# CORRESPONDENCE.

## ASSURANCES ON $x$ AGAINST $y$ AND $t$ YEARS LONGER.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—As premiums for assurances on $x$ provided he die before $y$ or within $t$ years after him, are frequently required in connection with reversionary transactions, I thought it would be useful to have a table by which the addition to be made to the ordinary survivorship net premium to cover the risk of $x$ dying within $t$ years after $y$, could be readily determined, instead of making an arbitrary addition as is sometimes done in consequence of the labour of computing it. I have therefore calculated that addition by the $H^M$ Table at 4 per-cent interest for decennial ages, and $t=1$, 3, 5, 7, and 10, (1) when the premium is payable until the risk determines, and (2) when it is payable during the joint existence of the lives only, and trust that you will find space for the tables in your valuable *Journal*. For the calculation of the annual premium for $x$ before $y$ or within $t$ years after him, when it is payable until the risk determines, I used the formula

$$\frac{A_x - \dfrac{D_{x+t}}{D_x}\{A_{x+t} - A^{\,1}_{x+t.\,y}\}}{1 + a_x - \dfrac{D_{x+t}}{D_x}\{a_{x+t} - a_{x+t.\,y}\}},$$

and the substitution of $\mathbf{a}_{xy}$ for the expression in the denominator gave the annual premium payable during the joint existence of the lives only. I calculated the survivorship premiums by the ordinary formula; and as I believe these have not hitherto been published, I append a table of them also for decennial ages.

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I should have liked, if possible, to have made the calculation by Mr. Sprague's Select Tables in combination with the $H^{M(5)}$; but as the joint-life annuities by these tables are not tabulated, and the calculation had to be kept within practicable limits, I had to abandon the idea. I may mention that I had 216 values of $A_{x+t.y}^{\;1}$ to compute before I was in a position to commence the calculation of the addition to the survivorship premium, and I also append a table of some of these values in the hope that they will prove serviceable. I have made some calculations, however, with the view of ascertaining what addition would have been made to the Select and $H^{M(5)}$ survivorship premiums had these tables been used in the calculation; and it may be interesting to give a few of the results, and compare them with those by the $H^M$ Table. The Select premium for 40 against 70 is ·01098 and the $H^M$ ·01149; while for 70 against 40 they are ·08991 and ·09541 respectively, and for 30 against 30, ·01354 and ·01315. The Select premiums if 40 die before 70 or within 3, 7, and 10 years after him, payable until the risk determines, are ·01196, ·01331, and ·01435 respectively, the additions to the survivorship premium being therefore ·00098, ·00233, and ·00337 respectively, which are considerably greater than those given in my table, namely, ·00072, ·00182, and ·00273. The Select premiums for 40 against 70 or within 3, 7, and 10 years after him, payable during the joint existence of the lives only, are ·01500, ·02038, and ·02437 respectively, the additions to the survivorship premium being therefore ·00402, ·00940, and ·01339 respectively, which agree very closely with those given in my table, namely, ·00401, ·00934, and ·01333. I believe that, when the premium is payable during the joint existence of the lives only, the addition will be almost exactly the same as that given in my table, but that, when the premium is payable until the risk determines, the addition will be greater, except when one—or both—of the lives is young.

Mr. Meikle has given (*J.I.A.*, iv, 134) a method of approximating to the premium for $x$ against $y$ and $t$ years longer. His formula is (using modern notation) $P_{xy}^{1} + P_{xy.}^{\;1}{}_{|t}A_{x+z}$, where $z = e_{xy}$. Here it will be noticed that, when the premium is payable during the joint existence of the lives only, the addition to the ordinary survivorship premium is $P_{xy.}^{\;1}{}_{|t}A_{x+z}$, that is to say, the annual premium which will provide a temporary insurance on $x$ for $t$ years after the joint existence has failed, provided it is dissolved by the death of $y$. I give some of the additions calculated in this way, and it will be observed that the results by his approximate method agree fairly well with the exact values given in my table; but as the formula assumes a table of the expectation of two joint lives to have been formed, it cannot be readily applied. For 40 against 70, and $t$=1, 3, 5, 7, and 10, the additions, using first differences, would be ·00128, ·00384, ·00637, ·00887, and ·01266 respectively, as against ·00134, ·00401, ·00668, ·00934, and ·01333 by my table. When the premium is payable until the risk determines, however, the above formula should not be multiplied by $\frac{1+a_{xy}}{1+a_x}$, as stated by him, but by $\frac{1+a_{xy}}{1+a_{y\overline{(t)}.x}}$.

