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SHARP GRADIENT ESTIMATES FOR EIGENFUNCTIONS ON RIEMANNIAN MANIFOLDS

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Sharp gradient estimates are derived for positive eigenfunctions on complete Riemannian manifolds with Ricci curvature bounded below.

0. INTRODUCTION

The aim of this note is to derive, in a simple and rather elementary way, sharp gradient estimates for positive eigenfunctions on complete Riemannian manifolds with Ricci curvature bounded from below by a constant. One can view these inequalities as the infinitesimal form of the Harnack inequality for positive eigenfunctions on complete Riemannian manifolds.

Estimates of this type were obtained in [**2**]. In case of harmonic functions the estimate is sharp [**2**, Theorem 3″], but for eigenfunctions in general [**2**, Theorem 3′] the constant involved in the estimate was not explicitely computed.

For an eigenfunction f with a negative eigenvalue, a sharp lower bound on the $\sup |\nabla f| / f$ is obtained as well.

The main result is the following.

THEOREM. *Let $M = M^n$ be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $-(n-1)K^2$, $(K \geqslant 0)$ and let f be a positive eigenfunction, that is, $\Delta f = \lambda f$ for some λ. Then $\lambda \geqslant -(1/4)(n-1)^2 K^2$ and*

$$\frac{(n-1)K - \sqrt{\left((n-1)K\right)^2 + 4\lambda}}{2} \leqslant \sup \frac{|\nabla f|}{f} \leqslant \frac{(n-1)K + \sqrt{\left((n-1)K\right)^2 + 4\lambda}}{2}.$$

If, at some point q, $|\nabla f|/f = \sup |\nabla f|/f$ and $|\nabla f|/f = (1/2)\Big((n-1)K + \sqrt{\left((n-1)K\right)^2 + 4\lambda}\Big)$ or $|\nabla f|/f = (1/2)\Big((n-1)K - \sqrt{\left((n-1)K\right)^2 + 4\lambda}\Big)$ with $\lambda < 0$, then, at this point $\mathrm{Ric}\big((\nabla f)/|\nabla f|, (\nabla f)/|\nabla f|\big) = -(n-1)K^2$ and ∇f is an eigenvector and $\nabla f^{\perp}$ is an eigenspace of $\mathrm{Hess}\, f$.

To see that the estimates above are sharp, one takes the constant curvature model M_K, a simply connected space of constant sectional curvature $-K^2$. Set

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$f = \exp(-\alpha b)$, where b denotes a Busemann function on M_K. Then, a simple computation shows that f is an eigenfunction with eigenvalue $\lambda = \alpha^2 - \alpha(n-1)K$ and equality is achieved in the above inequalities.

The situation is quite different if the Ricci curvature is non-negative. From the proof of the Theorem one obtains easily the following.

Corollary. *Let $M = M^n$ be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $C > 0$. Then there is no positive function f defined on M with $\Delta f = \lambda f$ for any $\lambda \neq 0$. If $\lambda = 0$, then the only solution of $\Delta f = \lambda f$ is the constant function.*

The case of harmonic functions ($\lambda = 0$) was already covered in [**2**] but we added it for the sake of completeness.

1. Preliminaries

For the convenience of the reader, in this section, we collect some basic formulas which will be used later on.

Let $M = M^n$ be an n-dimensional complete Riemannian manifold. Then, for any $f \in C^\infty(M)$ the Bochner-Lichnerowicz formula [**1**, Propisition 4.15] states that

$$\frac{1}{2}\Delta\left(|\nabla f|^2\right) = |\mathrm{Hess}\, f|^2 + \langle \nabla f, \nabla(\Delta f)\rangle + \mathrm{Ric}\,(\nabla f, \nabla f).$$

For a linear map $A : T_pM \to T_pM$ we define $|A|^2$ by

$$|A|^2 = \mathrm{tr}\, A^T A = \sum_{i=1}^{n} |AE_i|^2 = \sum_{i,j=1}^{n} \langle AE_i, E_j\rangle^2, \tag{1.1}$$

for any orthonormal system $E_1, \dots, E_n$.

The following elementary estimate will play an important role.

Proposition 1.1. *Let $U \in T_pM$ be any unit vector and set $\mathrm{tr}\, A = \lambda$. Then we have*

$$|A|^2 \geqslant \langle AU, U\rangle^2 + \frac{1}{n-1}\left(\langle AU, U\rangle - \lambda\right)^2.$$

Equality occurs if and only if U is an eigenvector and $U^\perp$ is an eigenspace.

