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PERIODIC AND FIXED POINT THEOREMS IN A QUASI-METRIC SPACE

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Abstract

General periodic and fixed point theorems are proved for a class of self maps of a quasi-metric space which satisfy the contractive definition (A) below. Two examples are presented to show that the class of mappings which satisfy (A) is indeed wider than a class of selfmaps which satisfy Caristi's contractive definition (C) below. Also a common fixed point theorem for a pair of maps which satisfy a contractive condition (D) below is established.

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1. Introduction

Let X be a non-void set and $T: X \to X$ a selfmap. A point $x \in X$ is called a periodic point for T iff there exists a positive integer k such that $T^k x = x$. If $k = 1$, then x is called a fixed point for T.

Caristi [4] proved the following very general contraction fixed point theorem.

THEOREM 1 (Caristi [4]). *Suppose $T: X \to X$ and $\Phi: X \to [0, \infty)$, where X is a complete metric space and Φ is lower semicontinuous. If for each x in X*

$$d(x, Tx) \le \Phi(x) - \Phi(Tx), \tag{C}$$

then T has a fixed point.

Caristi's proof, based on the work of Brondsted [3], is not elementary, as well as the other new proofs of Theorem 1 ([10, 13]). Bhakta and Basu [1] observed that the proof of Theorem 1 becomes much simpler by adding the hypotheses of orbital continuity of a selfmap T (compare [5]). Moreover, they pointed out that the hypotheses of the lower semi-continuity of a function Φ in that case may be dropped.

Recently Bollenbacher and Hicks [2] obtained a version of Caristi's Theorem 1 by using the concept of T-orbitally lower semi-continuity of a real function $G: X \to [0, \infty)$ (defined by $G(x) = d(x, Tx)$), which was introduced in [9] (compare [12]). Hicks [8] extended this version for a metric space to one for a quasi-metric space, which need not satisfy $d(y, x) = d(x, y)$.

The purpose of this note is to relax Caristi's contractive definition (C), slightly relax the concept of T-orbital lower semi-continuity introduced in [9] and to obtain a periodic and a fixed point theorem which extend and generalize main results of [2], [7] and [8]. We shall also prove a common fixed point theorem having the fixed point theorems of [2, 7, 8] as corollaries.

2. Main results

Let (X, d) be a quasi-metric space and $T: X \to X$ a mapping of X. A set $0(x, \infty) = \{x, Tx, T^2x, \dots, \}$ is called the orbit of x.

DEFINITION 2.1. A real-valued function $G: X \to [0, \infty)$ is said to be T-orbitally weak lower semi-continuous (w.l.s.c.) relative to x iff $\{x_n\}$ is a sequence in $0(x, \infty)$ and

$$(1) \qquad \lim_{n\to\infty} x_n = p \quad \text{implies} \quad G(p) \le \lim_{n\to\infty} \sup G(x_n).$$

Clearly, every function G that is T-orbitally lower semi-continuous (l.s.c.) relative to $x \in X$ (that is, $\{x_n\} \subseteq 0(x, \infty)$ and $\lim x_n = p$ imply $G(p) \le \liminf G(x_n)$ (see [9, 2])) is also T-orbitally w.l.s.c. relative to x, but the implication is not reversible.

Note that the condition (1) was used in [6], but there it was supposed that (1) was true for every sequence $\{x_n\}$ in X.

THEOREM 2. *Suppose $T: X \to X$, $n: X \to N$ and $\Phi: X \to [0, \infty)$, where X is a complete quasi-metric space. If for some $x_0 \in X$ there exists a subsequence $S = \{x_n\}_{n=0}^{\infty}$ in $0(x_0, \infty)$ such that $T^{n(x_n)}x_n \in S$ and*

$$(\mathrm{A}) \qquad d(y, T^{n(y)}y) \le \Phi(y) - \Phi(T^{n(y)}y)$$

holds for each $y \in S$, then we have

(a) $\lim x_n = p$ *exists,*

(b) $T^{n(p)}p = p$ *if and only if* $G(x) = d(x, T^{n(x)}x)$ *is* T*-orbitally w.l.s.c. relative to* x_0,

(c) $d(x_0, x_n) \le \Phi(x_0)$,

(d) *If* $y \to d(z, y)$ *is* T*-orbitally w.l.s.c. relative to* x_0 *for* $z \in S$ *then* $d(x_n, p) \le \Phi(x_n)$ *and* $d(x_0, p) \le \Phi(x_0)$.

