $D(P)$ be the discriminant of the polynomial $P(x) \in \mathcal{P}_n(Q)$ with roots $\alpha_1, \ldots, \alpha_n$, then it is well know that

$$D(P) = a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.$$

It can also be shown that $D(P)$ is expressible as the determinant of a $(2n-1) \times (2n-1)$ Sylvester matrix; see [4] for details. This, in particular, implies that $D(P) \in \mathbb{Z}$.
Define $\mathcal{P}_n^v(Q)$ for $0 \leq v \leq n-1$ as follows:

$$\mathcal{P}_n^v(Q) := \left\{P(x) \in \mathcal{P}_n(Q) : 1 \leq |D(P)| < Q^{2n-2-2v}\right\}.$$

Letting $\#U$ represent the cardinality of some set U, we are interested in finding bounds for $\#\mathcal{P}_n^v(Q)$. In 2010, Koleda [14] obtained both upper and lower bounds for the cardinality of $\mathcal{P}_n^v(Q)$ in the case $n = 3$ and $0 \leq v < 3/5$. In particular, it was shown that for $0 \leq v < 3/5$ and c_1, a positive constant that depends only on n and is independent of Q,

$$\#P_3(Q, v) = c_1 Q^{4-\frac{5}{3}v}(1 + o(1)).$$

In 2013, Koleda and Korlukova [15] showed that for $0 \leq v < \frac{1}{2}$,

$$\#P_2(Q, v) = \lambda Q^{3-2v}(1 + o(1)), \ \ \lambda = 20(1 + \ln 2).$$

It was shown by Beresnevich, Bernik and Götze [4] in 2016 that for $0 \leq v \leq n-1$,

$$\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v}.$$

Using (2) of Lemma 2, it will now be shown that the upper bound is in fact of the same order of the lower bound in a very particular case. The result is believed to hold true in general and the proof of this will be the subject of future work.

Consider Lemma 2 in the one-dimensional setting. Just as was done in [4], we will consider only the unit interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ as all results may be extended to any arbitrary interval in $\mathbb{R}$; see [3] for appropriate techniques. We begin by assuming that the upper bound is not of the same order as the lower bound, in particular, assume that

$$\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v+\epsilon}.$$

Then there must exist an interval I of size $Q^{-\rho_2(P)}$ containing a root of $P(x)$ such that

$$\#\mathcal{P}_n^v(Q, I) \gg Q^{n+1-\frac{n+2}{n}v-\rho_2(P)+\epsilon},$$

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where $\mathcal{P}_n^v(Q, I)$ is the set of polynomials $P \in \mathcal{P}_n^v(Q)$ which have a root in the interval I. If not, we would have that $\#\mathcal{P}_n^v(Q, I) \ll Q^{n+1-\frac{n+2}{n}v-\rho_2(P)+\epsilon}$ in all $Q^{\rho_2(P)}$ subintervals I, but this contradicts the assumption that $\#\mathcal{P}_n^v(Q) \gg Q^{n+1-\frac{n+2}{n}v+\epsilon}$. Note that choosing the interval to be of size $Q^{-\rho_2(P)}$ is for convenience only.

Let $m = n+1-\frac{n+2}{n}v - \rho_2(P)$. Using (2) of Lemma 2, a contradiction to the assumption that $\#\mathcal{P}_n^v(Q, I) \gg Q^{m+\epsilon}$ will now be obtained in the case that $m > 0$.

Using Taylor series, it is not difficult to show that on the interval I

$$|P(x)| \ll Q^{1-q_1(P)-\rho_2(P)}.$$

In [4], it was shown that if

$$\rho_2(P) + 2\rho_3(P) + \cdots + (n-1)\rho_n(P) \geq v,$$

then $P(x) \in \mathcal{P}_n^v(Q)$ (see how equation 40 was obtained in [4] for details). Moving forward, we will assume that $\rho_2(P) + 2\rho_3(P) + \cdots + (n-1)\rho_n(P) \geq v$.

