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A NOTE ON THE DIRICHLET CONDITION FOR SECOND-ORDER DIFFERENTIAL EXPRESSIONS

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1. Let M denote the formally symmetric, second-order differential expression given by, for suitably differentiable complex-valued functions f,

$$M[f] = -(pf')' + qf \quad \text{on } [a, b) \quad ('\equiv d/dx). \tag{1.1}$$

The coefficients p and q are real-valued, Lebesgue measurable on the half-closed, half-open interval $[a, b)$ of the real line, with $-\infty < a < b \leqq \infty$, and satisfy the basic conditions:

(1.2) (i) $p(x) > 0$ (almost all $x \in [a, b)$) and p^{-1} is locally Lebesgue integrable on $[a, b)$, and
(ii) q is locally Lebesgue integrable on $[a, b)$.

A property is said to be 'local' on $[a, b)$ if it is satisfied on all compact sub-intervals of $[a, b)$. $L(a, b)$ and $L^2(a, b)$ denote the classical Lebesgue, complex integration spaces.

Consider the differential equation

$$M[y] = 0 \quad \text{on } [a, b]. \tag{1.3}$$

The function y is said to be a solution of (1.3) on $[a, b)$ if both y and py' are locally absolutely continuous on $[a, b)$ and

$$M[y](x) = -(p(x)y'(x))' + q(x)y(x) = 0 \text{ (almost all } x \in [a, b)). \tag{1.4}$$

With the basic conditions (1.2) satisfied the differential expression M is *regular* at all points of $[a, b)$, i.e. if $\xi \in [a, b)$ then the initial value problem

$$y(\xi) = \alpha \quad (py')(\xi) = \beta \quad M[y] = 0 \quad \text{on } [a, b) \tag{1.5}$$

can be solved for arbitrary complex numbers α, β; for this result see the existence theorem in [**8**, Section 16.1].

M is said to be *singular* at the open end-point b if either $b = \infty$, or if $b < \infty$ then the initial value problem (1.5) cannot be solved at b for arbitrary α and β. We note that if $b < \infty$ and the conditions (1.2) hold then M is singular at b if and only if

$$\text{either } p^{-1} \notin L(a, b), \text{ or } q \notin L(a, b) \text{ or both;} \tag{1.6}$$

this result follows from an examination of the theorem in [**8**, Section 16.1].

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If M is singular at b then M is classified as either *limit-point* (LP) or *limit-circle* (LC) at b; for this now standard terminology see [**8**, Section 17.5]. If M is LP (LC) at b then the differential equation (1.3) has at least one solution (all solutions) not in (in) the space $L^2(a, b)$.

Let the linear manifold $\Delta = \Delta(p, q)$ of $L^2(a, b)$ (Δ depends on the coefficients p and q) be defined by: $f \in \Delta$ if (i) $f \in L^2(a, b)$, (ii) f and pf' are both locally absolutely continuous on $[a, b)$, and (iii) $M[f] \in L^2(a, b)$.

When $f, g \in \Delta$ it is known, from Green's formula, that the limit

$$(1.7) \qquad \lim_{b-} p(fg' - f'g) = \lim_{x \to b-} p(x)(f(x)g'(x) - f'(x)g(x))$$

exists and is finite. A necessary and sufficient condition for M to be LP at b is that the limit (1.7) should be zero for all $f, g \in \Delta$; for this result see [**2**] or [**8**, Section 17.4].

M is said to be a *strong limit-point* (SLP) at b if

$$(1.8) \qquad \lim_{b-} pfg' = 0 \ (f, g \in \Delta).$$

For this definition see [**3**] but, in particular, [**6**, Section 1]. Clearly SLP at b implies LP at b, but it is known that the converse result is false; for these results see [**6**, Sections 2 and 8].