PROOF. Without loss of generality we may suppose that $\{x_n\}$ has the property that $x_{n+1} = T^{n(x_n)}x_n$; $n = 0, 1, 2, \ldots$. Then we have, for $n = 0, 1, 2, \ldots$,

$$d(x_n, x_{n+1}) = d(x_n, T^{n(x_n)}x_n) \le \Phi(x_n) - \Phi(x_{n+1}).$$

For $m \ge 0$,

$$\begin{aligned}\sigma_m = \sum_{n=0}^{m} d(x_n, x_{n+1}) &\le \sum_{n=0}^{m}[\Phi(x_n) - \Phi(x_{n+1})] = [\Phi(x_0) - \Phi(x_1)] \\ &\quad + [\Phi(x_1) - \Phi(x_2)] + \cdots + [\Phi(x_m) - \Phi(x_{m+1})] \\ &= \Phi(x_0) - \Phi(x_{m+1}) \le \Phi(x_0).\end{aligned}$$

The sequence $\{\sigma_m\}_{m=0}^{\infty}$ of partial sums of the infinite series $\sum d(x_n, x_{n+1})$ is a nondecreasing sequence bounded above by $\Phi(x_0)$ and therefore converges. This implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Since (X, d) is complete, we have (a).

Assume that $G(p) \le \lim_{n\to\infty} \sup G(x_n)$. Then by definition of $G(x)$ and x_{n+1} we have $G(x_n) = d(x_n, x_{n+1})$. So $G(x_n)$ is a general term of a convergent series $\sum d(x_n, x_{n+1})$, and hence $G(x_n) \to 0$, as $n \to \infty$. Therefore, $G(p) = d(p, T^{n(p)}p) = 0$. Hence p is a periodic point of T. This shows (b). Clearly (c) holds.

To prove (d), let $n \ge 0$. Then

$$\begin{aligned}d(x_n, x_{n+k}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k}) \\ &= \sum_{m=n}^{n+k-1} d(x_m, x_{m+1}) \le \sum_{m=n}^{n+k-1} [\Phi(x_m) - \Phi(x_{m+1})] \\ &= \Phi(x_n) - \Phi(x_{n+k}) \le \Phi(x_n).\end{aligned}$$

Assume that $y \to d(x_n, y)$ is T-orbitally w.l.s.c. relative to x_0 for each $n = 0, 1, 2, \ldots$. Then

$$d(x_n, p) \le \lim_{k\to\infty} \sup d(x_n, x_{n+k}) \le \lim_{k\to\infty} \sup \Phi(x_n) = \Phi(x_n).$$

This shows (d).

REMARK 1. Example 1 shows that in (b) need not be $n(p) = 1$, that is, T need not have a fixed point.

Theorem 3. *Suppose* $T: X \to X$, $n: X \to N$ *and* $\Phi: X \to [0, \infty)$, *where* X *is a complete quasi-metric space. If* T *satisfies all hypotheses of Theorem* 2 *and in addition for all* $y \in Cl[0(x_0, \infty)]$

$$y \neq Ty \quad \textit{implies} \quad \Phi(T^m y) < \Phi(y) \tag{1}$$

for some positive integer $m = m(y)$, *then* T *has a fixed point.*

Proof. From Theorem 2, there is p in X such that $T^k p = p$. Then $0(p, \infty)$ is a finite set of points in X. Let $y \in 0(p, \infty)$ be such that

$$\Phi(y) = \min\{\Phi(z): z \in 0(p, \infty)\}. \tag{2}$$

Assume that $y \neq Ty$. Then from (1) there is $m \in N$ such that $\Phi(T^m y) < \Phi(y)$. But, since $T^n y \in 0(p, \infty)$ for all $n \in N$, it follows from (2) that $\Phi(y) \leq \Phi(T^m y)$, a contradiction. Therefore, $y = Ty$, which completes the proof.

Corollary 1 (Hicks [8, Theorem 2]). *Let* X *and* Y *be quasi-metric spaces with* X *complete. Suppose* $T: X \to X$, $f: X \to Y$ *and* $\Phi: fX \to [0, \infty)$. *If there exists* $x_0 \in X$ *and* $c > 0$ *such that*

$$\max\{d(y, Ty), c \cdot d(fy, fTy)\} \leq \Phi(fy) - \Phi(fTy) \tag{B}$$

for all $y \in 0(x_0, \infty)$, *then*

(a′) $\lim T^n x_0 = p$ *exists,*

(b′) $Tp = p$ *if and only if* $G(x) = d(x, Tx)$ *is* T*-orbitally l.s.c. relative to* x_0,

(c′) $d(x_0, T^n x_0) \leq \Phi(fx_0)$,

(d′) *If* $y \to d(z, y)$ *is continuous for* $z \in 0(x_0, \infty)$, *then* $d(T^n x_0, p) \leq \Phi(fT^n x_0)$ *and* $d(x_0, p) \leq \Phi(fx_0)$.

Proof. It is clear that (B) implies that $d(y, Ty) \leq \Phi(fy) - \Phi(fTy)$. Put $\Phi_1 = \Phi f$. Then $\Phi_1: X \to [0, \infty)$ and

$$d(y, Ty) \leq \Phi_1(y) - \Phi_1(Ty).$$

Therefore, if T satisfies (B), then T satisfies (A) and (1) with $\Phi = \Phi_1$, $n(y) = 1$ and $m(y) = 1$ for all $y \in S = 0(x_0, \infty)$.

