NÖRLUND METHODS OF SUMMABILITY ASSOCIATED WITH POLYNOMIALS

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1. Introduction

Let s, s_n $(n = 0, 1, \ldots)$ be arbitrary complex numbers, and let

$$p(z) = p_0 + p_1 z + \ldots + p_j z^j$$

be a polynomial, with complex coefficients, which satisfies the normalizing condition

$$p(1) = 1.$$

Associated with such a polynomial is a Nörlund method of summability N_p: the sequence $\{s_n\}$ is said to be N_p-convergent to s, and we write $s_n \to s \ (N_p)$, if

$$\lim_{n\to\infty} \sum_{\nu=0}^{j} p_\nu s_{n-\nu} = s.$$

Evidently the method is regular, i.e. $s_n \to s \ (N_p)$ whenever $s_n \to s$.

Let $q(z) = q_0 + q_1 z + \ldots + q_k z^k$, $\quad q(1) = 1$.

For convenience, we suppose throughout that $p_n = 0$ for $n > j$ and $q_n = 0$ for $n > k$, so that

$$p(z) = \sum_{n=0}^{\infty} p_n z^n, \qquad q(z) = \sum_{n=0}^{\infty} q_n z^n,$$

$$\sum_{\nu=0}^{n} p_\nu s_{n-\nu} = \sum_{\nu=0}^{j} p_\nu s_{n-\nu} \text{ for } n > j,$$

and

$$\sum_{\nu=0}^{n} q_\nu s_{n-\nu} = \sum_{\nu=0}^{k} q_\nu s_{n-\nu} \text{ for } n > k.$$

The object of this note is to investigate some of the properties of Nörlund methods associated with polynomials. We shall also be concerned with the Cesàro method (C, α), the Abel method A, and the "product" methods $(C, \alpha)N_p$ and AN_p; the latter two methods being defined as follows. The sequence $\{s_n\}$ is $(C, \alpha)N_p$-convergent to s if $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu} \to s \ (C, \alpha)$; it is AN_p-convergent to s if $t_n \to s \ (A)$.

A summability method X is said to include a method Y if the Y-convergence of any sequence to s implies its X-convergence to s. The methods are said to be equivalent if each includes the other.

Throughout the note it should be borne in mind that the Nörlund methods N_p and N_q, being associated with the *polynomials* $p(z)$ and $q(z)$, are not of the most general type (see (**2**), § 4.1).

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2. Simple Theorems Concerning Inclusion

We defer the statement of the main theorems till § 3 and proceed to prove some simpler results.

Theorem 1. *There is a sequence which is (C, α)-convergent for every $\alpha>0$ but not N_p-convergent.*

Proof. If $|z| = 1$, $z \neq 1$, $p(1/z) \neq 0$, then, for $n>j$,

$$\sum_{\nu=0}^{n} p_\nu z^{n-\nu} = z^n p(1/z)$$

which oscillates as n tends to infinity; and so the sequence $\{z^n\}$ is not N_p-convergent, but as is well known, it is (C, α)-convergent to 0 for every $\alpha>0$.

Corollary. *N_p does not include (C, α) for any $\alpha>0$.*

Theorem 2. *The method N_f, associated with the polynomial $f(z) = p(z)q(z)$, includes both N_p and N_q.*

Proof. (Cf. the proof of Theorem 17 in (**2**)). Let $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$, and note that $f(z) = \sum_{n=0}^{\infty} f_n z^n$ where $f_n = \sum_{\nu=0}^{n} p_\nu q_{n-\nu}$. Then

$$\sum_{\nu=0}^{n} f_\nu s_{n-\nu} = \sum_{\nu=0}^{n} q_\nu t_{n-\nu}$$

which tends to s whenever $t_n \to s$, i.e. N_f includes N_p. Similarly, N_f includes N_q.