M is said to have the *Dirichlet* (D) *property* at b if

$$(1.9) \qquad p^{1/2}f' \text{ and } |q|^{1/2}f \in L^2(a, b) \quad (f \in \Delta),$$

and the *conditional Dirichlet* (CD) *property* at b if

$$(1.10) \quad p^{1/2}f' \in L^2(a, b) \text{ and } \lim_{x \to b-} \int_a^x qfg \text{ exists and is finite,} \quad (f, g \in \Delta).$$

For these definitions see [**5**, Section 1] and [**4**, Sections 1 and 2]; in particular [**4**] discusses earlier work in this field. Clearly D at b implies CD at b but the converse is known to be false; for this result see the remarks in [**4**, Sections 1 and 2] and [**1**, Sections 8, 9 and 10].

A general survey of Dirichlet type results at both finite and infinite singularities is given in [**7**].

It is known that when $b = \infty$ it is possible for M to be SLP at ∞ but not D or even CD at ∞. An example to illustrate this phenomena is given in [**6**, Sections 3 and 4].

This note concerns the problem of the relationship between the LP, SLP classification of M at b, and the D, CD property of M at b. In an addendum to [**7**] it is shown that if M is D at b, with $b \leqq \infty$, then M is SLP at b. Here we give a more direct proof of this result and additionally prove that if M is CD at ∞ then M is SLP at ∞.

The results are contained in the following theorems.

THEOREM 1. *Let the differential expression M be defined on the interval $[a, b)$ by* (1.1); *let the real-valued coefficients p and q satisfy the basic conditions* (1.2);

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let the definitions of regular and singular points, LP, SLP, D and CD of M at b hold, as given above. Then

(i) *if* $b = \infty$ *then M is CD at* ∞ *implies M is SLP at* ∞, *and*

(ii) *if* $b < \infty$ *and M is singular at b then M is D at b implies M is SLP at b.*

Proof. This is given in Sections 3 and 4 below.

Remarks 1. Note that when $b = \infty$, i.e. M is defined on the half-line $[a, \infty)$, we have the following chain of (strict) implications

$$(1.11) \quad D \Rightarrow CD \Rightarrow SLP \Rightarrow LP.$$

From the examples referred to above it follows that all these implications are false, in general, if taken in the opposite direction.

2. Note that when $b < \infty$ it is necessary to stipulate that M is singular at b, since if the conditions (1.2) hold and M is regular at b it may be shown that M is D at b and this case has to be eliminated.

3. Two questions remain unanswered:

(i) Do examples exist to show that *if* $b < \infty$ then M can be CD but not D at b; also SLP but not D or CD at b?

(ii) *If* $b < \infty$ and if M is CD at b is it the case that M is SLP at b? It seems unlikely that there is an affirmative answer.

THEOREM 2. *Let all the conditions and definitions of Theorem* 1 *hold on the interval* $[a, \infty)$. *Then*

(i) *if M is SLP at* ∞ *then*

$$(1.12) \quad \lim_{x \to \infty} \int_a^x \{p|f'|^2 + q|f|^2\} \text{ exists and is finite for all } f \in \Delta.$$

If $p^{-1} \notin L(a, \infty)$ *and the limit condition* (1.12) *above is satisfied then M is SLP at* ∞.

(ii) *M is CD at* ∞ *if and only if*

$$\lim_{x \to \infty} \int_a^x q|f|^2 \text{ exists and is finite for all } f \in \Delta.$$

(iii) *M is D at* ∞ *if and only if*

$$\lim_{x \to \infty} \int_a^x |q||f|^2 < \infty, \text{ i.e. } |q|^{1/2} f \in L^2(a, \infty), \text{ for all } f \in \Delta.$$

Proof. This is given in Section 5 below.

Remarks 1. It is not clear, but it seems unlikely, that corresponding results hold at a finite singularity for the differential expression M.

2. In the case of a singular point at ∞ results (ii) and (iii) give necessary and sufficient conditions on the elements of the linear manifold Δ in order to classify M as CD or D at ∞. It is not clear if a similar condition always exists for M to be SLP at ∞ since in the sufficiency part of (i) the additional condition

$p^{-1} \notin L(a, \infty)$ is required; it is an open question as to whether or not the limit condition (1.12) is sufficient for M to be SLP at ∞ if $p^{-1} \in L(a, \infty)$.

