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PROJECTIVE LIMIT OF INFINITE RADON MEASURES

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Abstract

We show that for any self-consistent sequentially maximal system $\{\mu_\alpha\}$ of infinite (perhaps non-σ-finite) Radon measures, the projective limit of $\{\mu_\alpha\}$ exists.

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1. Introduction

Yamasaki (1975) has studied the extension problem of a self-consistent system of infinite measures to a σ-additive measure, under the assumption that every system of probability measures has a unique σ-additive extension. In this paper we consider a system of infinite Radon measures on arbitrary topological spaces. In Theorem 1, we prove that for every denumerable projective system $\{\mu_n\}$, there uniquely exists a Radon measure μ satisfying $p_n(\mu) = \mu_n$. In Theorem 2, for a general projective system, we show that Kolmogorov consistency theorem is valid.

Throughout this paper we assume every topological space is a Hausdorff space. Let T be a topological space. By the *Borel field* $\mathbf{B}(T)$, we mean the minimal σ-algebra generated by all open subsets of X. A *Radon measure* μ is a non-negative, extended real-valued Borel measure on $\mathbf{B}(T)$ such that

1) for x in T, there exists an open neighborhood U of x such that $\mu(U) < \infty$ (*locally finite*);

2) for every Borel set E in $\mathbf{B}(T)$,

$$\mu(E) = \sup\{\mu(K);\ K \subset E \text{ and } K \text{ is compact}\}.$$

Let $(A, \leqq)$ be a directed set. A system $\{(T_\alpha, p_{\alpha\beta});\ \alpha, \beta \in A, \alpha \leqq \beta\}$ of topological spaces $\{T_\alpha; \alpha \in A\}$ is said to be a *projective system* if for $\alpha \leqq \beta \leqq \gamma$, $p_{\alpha\beta}$ and $p_{\beta\gamma}$ are continuous ($p_{\alpha\alpha}$ = id) and $p_{\alpha\beta} \circ p_{\beta\gamma} = p_{\alpha\gamma}$ holds. We denote by $T = \lim_{\leftarrow \alpha} T_\alpha$ the projective limit $\{(x_\alpha) \in \Pi_{\alpha \in A} T_\alpha; p_{\alpha\beta}(x_\beta) = x_\alpha, \alpha \leqq \beta\}$. By p_α we mean the projection of T to T_α. A family $\{\mu_\alpha\}$ of measures on the projective system $\{(T_\alpha, p_\alpha, p_{\alpha\beta})\}$ is called self-consistent if μ_α is a non-negative, extended real-valued Borel measure on T_α and $p_{\alpha\beta}(\mu_\beta) = \mu_\alpha$ holds for $\alpha \leqq \beta$.

2. Kolmogorov consistency theorem

THEOREM 1. *Let $T = \lim_{\leftarrow n} T_n$ be the projective limit of a projective system $\{(T_n, p_n, p_{nm}); n \leqq m\}$ and $\{\mu_n\}$ be a self-consistent system of Radon measures on $\{T_n\}$. Then there exists a unique Radon measure μ on T such that $p_n(\mu) = \mu_n$ on T_n.*

PROOF. Let $\{U_\lambda; \lambda \in \Lambda\}$ be the family of all open subsets of T_1 such that $\mu_1(U_\lambda) < \infty$. Naturally we may assume Λ is directed. For any λ the system $\{(p_{1n}^{-1}(U_\lambda), q_{nm}); n \leqq m\}$ is a projective system, where $q_{nm} = p_{nm} \mid p_{1m}^{-1}(U_\lambda)$. If we put $\nu_\lambda^n = \mu_n \mid p_{1n}^{-1}(U_\lambda)$, then we have $q_{nm}(\nu_\lambda^m) = \nu_\lambda^n$ for $n \leqq m$, where $\mu_n \mid p_{1n}^{-1}(U_\lambda)$ is the restriction of μ_n to $p_{1n}^{-1}(U_\lambda)$. By Theorem 4.2 in Bourbaki (1969), there exists a Radon measure ν_λ on $V_\lambda = \lim_{\leftarrow n} p_{1n}^{-1}(U_\lambda)$ for every λ in Λ. From the construction of V_λ, it is clear $V_\lambda = p_1^{-1}(U_\lambda)$. We put

$$\mu(E) = \sup_\lambda \nu_\lambda(E \cap V_\lambda)$$

for every Borel set E in $\mathbf{B}(T)$. For every Borel set E_n in $\mathbf{B}(T_n)$, we have

$$\begin{aligned} p_n(\mu)(E_n) = \mu(p_n^{-1}(E_n)) &= \sup_\lambda \nu_\lambda(p_n^{-1}(E_n) \cap V_\lambda) \\ &= \sup_\lambda \nu_\lambda(p_n^{-1}(E_n \cap p_{1n}^{-1}(U_\lambda))) \\ &= \sup_\lambda \nu_\lambda^n(E_n \cap p_{1n}^{-1}(U_\lambda)) \\ &= \sup_\lambda \mu_n(E_n \cap p_{1n}^{-1}(U_\lambda)) \\ &= \sup_\lambda (\mu_n \mid E_n)(E_n \cap p_{1n}^{-1}(U_\lambda)) \\ &= \mu_n(E_n), \end{aligned}$$

since $\mu_n \mid E_n$ is a Radon measure on E_n. Therefore we have $p_n(\mu) = \mu_n$ for every n, which shows μ is locally finite. Since every ν_λ is a Radon measure, μ is a Radon measure on T.