Corollary. *The methods N_p and N_q are consistent, i.e. if $s_n \to s$ (N_p) and $s_n \to s'$ (N_q), then $s = s'$.*

From Theorem 2 we can at once deduce a result of Silverman and Szasz ((**4**), Theorem 14), namely that, if $p(z) = (1+z+\ldots+z^j)/(1+j)$, $q(z) = (1+z+\ldots+z^k)/(1+k)$, then a sufficient condition for N_q to include N_p is that $1+j$ should be a factor of $1+k$. Theorem I (below) shows that the condition is also necessary. The next theorem is a generalisation of another of their results ((**4**), Theorem 15).

Theorem 3. *If $h(z)$ is the highest common factor of $p(z)$ and $q(z)$, normalized so as to make $h(1) = 1$, then a necessary and sufficient condition for a sequence to be both N_p- and N_q-convergent is that it be N_h-convergent.*

Proof. That the condition is sufficient follows from Theorem 2. To prove that it is necessary, we observe that there are polynomials

$$a(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad b(z) = \sum_{n=0}^{\infty} b_n z^n$$

such that

$$h(z) = a(z)p(z) + b(z)q(z) = \sum_{n=0}^{\infty} h_n z^n$$

say. Hence if $t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu} \to s$ and $u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu} \to s$, then

$$\sum_{\nu=0}^{n} h_\nu s_{n-\nu} = \sum_{\nu=0}^{n} a_\nu t_{n-\nu} + \sum_{\nu=0}^{n} b_\nu u_{n-\nu} \to sa(1)+sb(1) = s,$$

since $h(1) = p(1) = q(1) = 1$. The required result follows.

3. The Main Theorems

It is to be supposed throughout the rest of the note that

$$p(0) \neq 0.$$

This restriction is not a serious one, since, if r is a positive integer and $f(z) = z^r p(z) = \sum_{\nu=0}^{j+r} f_\nu z^\nu$, then $\sum_{v=0}^{n+r} f_\nu s_{n+r-\nu} = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}$ so that N_p and N_f are equivalent.

Theorem I. *In order that N_q should include N_p it is necessary and sufficient that $q(z)/p(z)$ should not have poles on or within the unit circle.*

Theorem II. *If $q(z)/p(z)$ has poles of maximum order m on the unit circle and does not have poles within the unit circle, then $(C, m)N_q$ includes N_p, but, for any $\varepsilon > 0$, there is an N_p-convergent sequence which is not $(C, m-\varepsilon)N_q$-convergent.*

Theorem III. *If $q(z)/p(z)$ has a pole within the unit circle, then there is an N_p-convergent sequence which is not AN_q-convergent.*

Noting that $(C, 0)$ is identical with N_q when $q(z)$ is 1 (i.e. $q_0 = 1$, $q_n = 0$ for $n > 0$), and that N_p always includes $(C, 0)$, we obtain the following corollaries of the theorems.

I′. *In order that N_p should be equivalent to $(C, 0)$ it is necessary and sufficient that $p(z)$ should not have zeros on or within the unit circle.*

II′. *If $p(z)$ has zeros of maximum order m on the unit circle and does not have zeros within the unit circle, then (C, m) includes N_p, but, for any $\varepsilon > 0$, there is an N_p-convergent sequence which is not $(C, m-\varepsilon)$-convergent.*

III′. *If $p(z)$ has a zero within the unit circle, then there is an N_p-convergent sequence which is not A-convergent.*

Result I′ is essentially equivalent to a theorem due to Kubota (3).

Some of the principal results established by Boyd and myself in a recent paper (1) can be deduced from II′ by considering $p(z) = 2^{-m}(1+z)^m$ and $p(z) = \alpha + \beta z + (1-\alpha-\beta)z^2$ with α, β real.

4. Proof of Theorem III, and Lemmas

Proof of Theorem III. We start with this theorem because its proof is simpler than those of Theorems I and II.