I take this opportunity of submitting another solution of the problem, in the belief that it will be more readily followed by the younger readers of the *Journal*.

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Required the present value of an assurance of 1 payable if $x$ die before $y$ or within $t$ years after him.

(1) During the first $t$ years the insurance would be paid whether $x$ died before or after $y$, and we have ${}_{|t}A^1_{xy}+{}_{|t}A^2_{xy}={}_{|t}A_x$. (2) After $t$ years the insurance would be paid in any year, say the $(t+n)$th, if $x$ die in that year and $y$ be alive $t$ years previously, that is to say, on the average at the middle of the $n$th year. The value of the second part is therefore

$$\Sigma v^{n+t}\frac{d_{x+t+n-1}}{l_x}\cdot\frac{l_{y+n-\frac{1}{2}}}{l_y}$$

$$=v^t\frac{l_{x+t}}{l_x}\Sigma v^n\frac{d_{x+t+n-1}}{l_{x+t}}\cdot\frac{l_{y+n-\frac{1}{2}}}{l_y}$$

$$=v^t{}_tp_xA^{\ 1}_{x+t.y}.$$

The total value of the assurance is therefore

$$_{|t}A_x+v^t{}_tp_xA^{\ 1}_{x+t.y}.$$

It may be useful to point out that Mr. Curtis Otter, in solving this problem (*J.I.A.*, vii, 240), speaks of the payment in the $n$th year when it is evidently the payment in the $(t+n)$th year which is meant.

I am, Sir,
Your obedient servant,
JAMES CHATHAM.

*Scottish Equitable Life Assurance Socy., Edinburgh, 7th December,* 1885.

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*Table showing the Addition to* $100\varpi^1_{xy}$ *to cover the Risk of x dying within t Years after y, when the Premium is payable until the Risk determines,*—$100(\varpi^1_{x:y\overline{(t)}}-\varpi^1_{xy})$. $H^M$ 4 *per-cent.*

| $x$ | $y$ | $t=1$ | $t=3$ | $t=5$ | $t=7$ | $t=10$ | $x$ | $y$ | $t=1$ | $t=3$ | $t=5$ | $t=7$ | $t=10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 20 | ·008 | ·026 | ·042 | ·058 | ·080 | 50 | 20 | ·008 | ·024 | ·041 | ·058 | ·085 |
| | 30 | ·010 | ·030 | ·049 | ·070 | ·099 | | 30 | ·015 | ·043 | ·071 | ·098 | ·137 |
| | 40 | ·009 | ·027 | ·046 | ·067 | ·098 | | 40 | ·026 | ·077 | ·126 | ·173 | ·239 |
| | 50 | ·007 | ·022 | ·039 | ·057 | ·085 | | 50 | ·039 | ·118 | ·197 | ·273 | ·381 |
| | 60 | ·007 | ·021 | ·035 | ·051 | ·075 | | 60 | ·045 | ·142 | ·243 | ·347 | ·502 |
| | 70 | ·007 | ·021 | ·035 | ·049 | ·072 | | 70 | ·047 | ·148 | ·258 | ·373 | ·555 |
| 30 | 20 | ·011 | ·028 | ·045 | ·061 | ·084 | 60 | 20 | ·007 | ·025 | ·044 | ·066 | ·097 |
| | 30 | ·014 | ·041 | ·068 | ·094 | ·131 | | 30 | ·013 | ·040 | ·070 | ·100 | ·142 |
| | 40 | ·016 | ·048 | ·081 | ·114 | ·164 | | 40 | ·024 | ·074 | ·123 | ·172 | ·237 |
| | 50 | ·014 | ·044 | ·076 | ·109 | ·163 | | 50 | ·045 | ·137 | ·225 | ·310 | ·424 |
| | 60 | ·011 | ·038 | ·066 | ·095 | ·143 | | 60 | ·069 | ·216 | ·363 | ·505 | ·702 |
| | 70 | ·012 | ·037 | ·062 | ·089 | ·131 | | 70 | ·090 | ·284 | ·486 | ·690 | ·985 |
| 40 | 20 | ·008 | ·026 | ·042 | ·059 | ·082 | 70 | 20 | ·011 | ·034 | ·061 | ·088 | ·121 |
| | 30 | ·015 | ·045 | ·074 | ·102 | ·139 | | 30 | ·014 | ·048 | ·085 | ·119 | ·162 |
| | 40 | ·023 | ·069 | ·114 | ·158 | ·220 | | 40 | ·023 | ·077 | ·133 | ·183 | ·244 |
| | 50 | ·027 | ·081 | ·138 | ·194 | ·280 | | 50 | ·046 | ·146 | ·243 | ·329 | ·430 |
| | 60 | ·024 | ·077 | ·134 | ·193 | ·288 | | 60 | ·091 | ·279 | ·460 | ·620 | ·808 |
| | 70 | ·023 | ·072 | ·125 | ·182 | ·273 | | 70 | ·158 | ·490 | ·808 | 1·092 | 1·432 |