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2. We commence the proof of the above theorems by noting that it is sufficient throughout to argue only with real-valued elements $f, g \in \Delta$, since otherwise we work separately with the real and imaginary parts of f and g. This is a consequence of the assumption that the coefficients p and q are real-valued on $[a, b)$. We denote by Δ_R the set of all real-valued elements of Δ and work only with Δ_R in the proof of both Theorems 1 and 2.

It is helpful to begin with the following lemma which applies to both finite and infinite singular points.

LEMMA. *Suppose* $b \leqq \infty$ *and that all the conditions of Theorem* 1 *are satisfied; suppose that for some pair* $f, g \in \Delta_R$

(i) $\lim_{b-} pfg'$ *exists and is finite, and*

(ii) *there is a sequence* $\{b_n : n = 1, 2, 3, \ldots\}$ *for which* $a < b_n < b_{b+1} < b$ $(n = 1, 2, 3, \ldots)$ *and* $\lim b_n = b$ *such that* $\lim_{n\to\infty} f(b_n) = 0$ *or* $+\infty$ *or* $-\infty$. *Then* $\lim_{b-} pfg' = 0$.

Proof. Suppose the conclusion of the lemma is false; then from (i) $\lim_{b-} pfg' = \mu \neq 0$, i.e. $\lim_{b-} p|fg'| = |\mu| > 0$. Thus for some $b_0 \in [a, b)$ we have $|f(x)| > 0$ $(x \in [b_0, b))$ and so, without loss of generality, we may assume that $f(x) > 0$ $(x \in [b_0, b))$. Hence, with a possible change of b_0,

$$p(x)|g'(x)| \geqq \tfrac{1}{2}|\mu|/f(x) \quad (x \in [b_0, b)),$$

$$\text{i.e.}\quad p(x)|f'(x)g'(x)| \geqq \tfrac{1}{2}|\mu||f'(x)|/f(x) \quad (x \in [b_0, b)).$$

Integrating this last result gives

$$\int_{b_0}^{x} p|f'g'| \geqq \frac{1}{2}|\mu| \int_{b_0}^{x} |f'|/f \geqq \frac{1}{2}|\mu| \left| \int_{b_0}^{x} f'/f \right| = \frac{1}{2}|\mu| \left| \log\,(f(x)/f(b_0)) \right|$$

for all $x \in [b_0, b)$. The integral on the left of this inequality is bounded for all $x \in [b_0, b)$ for, from the assumptions in (i) and (ii) of Theorem 1, both $p^{1/2}f'$ and $p^{1/2}g' \in L^2(a, b)$, since M is either D or CD at b. The term on the extreme right of the inequality is however unbounded on the sequence $\{b_n; n = 1, 2, 3, \ldots\}$ in view of condition (ii) of the lemma. This gives a contradiction and so $\mu = 0$.

This completes the proof.

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3. In this section we give the proof of part (i) of Theorem 1. We note that $b = \infty$ and that M is defined on the half-line $[a, \infty)$.

We have the following identity

$$(3.1)\qquad \int_a^x \{pf'g' + qfg\} = (pfg')(x) - (pfg')(a) + \int_a^x fM[g]$$

valid for all $f, g \in \Delta_R$ and all $x \in [a, \infty)$. From the hypothesis in (i) of the theorem, i.e. M is CD at ∞, it follows from (1.10) that

$$(3.2)\qquad \lim_{x\to\infty} \int_a^x pf'g' \quad \text{and} \quad \lim_{x\to\infty} \int_a^x qfg$$

both exist and are finite. Also the integral on the right of (3.1) is convergent as $x \to \infty$ since both f and $M[g]$ are in $L^2(a, \infty)$. Thus

$$(3.3)\qquad \lim_{x\to\infty} p(x)f(x)g'(x) \text{ exists and is finite } (f, g \in \Delta_R).$$

Since $f \in L^2(a, \infty)$ it follows from known results that there is a monotonic increasing sequence $\{b_n;\, n = 1, 2, 3, \ldots\}$ such that

$$(3.4)\qquad \lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} f(b_n) = 0.$$

From (3.3) and (3.4) we see that conditions (i) and (ii), respectively, of the lemma of Section 2 are satisfied. Thus from that lemma

$$\lim_{\infty} pfg' = 0 \quad (f, g \in \Delta_R)$$

and M is SLP at ∞.