Since $1/p(z)$ is analytic in a neighbourhood U of the origin, there is a

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sequence $\{s_n\}$ such that, for z in U,

$$\sum_{n=0}^{\infty} s_n z^n = 1/p(z).$$

Let

$$t_n = \sum_{v=0}^{n} p_v s_{n-v}, \qquad u_n = \sum_{v=0}^{n} q_v s_{n-v}.$$

Then, for z in U,

$$\sum_{n=0}^{\infty} t_n z^n = p(z) \sum_{n=0}^{\infty} s_n z^n = 1$$

and

$$\sum_{n=0}^{\infty} u_n z^n = q(z) \sum_{n=0}^{\infty} s_n z^n = q(z)/p(z).$$

Hence $t_0 = 1$, $t_n = 0$ for $n>0$, and so $\{s_n\}$ is N_p-convergent to 0. On the other hand $\Sigma u_n z^n$ has radius of convergence less than unity, because, by hypothesis, $q(z)/p(z)$ has a pole within the unit circle. Consequently, $\{u_n\}$ is not A-convergent and so $\{s_n\}$ is not AN_q-convergent.

We now prove two lemmas.

Lemma 1. *If $q(z)/p(z)$ has poles $\lambda_1, \lambda_2, \ldots, \lambda_l$ (and no others) of orders $m_1, m_2, \ldots, m_l$, and if, for $n = 0, 1, \ldots,$*

$$t_n = \sum_{v=0}^{n} p_v s_{n-v}, \qquad u_n = \sum_{v=0}^{n} q_v s_{n-v},$$

then

$$u_n = \sum_{v=0}^{n} c_v t_{n-v} + \sum_{r=1}^{l} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{v=0}^{n} \binom{v+\rho-1}{\rho-1} \lambda_r^{-v} t_{n-v}$$

where the c's are constants, depending only on $p_0, p_1, \ldots, p_j, q_0, q_1, \ldots, q_k$, such that $c_n = 0$ for $n>k-j$ and $c_{r,m_r} \neq 0$.

Proof. Let N be any positive integer, and let

$$s'_n = \begin{cases} s_n \text{ for } 0 \leqq n \leqq N, \\ 0 \text{ for } n>N, \end{cases}$$

$$t'_n = \sum_{v=0}^{n} p_v s'_{n-v}, \qquad u'_n = \sum_{v=0}^{n} q_v s'_{n-v};$$

so that $t'_n = t_n$, $u'_n = u_n$ for $0 \leqq n \leqq N$, and $t'_n = u'_n = 0$ for $n>j+k+N$. Then

$$\sum_{n=0}^{\infty} t'_n z^n = p(z) \sum_{n=0}^{\infty} s'_n z^n, \qquad \sum_{n=0}^{\infty} u'_n z^n = q(z) \sum_{n=0}^{\infty} s'_n z^n,$$

and so, since 0 is not a pole of $q(z)/p(z)$,

$$\sum_{n=0}^{\infty} u'_n z^n = \frac{q(z)}{p(z)} \sum_{n=0}^{\infty} t'_n z^n$$

$$= \left\{ \sum_{n=0}^{\infty} c_n z^n + \sum_{r=1}^{l} \sum_{\rho=1}^{m_r} c_{r,\rho} \left(1 - \frac{z}{\lambda_r}\right)^{-\rho} \right\} \sum_{n=0}^{\infty} t'_n z^n$$

where $c_n = 0$ for $n > k-j$ and $c_{r,m_r} \neq 0$. Expanding $(1-z/\lambda_r)^{-\rho}$, with $|z| < \min(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_l|)$, and equating coefficients, we obtain the required identity for $0 \leqq n \leqq N$. Since N can be taken arbitrarily large it must hold for all n.

Lemma 2. *If* $|\lambda| > 1$, ρ *is any real number, and* $t_n \to 0$, *then*

$$\lim_{n\to\infty} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda^{-\nu} t_{n-\nu} = 0.$$

Since $\sum_{\nu=0}^{\infty} \binom{\nu+\rho-1}{\rho-1} \lambda^{-\nu}$ is absolutely convergent, the result is evident.