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*Table showing the addition to $100\varpi^1_{xy}$ to cover the Risk of $x$ dying within $t$ Years after $y$, when the Premium is payable during the joint existence of the Lives only,—$100\left\{\frac{A^1_{x:y(\overline{t})}}{a_{xy}} - \varpi^1_{xy}\right\}$. $H^M$ 4 per-cent.*

| $x$ | $y$ | $t=1$ | $t=3$ | $t=5$ | $t=7$ | $t=10$ | $x$ | $y$ | $t=1$ | $t=3$ | $t=5$ | $t=7$ | $t=10$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 20 | ·018 | ·055 | ·088 | ·120 | ·163 | 50 | 20 | ·033 | ·091 | ·143 | ·194 | ·258 |
| | 30 | ·023 | ·068 | ·111 | ·152 | ·212 | | 30 | ·047 | ·133 | ·210 | ·279 | ·368 |
| | 40 | ·026 | ·079 | ·129 | ·180 | ·253 | | 40 | ·074 | ·210 | ·333 | ·442 | ·584 |
| | 50 | ·032 | ·095 | ·157 | ·218 | ·308 | | 50 | ·114 | ·331 | ·532 | ·717 | ·961 |
| | 60 | ·045 | ·132 | ·216 | ·297 | ·415 | | 60 | ·163 | ·484 | ·794 | 1·092 | 1·507 |
| | 70 | ·069 | ·203 | ·331 | ·454 | ·630 | | 70 | ·232 | ·694 | 1·153 | 1·607 | 2·273 |
| 30 | 20 | ·024 | ·065 | ·104 | ·139 | ·187 | 60 | 20 | ·043 | ·124 | ·194 | ·255 | ·329 |
| | 30 | ·031 | ·091 | ·148 | ·200 | ·272 | | 30 | ·061 | ·169 | ·264 | ·345 | ·444 |
| | 40 | ·039 | ·117 | ·191 | ·264 | ·367 | | 40 | ·093 | ·261 | ·406 | ·529 | ·676 |
| | 50 | ·047 | ·142 | ·235 | ·326 | ·462 | | 50 | ·160 | ·450 | ·702 | ·918 | 1·179 |
| | 60 | ·061 | ·184 | ·304 | ·422 | ·597 | | 60 | ·269 | ·775 | 1·230 | 1·631 | 2·133 |
| | 70 | ·091 | ·270 | ·443 | ·613 | ·861 | | 70 | ·429 | 1·260 | 2·042 | 2·766 | 3·725 |
| 40 | 20 | ·026 | ·074 | ·118 | ·159 | ·213 | 70 | 20 | ·070 | ·184 | ·277 | ·349 | ·424 |
| | 30 | ·038 | ·111 | ·178 | ·239 | ·318 | | 30 | ·088 | ·239 | ·358 | ·450 | ·545 |
| | 40 | ·056 | ·164 | ·265 | ·360 | ·487 | | 40 | ·127 | ·345 | ·517 | ·647 | ·780 |
| | 50 | ·075 | ·222 | ·365 | ·503 | ·699 | | 50 | ·219 | ·594 | ·888 | 1·111 | 1·336 |
| | 60 | ·096 | ·289 | ·480 | ·669 | ·949 | | 60 | ·415 | 1·132 | 1·704 | 2·145 | 2·596 |
| | 70 | ·134 | ·401 | ·668 | ·934 | 1·333 | | 70 | ·773 | 2·154 | 3·300 | 4·212 | 5·183 |

*Table of $\varpi^1_{xy}$. $H^M$ 4 per-cent.*

| $x$ | $y$ | $\varpi^1_{xy}$ | $x$ | $y$ | $\varpi^1_{xy}$ | $x$ | $y$ | $\varpi^1_{xy}$ |
|---|---|---|---|---|---|---|---|---|
| 20 | 20 | ·01004 | 40 | 20 | ·02127 | 60 | 20 | ·05526 |
| | 30 | ·00913 | | 30 | ·02012 | | 30 | ·05455 |
| | 40 | ·00823 | | 40 | ·01823 | | 40 | ·05310 |
| | 50 | ·00748 | | 50 | ·01580 | | 50 | ·05010 |
| | 60 | ·00686 | | 60 | ·01344 | | 60 | ·04505 |
| | 70 | ·00635 | | 70 | ·01149 | | 70 | ·03858 |
| 30 | 20 | ·01432 | 50 | 20 | ·03333 | 70 | 20 | ·09697 |
| | 30 | ·01315 | | 30 | ·03239 | | 30 | ·09645 |
| | 40 | ·01168 | | 40 | ·03056 | | 40 | ·09541 |
| | 50 | ·01024 | | 50 | ·02746 | | 50 | ·09304 |
| | 60 | ·00905 | | 60 | ·02350 | | 60 | ·08812 |
| | 70 | ·00811 | | 70 | ·01963 | | 70 | ·07966 |