Proof: Let $U = E_1, E_2, \dots, E_n$ be an orthonormal system. Then, we have

$$\mathrm{tr}\,(A) = \sum_{i=1}^{n}\langle AE_i, E_i\rangle = \lambda.$$

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By the Cauchy-Schwarz inequality we see that

$$\big(\langle AE_1, E_1\rangle - \lambda\big)^2 \leqslant (n-1)\sum_{i=2}^{n}\langle AE_i, E_i\rangle^2.$$

Adding $(n-1)\langle AE_1, E_1\rangle^2$ to both sides we get that

$$(n-1)\langle AE_1, E_1\rangle^2 + \big(\langle AE_1, E_1\rangle - \lambda\big)^2 \leqslant (n-1)\sum_{i=1}^{n}\langle AE_i, E_i\rangle^2 \leqslant (n-1)\,|A|^2.$$

From this the proposition follows easily.

The case of equality is simple as well. From the last inequality we conclude that $E_1, \dots, E_n$ must all be eigenvectors and the equality case of Cauchy- Schwarz implies that $E_2, \dots, E_n$ must have the same eigenvalues. This concludes the proof. □

We also need a formula for $|\mathrm{Hess}\log f|^2$. From (1.1) we have

$$|\mathrm{Hess}\log f|^2 = \sum_{i,j=1}^{n}\left\langle \nabla_{E_i}\frac{\nabla f}{f}, E_j\right\rangle^2 = \sum_{i,j=1}^{n}\left(\frac{1}{f}\langle \nabla_{E_i}\nabla f, E_j\rangle - \frac{1}{f^2}\langle E_i f\nabla f, E_j\rangle\right)^2$$
$$= \sum_{i,j=1}^{n}\frac{1}{f^2}\langle \nabla_{E_i}\nabla f, E_j\rangle^2 + \frac{1}{f^4}\langle E_i f\nabla f, E_j\rangle^2 - \frac{2}{f^3}\langle \nabla_{E_i f E_i}\nabla f, E_j f E_j\rangle.$$

Hence we have

$$|\mathrm{Hess}\log f|^2 = \frac{|\mathrm{Hess}\, f|^2}{f^2} + \left|\frac{\nabla f}{f}\right|^4 - 2\frac{\mathrm{Hess}\, f}{f}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right). \tag{1.2}$$

We now establish the crucial inequality about eigenfunctions.

Proposition 1.2. *Let M^n be an n-dimensional complete Riemannian manifold and let f be a positive eigenfunction with $\Delta f = \lambda f$. Then*

$$\frac{n-1}{2}\Delta\left|\frac{\nabla f}{f}\right|^2 \geqslant \left(\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\left(\frac{1}{2}|\nabla f|^2\right)}{|\nabla f|\, f}\right\rangle - \lambda\right)^2 + (n-1)\,\mathrm{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right)$$
$$+ 2(n-1)\left|\frac{\nabla f}{f}\right|^2\left(\left|\frac{\nabla f}{f}\right|^2 - \left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\left(\frac{1}{2}|\nabla f|^2\right)}{|\nabla f|\, f}\right\rangle\right).$$

If at some point $p \in M^n$ equality occurs then at that point ∇f is an eigenvector and $\nabla f^{\perp}$ is an eigenspace of Hess f.

Proof: First we apply the Bochner-Lichnerowicz formula to $\log f$.

$$\frac{1}{2}\Delta\big(|\nabla \log f|^2\big) = |\mathrm{Hess}\log f|^2 + \big\langle \nabla\log f, \nabla(\Delta\log f)\big\rangle + \mathrm{Ric}\,(\nabla\log f, \nabla\log f).$$

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Taking into account (1.2) and the following simple identities

$$\Delta f = \lambda f, \qquad \nabla \log f = \frac{\nabla f}{f}, \qquad \Delta \log f = \frac{\Delta f}{f} - \left|\frac{\nabla f}{f}\right|^2,$$

we get

$$\begin{aligned}
\frac{1}{2}\Delta\left|\frac{\nabla f}{f}\right|^2 &= \frac{|\mathrm{Hess}\, f|^2}{f^2} + \left|\frac{\nabla f}{f}\right|^4 - 2\frac{\mathrm{Hess}\, f}{f}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) \\
&\quad - 2\left\langle \frac{\nabla f}{f}, \nabla\Big(\frac{\frac{1}{2}|\nabla f|^2}{f^2}\Big)\right\rangle + \mathrm{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) \\
&= \frac{|\mathrm{Hess}\, f|^2}{f^2} + 3\left|\frac{\nabla f}{f}\right|^4 - 2\frac{\mathrm{Hess}\, f}{f}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) \\
&\quad - 2\left\langle \frac{\nabla f}{f}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{f^2}\right\rangle + \mathrm{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right).
\end{aligned}$$