5. Proof of Theorem I, and Lemmas

Proof of Theorem I (sufficiency). The hypothesis is that the function $q(z)/p(z)$ does not have poles on or within the unit circle. If it does not have any poles at all it must be a polynomial and so, by Theorem 2, N_q includes N_p. Otherwise, it follows from Lemmas 1 and 2 that $s_n \to 0$ (N_q) whenever $s_n \to 0$ (N_p), and hence that $s_n \to s$ (N_q) whenever $s_n \to s$ (N_p).

The necessity part of Theorem I is a consequence of Theorems II and III. It remains only to prove Theorem II and for this we require three additional lemmas.

Lemma 3. *If* $|\lambda| = 1$, $\lambda \neq 1$, $\alpha > -1$, $\beta > -1$, *then*

$$\sum_{\nu=0}^{n} \binom{n-\nu+\beta}{\beta}\binom{\nu+\alpha}{\alpha} \lambda^{-\nu} = \binom{n+\beta}{\beta}(1-1/\lambda)^{-\alpha-1} + \binom{n+\alpha}{\alpha} \lambda^{-n}(1-\lambda)^{-\beta-1} + O(n^{\beta-1}+n^{\alpha-1}).$$

Here and elsewhere it is to be assumed that powers of complex numbers have their principal values.

A proof of the above lemma is given in (**2**), §6.9. Using a similar method of proof we shall establish

Lemma 4. *If* $|\lambda| = |\mu| = 1$, $\lambda \neq 1$, $\mu \neq 1$, $\lambda \neq \mu$, $\alpha > -1$, $\beta > -1$, $\gamma > -1$, *and*

$$v_n = \sum_{\nu=0}^{n} \binom{n-\nu+\beta}{\beta} \mu^{\nu-n} \binom{\nu+\alpha}{\alpha} \lambda^{-\nu},$$

then

$$\begin{aligned} w_n = \sum_{r=0}^{n} \binom{r+\gamma}{\gamma} v_{n-r} = \binom{n+\gamma}{\gamma} &(1-1/\lambda)^{-\alpha-1}(1-1/\mu)^{-\beta-1} \\ &+ \binom{n+\alpha}{\alpha} \lambda^{-n}(1-\lambda)^{-\gamma-1}(1-\lambda/\mu)^{-\beta-1} \\ &+ \binom{n+\beta}{\beta} \mu^{-n}(1-\mu)^{-\gamma-1}(1-\mu/\lambda)^{-\alpha-1} \\ &+ O(n^{\gamma-1}+n^{\alpha-1}+n^{\beta-1}). \end{aligned}$$

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Proof. Note that, within the unit circle,

$$\sum_{n=0}^{\infty} w_n z^n = (1-z/\lambda)^{-\alpha-1}(1-z/\mu)^{-\beta-1}(1-z)^{-\gamma-1}$$
$$= w(z)$$

say, so that

$$2\pi i w_n = \int_C w(z) z^{-n-1} dz$$

where C is the circle $|z| = \rho < 1$. Let $z_1 = 1$, $z_2 = \lambda$, $z_3 = \mu$, and let $n > 1/\delta$ where $\delta = \min(|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|)$. Then, by Cauchy's theorem,

$$2\pi i w_n = \sum_{r=1}^{3} \int_{C_r} w(z) z^{-n-1} dz, \tag{1}$$

where C_r is the contour formed by the circle $|z - z_r| = 1/n$ and the infinite segment $z = z_r \tau$, $\tau \geqq 1 + 1/n$, the latter being described twice.

Let $$u(z) = (1-1/\lambda)^{-\alpha-1}(1-1/\mu)^{-\beta-1}(1-z)^{-\gamma-1};$$

so that

$$\int_{C_1} u(z) z^{-n-1} dz = \int_C u(z) z^{-n-1} dz$$
$$= 2\pi i (1-1/\lambda)^{-\alpha-1}(1-1/\mu)^{-\beta-1}\binom{n+\gamma}{\gamma}.$$