Consider first the case in which $m \in \mathbb{N}$ and define the set

$$M\big(Q^{1-q_1(P)-\rho_2(P)}, a_n, \ldots, a_{n-l+1}\big) := \big\{P \in \mathcal{P}_n(Q) : |P(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } a_j(P) = a_j, j = l, \ldots, n\big\},$$

so that $M(Q^{1-q_1(P)-\rho_2(P)}, a_n, \ldots, a_{n-l+1})$ is the set of polynomials in $P_n(Q)$ with the $n-l+1$ coefficients $a_l, \ldots, a_n$ equal that satisfy $|P(x)| \ll Q^{1-q_1(P)-\rho_2(P)}$. Now fix $P_0(x) \in M(Q^{1-q_1(P)-\rho_2(P)}, a_n, \ldots, a_{n-l+1})$ and for $P_j(x) \in M(Q^{1-q_1(P)-\rho_2(P)}, a_n, \ldots, a_{n-l+1})$ construct the polynomials $R_j(x) = P_j(x) - P_0(x)$ with

$$|R_j(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } \deg(R_j(x)) = n - m = \frac{n+2}{n}v + \rho_2(P) - 1.$$

If there exist at least two $R_j(x)$ without common roots, then by (2) with

$$\tau_1 + 1 := \tau + 1 = q_1(P) + \rho_2(P) + \frac{\epsilon}{2},$$

one has

$$\tau_1 + 1 + 2(q_1(P) + \cdots + q_{n-1}(P)) < 2\deg(R_j) + \delta.$$

Suppose $\rho_2(P) = v$ and $\delta < \frac{\epsilon}{2}$. Then by the definition of m

$$1 \leq m = n + 1 - 2v - \frac{2v}{n} < n + 1 - 2v,$$

i.e. $v < \frac{n}{2}$ and by (2) of Lemma 2

$$\begin{aligned}
\tau_1 &+ 1 + 2(q_1(P) + \cdots + q_{n-1}(P)) \\
&= \frac{\epsilon}{2} + 3q_1(P) + \rho_2(P) + 2(q_2(P) + \cdots + q_{n-1}(P)) \\
&= \frac{\epsilon}{2} + 4\rho_2(P) + 5\rho_3(P) + 7\rho_4(P) + \cdots + (2n-1)\rho_n(P) \\
&> 2\rho_2(P) + 2v + \frac{4}{n}v - 2 + \delta = 2\deg(R_j) + \delta,
\end{aligned}$$

since $v \leq \rho_2(P) + 2\rho_3(P) + \cdots + (n-1)\rho_n(P)$. Thus we have a contradiction to Lemma 2.

If $m \notin \mathbb{N}$ define $\tilde{m} \in \mathbb{N}$ such that $m+1 > \tilde{m} > m$. Then we construct polynomials $R_j(x) = P_j(x) - P_0(x)$ with

$$|R_j(x)| \ll Q^{1-q_1(P)-\rho_2(P)} \text{ and } \deg(R_j(x)) = n - \tilde{m} < n - m = \frac{n+2}{n}v + \rho_2(P) - 1$$

and the proof follows as it did before. Thus it has been shown that

$$\#\mathcal{P}_n^v(Q) \ll Q^{n+1-\frac{n+2}{n}v},$$

provided there exist at least two $R_j(x)$ without common roots and $m = n+1-\frac{n+2}{n}v - \rho_2(P) > 0$.

5.1. Remarks.

Remark 1. By using (2) of Lemma 2 in the above, a contradiction was obtained. It can easily be seen that, although easier to use, (3) is weaker than (2) since it does not always guarantee a contradiction. To see this recall that by (3)

$$\tau + 1 + 2\sum_{j=1}^{n}(\tau + 1 - j\eta) < 2(\deg(R_j(x)) + \delta = 2\rho_2(P) + 2v + \frac{4}{n}v - 2 + \delta.$$

Then taking $\tau = 8$, $v = 9 = \rho_2(P)$ and $n = 12$, a contradiction is not obtained for (3).

Remark 2. It is clear that there is a lot left to do in order to completely prove

$$\#\mathcal{P}_n^v(Q) \ll Q^{n+1-\frac{n+2}{n}v}.$$

In particular, the case in which there does not exist two polynomials $R_j(x)$ without common roots must be considered. Completing this proof will be the subject of future work. It would appear that we can follow a similar method of proof to that used in [7] when dealing with reducible polynomials, with some slight modifications. Also the case when $m = n+1-\frac{n+2}{n}v - \rho_2(P) \leq 0$ must be dealt with.