A simple computation shows that

$$\frac{\mathrm{Hess}\, f}{f}\left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right) = \left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle$$

and

$$\frac{\mathrm{Hess}\, f}{f}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) = \left|\frac{\nabla f}{f}\right|^2\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle.$$

Applying now Proposition 1.1 to $(\mathrm{Hess}\, f)/f$ we have

$$\begin{aligned}
\frac{1}{2}\Delta\left|\frac{\nabla f}{f}\right|^2 \geqslant{} & \left(\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle^2 + \left|\frac{\nabla f}{f}\right|^4 - 2\left\langle \frac{(\nabla f)|\nabla f|}{f^2}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle\right) \\
& + \frac{1}{n-1}\left(\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle - \lambda\right)^2 + \mathrm{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) \\
& + 2\left|\frac{\nabla f}{f}\right|^2\left(\left|\frac{\nabla f}{f}\right|^2 - \left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle\right).
\end{aligned}$$

It is easy to see that on the right hand side the first term in parenthesis is always non-negative, therefore we have

$$\begin{aligned}
\frac{1}{2}\Delta\left|\frac{\nabla f}{f}\right|^2 \geqslant{} & \frac{1}{n-1}\left(\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle - \lambda\right)^2 + \mathrm{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right) \\
& + 2\left|\frac{\nabla f}{f}\right|^2\left(\left|\frac{\nabla f}{f}\right|^2 - \left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla\big(\frac{1}{2}|\nabla f|^2\big)}{|\nabla f|\, f}\right\rangle\right).
\end{aligned}$$

This implies the proposition. The case of equality follows easily from the equality case of Proposition 1.1. ∎

2. Proof of the Theorem.

Let $M = M^n$ be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $-(n-1)K^2$ and let f be a positive eigenfunction, that is, $\Delta f = \lambda f$ for some λ. By [2, Theorem 3′] we know that $|\nabla f|/f$ is bounded. Set

$$\alpha = \sup \frac{|\nabla f|}{f}.$$

For any $0 < \varepsilon < 1$ there is a point $q \in M$ such that

$$\frac{|\nabla f|}{f}(q) > \alpha(1-\varepsilon).$$

For some $\delta > 0$ (which will be choosen later) let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth, decreasing function such that $h(t) = 1$ for $0 \leqslant t < 1$, $\lim_{t\to\infty} h(t) = 1-\varepsilon$ and $|h'|, |h''| < \delta$. Clearly such a function must exist.

Let $g : M \to \mathbb{R}^+$ be defined by $g(p) = h(d_q(p))$, where d_q is the distance function from q. From [2, Lemma 1] we know that $\Delta d_g < C$ for $d_g > 1$, where C is some constant depending only on the dimension and the lower bound on the Ricci curvature. Taking into account that $\Delta g = h'\Delta d_q + h''\,|\nabla d_q|^2$, for a small enough δ we get that

$$\frac{|\nabla g|}{g} < \varepsilon, \qquad \frac{\Delta g}{g} > -\varepsilon. \tag{2.1}$$

Consider now the function $g\left|\nabla f/f\right|^2$. This must assume its maximum at some point q'. Hence, at the point q', we have

$$\nabla\left(g\left|\frac{\nabla f}{f}\right|^2\right) = 0, \quad \Delta\left(g\left|\frac{\nabla f}{f}\right|^2\right) \leqslant 0, \quad \text{and} \quad \left|\frac{\nabla f}{f}\right|^2 \geqslant \alpha^2(1-\varepsilon)^2. \tag{2.2}$$

From the first equality we get

$$\nabla\left(\left|\frac{\nabla f}{f}\right|^2\right) = -\frac{\nabla g}{g}\left|\frac{\nabla f}{f}\right|^2. \tag{2.3}$$

Applying this, the first inequality in (2.2) yields

$$\Delta\left(\left|\frac{\nabla f}{f}\right|^2\right) \leqslant \left(2\left|\frac{\nabla g}{g}\right|^2 - \frac{\Delta g}{g}\right)\left|\frac{\nabla f}{f}\right|^2.$$

Combining (2.1) and Proposition 1.2 we get that

$$\frac{n-1}{2}(2\varepsilon^2+\varepsilon)\left|\frac{\nabla f}{f}\right|^2 \geqslant \left(\left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla(\frac{1}{2}|\nabla f|^2)}{f\,|\nabla f|}\right\rangle - \lambda\right)^2 + (n-1)\operatorname{Ric}\left(\frac{\nabla f}{f}, \frac{\nabla f}{f}\right)$$
$$+ 2(n-1)\left|\frac{\nabla f}{f}\right|^2\left(\left|\frac{\nabla f}{f}\right|^2 - \left\langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla(\frac{1}{2}|\nabla f|^2)}{|\nabla f|\,f}\right\rangle\right),$$