Further, for z on C_1,

$$w(z) - u(z) = (1-z)^{-\gamma-1}\int_1^z \{\lambda^{-1}(\alpha+1)(1-t/\lambda)^{-\alpha-2}(1-t/\mu)^{-\beta-1}$$
$$+\mu^{-1}(\beta+1)(1-t/\lambda)^{-\alpha-1}(1-t/\mu)^{-\beta-2}\}dt$$
$$= O(|z-1|^{-\gamma}).$$

Consequently, the contribution of the circle to

$$\int_{C_1} \{w(z) - u(z)\} z^{-n-1} dz$$

is $O\{(1/n)^{-\gamma}(1/n)\} = O(n^{\gamma-1})$, and that of the rest of C_1 (see (**2**), 138) is

$$O\left\{\int_{1+1/n}^{\infty} (x-1)^{-\gamma} x^{-n-1} dx\right\} = O(n^{\gamma-1}).$$

Hence

$$\int_{C_1} w(z) z^{-n-1} dz = \int_{C_1} (1-z/\lambda)^{-\alpha-1}(1-z/\mu)^{-\beta-1}(1-z)^{-\gamma-1} z^{-n-1} dz$$
$$= 2\pi i (1-1/\lambda)^{-\alpha-1}(1-1/\mu)^{-\beta-1}\binom{n+\gamma}{\gamma} + O(n^{\gamma-1}). \tag{2}$$

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Now

$$\int_{C_2} w(z)z^{-n-1}dz = \int_{C_1} w(\lambda z)(\lambda z)^{-n-1}\lambda dz$$
$$= \lambda^{-n}\int_{C_1}(1-z)^{-\alpha-1}(1-\lambda z/\mu)^{-\beta-1}(1-\lambda z)^{-\gamma-1}z^{-n-1}dz$$
$$= 2\pi i\lambda^{-n}(1-\lambda)^{-\gamma-1}(1-\lambda/\mu)^{-\beta-1}\binom{n+\alpha}{\alpha}+O(n^{\alpha-1}) \qquad (3)$$

by (2), since $|\mu/\lambda| = 1$, $\mu/\lambda \neq 1$, $1/\lambda \neq 1$, $\mu/\lambda \neq 1/\lambda$.

Similarly,

$$\int_{C_3} w(z)z^{-n-1}dz = 2\pi i\mu^{-n}(1-\mu)^{-\gamma-1}(1-\mu/\lambda)^{-\alpha-1}\binom{n+\beta}{\beta}+O(n^{\beta-1}). \qquad (4)$$

The required conclusion follows from the numbered identities.

Lemma 5. *If* $|\lambda| = 1$, $\lambda \neq 1$, $\alpha > -1$ *and* $t_n \to 0$, *then*

$$v_n = \sum_{\nu=0}^{n}\binom{\nu+\alpha}{\alpha}\lambda^{-\nu}t_{n-\nu} \to 0\ (C, \alpha+1).$$

Proof. We have

$$\sum_{r=0}^{n}\binom{r+\alpha}{\alpha}v_{n-r} = \sum_{r=0}^{n} t_{n-r}\sum_{\nu=0}^{r}\binom{r-\nu+\alpha}{\alpha}\binom{\nu+\alpha}{\alpha}\lambda^{-\nu}$$

which, by Lemma 3, is

$$O\left\{\sum_{r=0}^{n}|t_{n-r}|\binom{r+\alpha}{\alpha}\right\} = o\left\{\binom{n+\alpha+1}{\alpha+1}\right\};$$

and this is the required conclusion.

6. Proof of Theorem II

Let

$$t_n = \sum_{\nu=0}^{n} p_\nu s_{n-\nu}, \qquad u_n = \sum_{\nu=0}^{n} q_\nu s_{n-\nu}.$$

Our hypothesis is that the function $q(z)/p(z)$ has poles of maximum order m on the unit circle and that its other poles (if any) lie outside the unit circle. Also $p(1) = 1$ and so $z = 1$ is not a pole of $q(z)/p(z)$. Hence, by Lemmas 1, 2 and 5, if $s_n \to 0$ (N_p), i.e. if $t_n \to 0$, then $u_n \to 0$ (C, m), i.e. $s_n \to 0$ $(C, m)N_q$. Since all the summability methods concerned are regular, it follows that $s_n \to s$ $(C, m)N_q$ whenever $s_n \to s$ (N_p), i.e. that $(C, m)N_q$ includes N_p.