*Table of* $A^{1}_{xy}$. $H^{M}$ 4 *per-cent.*

| $x$ | $y$ | $A^{1}_{xy}$ | $x$ | $y$ | $A^{1}_{xy}$ | $x$ | $y$ | $A^{1}_{xy}$ | $x$ | $y$ | $A^{1}_{xy}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 20 | ·1715 | 35 | 20 | ·2693 | 50 | 20 | ·4205 | 65 | 20 | ·6166 |
| | 30 | ·1475 | | 30 | ·2413 | | 30 | ·3994 | | 30 | ·6045 |
| | 40 | ·1211 | | 40 | ·2002 | | 40 | ·3602 | | 40 | ·5813 |
| | 50 | ·0944 | | 50 | ·1517 | | 50 | ·2940 | | 50 | ·5310 |
| | 60 | ·0682 | | 60 | ·1051 | | 60 | ·2097 | | 60 | ·4385 |
| | 70 | ·0448 | | 70 | ·0661 | | 70 | ·1299 | | 70 | ·3095 |
| 25 | 20 | ·1979 | 40 | 20 | ·3129 | 55 | 20 | ·4828 | 70 | 20 | ·6839 |
| | 30 | ·1712 | | 30 | ·2864 | | 30 | ·4650 | | 30 | ·6744 |
| | 40 | ·1395 | | 40 | ·2433 | | 40 | ·4305 | | 40 | ·6564 |
| | 50 | ·1069 | | 50 | ·1863 | | 50 | ·3661 | | 50 | ·6156 |
| | 60 | ·0760 | | 60 | ·1280 | | 60 | ·2723 | | 60 | ·5335 |
| | 70 | ·0489 | | 70 | ·0791 | | 70 | ·1734 | | 70 | ·4028 |
| 30 | 20 | ·2313 | 45 | 20 | ·3639 | 60 | 20 | ·5494 | 75 | 20 | ·7468 |
| | 30 | ·2031 | | 30 | ·3399 | | 30 | ·5345 | | 30 | ·7395 |
| | 40 | ·1662 | | 40 | ·2975 | | 40 | ·5057 | | 40 | ·7258 |
| | 50 | ·1262 | | 50 | ·2341 | | 50 | ·4471 | | 50 | ·6942 |
| | 60 | ·0887 | | 60 | ·1626 | | 60 | ·3504 | | 60 | ·6261 |
| | 70 | ·0567 | | 70 | ·1000 | | 70 | ·2336 | | 70 | ·5051 |

## ON THE ANALOGY BETWEEN AN ANNUITY-CERTAIN AND A LIFE ANNUITY.

*To the Editor of the Journal of the Institute of Actuaries.*

SIR,—The analogy existing between an annuity-certain and a life annuity has been remarked upon by Mr. G. King, in his interesting note in Volume xx of the *Journal* (p. 435), and Mr. James Chisholm in the preface to his recently-published *Tables of Policy-Values.* Both of these writers have investigated the subject on the assumption that the annuities were payable once a year, and little remains to be said upon the matter from this point of view. But the assumption of annual intervals makes it necessary that the functions should be manipulated before their similarity can be demonstrated; while even then, the analogy, in one respect (compare expressions (3) and (4), *J.I.A.*, xx, 436), appeals to the intelligence rather than to the eye. A far stricter resemblance—indeed, a complete and exact coincidence—will, however, be found to exist between the two functions, and between other cognate functions depending upon the same elements, when we regard them as being payable continuously, or by momently instalments. The following formulas attest the truth of this assertion, and may be considered of some interest to students of actuarial science.

On the assumption that the interest is convertible, and the annuity payable, momently, we have

$$\bar{a}_{\overline{n}|} = \frac{1-\epsilon^{-n\delta}}{\delta}.$$

Here $\epsilon^{-n\delta}$ represents the present value of 1 to be received at the end of $n$ years on the conditions specified, and is really the single payment necessary to secure the unit at the expiration of this period. As