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at the point q'. From (2.3) we also have

$$\frac{\nabla\left(\frac{1}{2}|\nabla f|^2\right)}{f\,|\nabla f|} = \frac{\nabla f}{f}\frac{|\nabla f|}{f} - \frac{\nabla g}{g}\left|\frac{\nabla f}{f}\right|^2.$$

Substituting this into the above inequality and using the fact that the Ricci curvature is bounded below by $-(n-1)K^2$, we have (at the point q')

$$\left((1-\varepsilon_1)\left|\frac{\nabla f}{f}\right|^2 - \lambda\right)^2 \leqslant (n-1)^2K^2\left|\frac{\nabla f}{f}\right|^2 + \varepsilon_2^2\left|\frac{\nabla f}{f}\right|^2, \tag{2.4}$$

where $\varepsilon_1 = \langle\nabla f, \nabla g\rangle/(|\nabla f|\,g)$ and $\varepsilon_2 = \sqrt{((n-1)/2)(2\varepsilon^2+\varepsilon) + 2(n-1)\varepsilon\alpha^2}$. It is clear from (2.1) that both ε_1 and ε_2 approach 0 when ε does.

Taking square roots, we have

$$(1-\varepsilon_1)\left|\frac{\nabla f}{f}\right|^2 - \lambda \leqslant \left((n-1)K + \varepsilon_2\right)\frac{|\nabla f|}{f}.$$

This implies that at the point q'

$$\frac{|\nabla f|}{f} \geqslant \frac{(n-1)K + \varepsilon_2 - \sqrt{\left((n-1)K+\varepsilon_2\right)^2 + 4\lambda}}{2(1-\varepsilon_1)}$$

and, by the last inequality of (2.2),

$$\alpha(1-\varepsilon) \leqslant \frac{|\nabla f|}{f} \leqslant \frac{(n-1)K + \varepsilon_2 + \sqrt{\left((n-1)K+\varepsilon_2\right)^2 + 4\lambda}}{2(1-\varepsilon_1)}.$$

Now, the inequalities in the theorem follow by letting ε go to 0. Let us remark that the same argument also shows that $\lambda \geqslant -(1/4)(n-1)^2K^2$.

If, at some point $q \in M$, $|\nabla f|/f = \sup|\nabla f|/f = \alpha$, then at this point we have

$$\nabla\left(\left|\frac{\nabla f}{f}\right|^2\right) = 0 \quad \text{and} \quad \Delta\left(\left|\frac{\nabla f}{f}\right|^2\right) \leqslant 0.$$

From the first equality we get

$$\frac{\nabla\left(\frac{1}{2}|\nabla f|^2\right)}{f\,|\nabla f|} = \frac{\nabla f}{f}\frac{|\nabla f|}{f}.$$

Combining these with Proposition 1.2, we have (at the point q)

$$\left(\alpha^2-\lambda\right)^2 + \alpha^2(n-1)\operatorname{Ric}\left(\frac{\nabla f}{|\nabla f|}, \frac{\nabla f}{|\nabla f|}\right) \leqslant 0.$$

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Now, if $\alpha = (1/2)\left((n-1)K + \sqrt{((n-1)K)^2 + 4\lambda}\right)$ or $\alpha = (1/2)\left((n-1)K - \sqrt{((n-1)K)^2 + 4\lambda}\right)$ with $\lambda < 0$, then a simple computation shows that the inequality above can hold only if, at the point q, $\mathrm{Ric}\left((\nabla f)/|\nabla f|, (\nabla f)/|\nabla f|\right) = -(n-1)K^2$. In this case, the inequality becomes an equality and the Theorem follows from the equality case of Proposition 1.2.

PROOF OF THE COROLLARY: Up to (2.4) the argument is the same as in the proof of the Theorem. But if the Ricci curvature is bounded below by $C > 0$, then the inequality (2.4) will be the following. At the point q' we have

$$\left((1-\varepsilon_1)\left|\frac{\nabla f}{f}\right|^2 - \lambda\right)^2 \leqslant -C\left|\frac{\nabla f}{f}\right|^2 + \varepsilon_2^2\left|\frac{\nabla f}{f}\right|^2. \tag{2.4'}$$

In case $\lambda \neq 0$, for a small enough ε it is clearly impossible. If $\lambda = 0$, then, by letting ε go to 0, we conclude that $\sup|\nabla f|/f = 0$. This completes the proof of the Corollary. ∎

REFERENCES

[1] S. Gallot, D. Hulin and J. Lafontaine, *Riemannian geometry* (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

[2] S.T. Yau, 'Harmonic functions on complete Riemannian manifolds', *Comm. Pure Appl. Math.* **28** (1975), 201-228.

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