We have thus established the first part of Theorem II. To prove the remainder, suppose, as we may without loss in generality, that

$$0 < \varepsilon < 1.$$

Let the poles of $q(z)/p(z)$ be $\lambda_1, \lambda_2, \ldots, \lambda_l$ with orders $m_1, m_2, \ldots, m_l$. Suppose

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the numbering to be such that the first l' of these are the ones on the unit circle and that

$$m_1 = m = \max(m_1, m_2, \dots, m_{l'}).$$

Let $\{s_n\}$ be the sequence for which

$$t_n = \lambda_1^{-n}\binom{n-\varepsilon}{-\varepsilon};$$

the existence (and uniqueness) of the sequence $\{s_n\}$ being ensured by the condition $p_0 = p(0) \neq 0$. Then, taking $c_{r,\rho}$ to be 0 if $\rho > m_r$, we have, by Lemma 1,

$$u_n = \sum_{\tau=1}^{4} u_n^{(\tau)},$$

where

$$u_n^{(1)} = \begin{cases} \sum_{\nu=0}^{n} c_\nu t_{n-\nu} + \sum_{r=l'+1}^{l} \sum_{\rho=1}^{m_r} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu} \text{ if } l > l', \\ \sum_{\nu=0}^{n} c_\nu t_{n-\nu} \text{ if } l = l'; \end{cases}$$

$$u_n^{(2)} = \begin{cases} \sum_{r=1}^{l'} \sum_{\rho=1}^{m-1} c_{r,\rho} \sum_{\nu=0}^{n} \binom{\nu+\rho-1}{\rho-1} \lambda_r^{-\nu} t_{n-\nu} \text{ if } m > 1, \\ 0 \text{ if } m = 1; \end{cases}$$

$$u_n^{(3)} = \begin{cases} \sum_{r=2}^{l'} c_{r,m} \sum_{\nu=0}^{n} \binom{\nu+m-1}{m-1} \lambda_r^{-\nu} \binom{n-\nu-\varepsilon}{-\varepsilon} \lambda_1^{\nu-n} \text{ if } l' > 1, \\ 0 \text{ if } l' = 1; \end{cases}$$

$$u_n^{(4)} = c_{1,m}\,\lambda_1^{-n} \sum_{\nu=0}^{n} \binom{\nu+m-1}{m-1}\binom{n-\nu-\varepsilon}{-\varepsilon} = c_{1,m}\,\lambda_1^{-n}\binom{n+m-\varepsilon}{m-\varepsilon}.$$

Now $t_n \to 0$, $c_\nu = 0$ for $\nu > k-j$, and $|\lambda_r| > 1$ if $l \geqq r > l'$: hence, by Lemma 2,

$$u_n^{(1)} \to 0.$$

Further, $|\lambda_r| = 1$, $\lambda_r \neq 1$ for $r = 1, 2, \dots, l'$, so that, by Lemma 5,

$$u_n^{(2)} \to 0\ (C, m-1);$$

and, by Lemma 4,

$$u_n^{(3)} \to 0\ (C, m-\varepsilon),$$

since $m-\varepsilon > \max(m-1, -\varepsilon)$ and $m-\varepsilon-1 > -1$.

Consequently $u_n - u_n^{(4)} \to 0\ (C, m-\varepsilon)$; but, by Lemma 3 (or by Theorem 46 in (**2**), since $u_n^{(4)} \neq o(n^{m-\varepsilon})$), $u_n^{(4)}$ does not tend to a limit $(C, m-\varepsilon)$. The sequence $\{u_n\}$ is therefore not $(C, m-\varepsilon)$-convergent; so that the sequence $\{s_n\}$ is not $(C, m-\varepsilon)N_q$-convergent though it is N_p-convergent to 0.

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