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REFERENCES

1. A. Baker and W. M. Schmidt, Diophantine approximation and Hausdorff dimension, *Proc. Lond. Math. Soc.* **21** (1970), 1–11.

2. R. C. Baker, Sprindžuk's theorem and Hausdorff dimension, *Mathematika* **23**(2) (1976), 184–197.

3. V. Beresnevich, On approximation of real numbers by real algebraic numbers, *Acta Arith.* **90** (1999), 97–112.

4. V. Beresnevich, V. I. Bernik, and F. Götze, Integral polynomials with small discriminants and resultants, *Adv. Math.* **298** (2016), 393–412.

5. V. I. Bernik, A metric theorem on the simultaneous approximation of zero by the values of integral polynomials, *Math. USSR-Izv.* **16**(1) (1981), 21–40.

6. V. I. Bernik, Application of the Hausdorff dimension in the theory of Diophantine approximation, *Acta Arith.* **42**(3) (1983), 219–253. (in Russian)

7. V. I. Bernik, N. Budarina, D. Dickinson, and S. Mc Guire, The distribution of algebraic conjugate points, in preparation.

8. V. I. Bernik and N. I. Kalosha, Approximation of zero by integer polynomials in space $\mathbb{R} \times \mathbb{C} \times \mathbb{Q}_p$, *Proc. Nat. Acad. Sci. Belarus Phis. Math. Ser.* **1** (2004), 121–123.

9. A. Besicovitch, Sets of fractional dimensions. IV: On rational approximation to real numbers, *J. Lond. Math. Soc.* **9** (1934), 126–131.

10. N. Budarina and D. Dickinson, Simultaneous Diophantine approximation of integral polynomials in the different metrics, *Chebyshevskii Sbornik* **9**(1) (2008), 169–184.

11. N. I. Feldman, Approximation of certain transcendental numbers, *I. Izv. Akad. Nauk SSSR Ser. Mat.* **15** (1951), 53–79; English transl. in Amer. Math. Soc. Transl. 59(2) (1966).

12. V. Jarník. Diophantische approximationen und hausdorffsches maß, *Rec. Math. Moscou* **36** (1929), 371–382.

13. F. Kasch and B. Volkmann, Zur Mahlerschen Vermutung über S-Zahlen, *Math. Ann.* **136** (1958), 442–453. (in German)

14. D. V. Koleda, An upper bound for the number of integral polynomials of third degree with a given bound for discriminants, *Vestsí Nats. Akad. Navuk Belarusí Ser. Fíz.-Mat. Navuk* **3** (2010), 10–16. (in Russian)

15. D. V. Koleda and I. A. Korlukova, Asymptotic quantity of integral quadratic polynomials with bounded discriminants, *Vesnik of Yanka Kupala State University of Grodno, Series 2*. **2**(151) (2013), 6–10.

16. E. Kovalevskaya, A metric theorem on the exact order of approximation of zero by values of integer polynomials in $\mathbb{Q}_p$, *Dokl. Nats. Akad. Nauk Belarusi* **43**(5) (1999), 34–36.

17. K. Mahler, Über das Mass der Menge aller S-Zahlen, *Math. Ann.* 106 (1932), 131–139.

18. N. A. Pereverzeva, The distribution of vectors with algebraic coordinates in $\mathbb{R}^2$, *Vestsi Akad. Naavuk BSSR. Ser. Fiz.-Mat. Navuk* **4** (1987), 114–116, 128. (in Russian)

19. W. M. Schmidt, Bounds for certain sums; a remark on a conjecture of Mahler, *Trans. Amer. Math. Soc.* **101** (1961), 200–210.

20. V. G. Sprindžuk, *Mahler's problem in the metric theory of numbers*, vol. 25 (American Mathematical Society, Providence, RI, 1969).

21. B. Volkmann, Zur metrischen Theorie der S-Zahlen, *J. Reine Angew. Math.* **209** (1962), 201–210.