This completes the proof of part (i) of Theorem 1.

4. In this section we give the proof of part (ii) of Theorem 1. We note that $a < b < \infty$ and that M is singular and D at b.

Firstly suppose that additionally

(4.1) $\quad M$ is LP at b.

Then to show that M is SLP at b it is sufficient to prove (see also [**6**, Section 4])

$$(4.2)\qquad \lim_{b-} pff' = 0 \quad (f \in \Delta_R).$$

For suppose (4.2) holds; then given any pair $f, g \in \Delta_R$

$$0 = \lim_{b-} p(f+g)(f+g)' = \lim_{b-} (pff' + pfg' + pf'g + pgg') = \lim_{b-} p(fg' + f'g).$$

Now (see (4.1)) since M is LP at b it follows from (1.7) that

$$\lim_{b-} p(fg' - f'g) = 0$$

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and from these two results

$$\lim_{b-} pfg' = 0 \quad (f, g \in \Delta_R).$$

Thus (4.1) and (4.2) imply that M is SLP at b.

We now prove that (4.2) follows from (4.1) and the assumption that M is D at b.

We note that the identity (3.1), taken over $[a, b)$, and M is D at b imply that

(4.3) $\lim_{b-} pff'$ exists and is finite $(f \in \Delta_R)$.

Suppose now that (4.2) does not hold; then for some $f \in \Delta_R$ and some $\mu > 0$

(4.4) (a) $\lim_{b-} pff' = \mu > 0$ or (b) $\lim_{b-} pff' = -\mu > 0$.

If (a) of (4.4) holds then from (i) of (1.2) it follows that $ff' > 0$ near b and hence that f^2 is monotonic increasing near b, i.e. $\lim_{b-} f^2 = L$, say, where $0 < L \leqq \infty$.

If $L = \infty$ then an application of the lemma of Section 2 shows that (4.2) holds, i.e. $\mu = 0$, and this is a contradiction.

If $0 < L < \infty$ then from the assumption that M is D at b, which implies

$$\int_a^b |q| f^2 < \infty,$$

it follows that $q \in L(a, b)$. Also from (a) of (4.4) we have, for some $b_0 \in [a, b)$,

$$p(x)f(x)f'(x) \geqq \frac{1}{2}\mu \quad (x \in [b_0, b)), \text{ that is,}$$

$$f(x)^2 - f(b_0)^2 = 2\int_{b_0}^x ff' \geqq \mu \int_{b_0}^x p^{-1} \quad (x \in [b_0, b));$$

since $\lim_{b-} f^2 = L < \infty$ we now obtain $p^{-1} \in L(a, b)$. Thus both p^{-1} and $q \in L(a, b)$; however this implies from (1.6) that M is regular at b in contradiction to (4.1).

Thus (a) of (4.4) is impossible.

If (b) of (4.4) holds then $ff' < 0$ near b and this implies that $\lim_{b-} f^2 = L$ with $0 \leqq L < \infty$. As before both the cases $L = 0$ (using the lemma of Section 2) and $0 < L < \infty$ (using (4.1)) lead to contradictions.

Thus (4.4) is impossible and consequently (4.2) must hold. As we have seen, taken with (4.1), this implies that M is SLP at b.

Secondly suppose, and since M is singular at b this is the only alternative to (4.1), that additionally M is LC at b. With M in D at b we shall show that this case is impossible.

Let ϕ and ψ be any two linearly independent solutions of the differential equation (1.3) such that

(4.5) $p(x)(\phi(x)\psi'(x) - \phi'(x)\psi(x)) = 1 \quad (x \in [a, b)).$

(Note the left-hand side of (4.5) is always constant on $[a, b)$ for any two solutions of (1.3)). Both ϕ and ψ are in $L^2(a, b)$, since M is LC at b, and hence both are in Δ_R.