Assume ν is another Radon measure on T such that $p_n(\nu) = \mu_n$. For every compact subset K of T, it holds $K = \bigcap_{n=1}^{\infty} p_n^{-1} p_n(K)$ (see Proposition 4.2 in Bourbaki (1969)). Thus we have

$$\begin{aligned}\nu(K) &= \lim_{n\to\infty} \nu(p_n^{-1} p_n(K)) \\ &= \lim_{n\to\infty} \mu_n(p_n(K)) \\ &= \mu(K),\end{aligned}$$

which shows ν is identical to μ. This proves the theorem.

COROLLARY. *If μ_n is σ-finite for some n, then the Radon measure μ is also σ-finite, moreover μ is outer regular.*

PROOF. Clearly μ is σ-finite and satisfies the conditions of Theorem A in Amemiya, Okada and Okazaki (to appear).

Next we deal with the general projective system $\{(T_\alpha, p_\alpha, p_{\alpha\beta});\ \alpha \leqq \beta\}$. If each μ_α is a probability measure, then $\{\mu_\alpha\}$ has a unique σ-additive extension by Theorem 5.1.1 in Bochner (1955). We put

$$D(A) = \{(\alpha_n)_{n=1}^{\infty};\quad \alpha_1 < \alpha_2 < \cdots, \alpha_n \in A\}.$$

For every $M = (\alpha_n)_{n=1}^{\infty}$ in $D(A)$, we denote by p_M the natural projection of T to $T_M = \lim_{\leftarrow n} T_{\alpha_n}$. We say the projective system $\{(T_\alpha, p_\alpha, p_{\alpha\beta});\ \alpha \leqq \beta\}$ is *sequentially maximal* if p_M is a surjection for every M in $D(A)$.

THEOREM 2. *Let $T = \lim_{\leftarrow \alpha} T_\alpha$ be the projective limit of a sequentially maximal projective system $\{(T_\alpha, p_\alpha, p_{\alpha\beta});\ \alpha \leqq \beta\}$ such that p_α is surjective. Let $\{\mu_\alpha\}$ be a self-consistent system of Radon measures on $\{T_\alpha\}$. Then there exists a unique σ-additive measure on the σ-algebra $\mathbf{B}_1 = \bigcup_{M\in D(A)} P_M^{-1}(\mathbf{B}(T_M))$ satisfying that $p_M(\mu)$ is a Radon measure on T_M and $p_\alpha(\mu) = \mu_\alpha$ on T_α for every α in A.*

PROOF. Let $\mathbf{F}$ be the algebra $\bigcup_{\alpha\in A} p_\alpha^{-1}(\mathbf{B}(T_\alpha))$. We define a finitely additive set function ρ on $\mathbf{F}$ by

$$\rho(p_\alpha^{-1}(E_\alpha)) = \mu_\alpha(E_\alpha)$$

for every E_α in $\mathbf{B}(T_\alpha)$. Since p_α is surjective, ρ is well defined.

For every $M = (\alpha_n)_{n=1}^{\infty}$ in $D(A)$, $\{\mu_{\alpha_n}\}$ is self-consistent on $\{(T_{\alpha_n}, q_{nM}, p_{\alpha_n\alpha_m})\}$, where q_{nM} is the natural projection of T_M to T_{α_n}. By Theorem 1 there exists a unique Radon measure μ_M on T_M such that $q_{nM}(\mu_M) = \mu_{\alpha_n}$. We introduce an order relation in $D(A)$ as follows: for $M = (\alpha_n)_{n=1}^{\infty}$, $N = (\beta_n)_{n=1}^{\infty}$ in $D(A)$,

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$$M \leqq N \text{ if and only if } \alpha_n \leqq \beta_n \text{ for every } n.$$

For (x_{β_n}) in T_N we set $p_{MN}((x_{\beta_n})) = (p_{\alpha_n\beta_n}(x_{\beta_n}))$. It follows that $p_M = p_{MN}p_N$, and $p_{MN}(\mu_N) = \mu_M$ since it holds $q_{nM}p_{MN}(\mu_N) = p_{\alpha_n\beta_n}q_{nN}(\mu_N) = p_{\alpha_n\beta_n}(\mu_{\beta_n}) = \mu_{\alpha_n}$. Thus we can define μ as follows:

$$\mu(p_M^{-1}(E_M)) = \mu_M(E_M)$$

for every E_M in $\mathbf{B}(T_M)$. Obviously μ is σ-additive on $\mathbf{B}_1$ and $p_M(\mu)$ is equal to the Radon measure μ_M on $\mathbf{B}(T_M)$ for every M in $D(A)$. For each α in A, there exists an M in $D(A)$ such that $\alpha_1 = \alpha$, which shows $p_\alpha(\mu) = q_{1M}p_M(\mu) = q_{1M}(\mu_M) = \mu_{\alpha_1} = \mu_\alpha$. Since $\mathbf{B}_1$ is a σ-algebra containing $\mathbf{F}$, μ is a σ-extension of ρ.

Suppose ν is another σ-additive extension of ρ on $\mathbf{B}_1$ such that $p_M(\nu)$ is a Radon measure on T_M for every $M = (\alpha_n)_{n=1}^\infty$ in $D(A)$. Then it follows that for every n, $q_{nM}(p_M(\nu)) = p_{\alpha_n}(\nu) = p_{\alpha_n}(\rho) = p_{\alpha_n}(\mu) = q_{nM}(p_M(\mu))$ on T_{α_n}. Thus by Theorem 1, we have $p_M(\nu) = p_M(\mu)$. From the definition of $\mathbf{B}_1$, ν is equal to μ. This proves the theorem.

REMARK. In Theorem 2, μ is not necessarily extended to a τ-smooth Borel measure even if every μ_α is a probability measure (see Theorem 4.6 in Moran (1968)).

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