Remark 2. Example 2 shows that Theorem 3 is a proper generalization of Hick's theorem [8], which is an extension of corresponding theorems for metric spaces given in [2, 7].

Remark 3. The proof of Corollary 1 shows that the condition $cd(fx, fTx) \leq \Phi(fx) - \Phi(fTx)$ in Theorems 2 and 3 in [7] and in Theorem 2 in [8] (that

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is, in our Corollary 1), which is included in (B), can be dropped. Theorem 2 and its Corollary 1 in [8] are equivalent.

Now we shall prove a common fixed point theorem for two maps.

THEOREM 4. *Suppose* $S, T: X \to X$ *and* $\Phi: X \to [0, \infty)$, *where* X *is a complete quasi-metric space. If there is* $x_0 \in X$ *such that*

$$\text{(D)} \qquad d(y, Ty) + d(Ty, STy) \leq \Phi(y) - \Phi(STy)$$

for all $y \in 0_{ST}(x_0, \infty) = \{x_0, Tx_0, STx_0, T(ST)x_0, \ldots, (ST)^n x_0, T(ST)^n x_0, \ldots\}$, *then we have*

(a$''$) $\lim_{n\to\infty}(ST)^n x_0 = \lim_{n\to\infty} T(ST)^n x_0 = p$ *exists.*

(b$''$) $Tp = p$, *if* $G_1(x) = d(x, Tx)$ *is* (S, T)*-orbitally w.l.s.c. relative to* x_0 (*that is,* (1) *is true if* $\{x_n\} \subseteq 0_{ST}(x_0, \infty)$).

(c$''$) $Tp = p = Sp$, *if* $G_1(x) = d(x, Tx)$ *and* $G_2(x) = d(x, Sx)$ *are* (S, T)*-orbitally w.l.s.c. relative to* x_0.

PROOF. Put $z_{2k} = (ST)^k x_0$, $z_{2k+1} = Tz_{2k}$ $(k = 0, 1, 2, \ldots,)$ and consider the sequence $\{z_n\}_{n=0}^{\infty}$. Just as in the proof of (a) of Theorem 2, by (D) we obtain $\sum_{n=0}^{\infty} d(z_n, z_{n+1}) \leq \Phi(x_0)$. Hence $\lim_{n\to\infty} z_n = p$ exists. Hence $\lim_{k\to\infty} z_{2k} = \lim_{k\to\infty}(ST)^k x_0 = p$ and $\lim_{k\to\infty} z_{2k+1} = \lim_{k\to\infty} T(ST)^k x_0 = p$. This shows (a$''$). Since $z_{2k} \to p$ and $G_1(z_{2k}) = d(z_{2k}, z_{2k+1}) \to 0$ as $k \to +\infty$, we have $G_1(p) = 0$. Hence $Tp = p$. This shows (b$''$). Statement (c$''$) clearly holds.

3. Examples

1. Let $X = [-2, -1] \cup [1, 2]$ with the usual metric. Define $T: X \to X$ by $Tx = -x$ and $\Phi: X \to [0, \infty)$ by $\Phi(x) = |x|$, for example. Then T satisfies (A) for all $y \in X$ with $n(y) = 2$ and $G(x) = d(x, T^2 x)$ is continuous on X.

2. Let $X = \{0\} \cup \{\pm 1/n: n = 1, 2, \ldots\}$ with the usual metric. Define $T: X \to X$ by $T(1/n) = -1/(n+1)$, $T(-1/n) = 1/(n+1)$ and $T(0) = 0$. Define $\Phi: X \to [0, \infty)$ by $\Phi(x) = d(x, Tx)$. Then for $x = \pm 1/n$ we have

$$d(x, Tx) = 1/n + 1/(n+1); \qquad d(x, T^2 x) = 1/n - 1/(n+2).$$

Hence

$$\begin{aligned} d(x, T^2 x) &= 1/n - 1/(n+2) < 1/n + 1/(n+1) - [1/(n+2) + 1/(n+3)] \\ &= \Phi(x) - \Phi(T^2 x). \end{aligned}$$

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Therefore, T satisfies (A) on X with $n(x) = 2$ for all $x \in X$. Since X is a complete metric space, there exists p ($p = 0$) such that (a) holds. Since $G(x) = d(x, T^2x) = 2x^2(1 + 2|x|)^{-1}$ is continuous and T satisfies (1) with $m(y) = 2$ for all $y \in X$, Theorem 3 can be applied.

We point out that Caristi's contractive condition (C), and hence (B), implies that $\sum_{n=0}^{\infty} d(T^n x, TT^n x)$ must be a convergent series. Since in our example, for any fixed $x = \pm 1/m_0$, we have

$$d(T^n x, T^{n+1}x) = 1/(n + m_0) + 1/(n + 1 + m_0) > 2/(n + m_0 + 1),$$

we conclude that the series diverges and so there is no functions $f: X \to Y$ and $\Phi: fX \to [0, \infty)$ such that (B) holds for any $x = \pm 1/n \in X$.

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