If Φ is any linearly independent solution of (1.3) then $\Phi \in \Delta_R$ and $\lim_{b-} p\Phi\Phi'$ exists and is finite; see (4.3). As in the proof of (4.2) the assumption that $\lim_{b-} p\Phi\Phi' \neq 0$ leads to a contradiction on repeating the analysis following (4.4). Now put $\Phi = \phi + \psi$ to obtain

$$0 = \lim_{b-} p\Phi\Phi' = \lim_{b-} p(\phi\phi' + \phi\psi' + \phi'\psi + \psi\psi') = \lim_{b-} p(\phi\psi' + \phi'\psi) \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\lim_{b-} p\phi\psi' = \frac{1}{2} \quad \text{and} \quad \lim_{b-} p\phi'\psi = -\frac{1}{2}. \tag{4.7}$$

This last result shows that ψ' must be of one sign in some neighbourhood of b and so $\lim_{b-} \psi = L$, say, where without loss of generality, we may assume that $0 \leqq L \leqq \infty$. If $L = 0$ or ∞ then an application of the lemma in Section 2 gives a contradiction to (4.7). If $0 < L < \infty$ then on repeating the analysis following (4.4) we find that M is regular at b in contradiction to the assumption that M is singular at b.

Thus it is impossible for M to be D and LC at b.

This completes the proof of part (ii) of Theorem 1, and so the theorem itself is now established.

5. In this section we give the proof of Theorem 2.

(i) Suppose that M is SLP at ∞; then it follows from a suitable application of the identity (3.1) that the limit condition (1.12) is satisfied for all $f \in \Delta$.

Conversely, suppose that the limit condition (1.12) is satisfied *and* that $p^{-1} \notin L(a, \infty)$. It follows from (1.12) and the identity (3.1) that

$$\lim_{\infty} pff' \text{ exists and is finite for all } f \in \Delta_R. \tag{5.1}$$

For any $f \in \Delta_R$ suppose that at $+\infty$ we have $\lim pff' = \mu \neq 0$. If $\mu > 0$ then, recalling (i) of (1.2), $ff' > 0$ in some neighbourhood of $+\infty$ and this is a contradiction on $f \in L^2(a, \infty)$. If $\mu < 0$ then for some $c \in (a, \infty)$

$$-f(x)f'(x) > \tfrac{1}{2}(-\mu)\{p(x)\}^{-1} \quad (x \in [c, \infty))$$

which gives, on integrating,

$$f(c)^2 - f(X)^2 \geqq (-\mu) \int_c^X p^{-1} \quad (X \in [c, \infty));$$

this implies that $p^{-1} \in L(a, \infty)$ and this is not the case. Thus

$$\lim_{\infty} pff' = 0 \ (f \in \Delta_R). \tag{5.2}$$

Assume now that M is LC at ∞; then on using (5.2) and repeating the argument in Section 4 from (4.5) onwards it follows that the two real-valued solutions ϕ and ψ of the differential equation (1.3) on $[a, \infty)$ satisfy

$$(5.3) \qquad \lim_{\infty} p\phi\psi' = \frac{1}{2} \quad \text{and} \quad \lim_{\infty} p\phi'\psi = -\frac{1}{2}$$

(compare with (4.7)). This result shows that all of ϕ, ψ, ϕ' and ψ' are of one sign in some neighbourhood (c, ∞) of $+\infty$; suppose, without loss of generality, that $\phi' < 0$ on (c, ∞); then ϕ is decreasing and so, since $\phi \in L^2(a, \infty)$, $\phi > 0$ on (c, ∞) and $\lim \phi = 0$ at $+\infty$. From the first part of (5.3) it follows that $\psi' > 0$ and from the second part that $\psi > 0$, i.e. $\psi\psi' > 0$ on (c, ∞), and this is a contradiction to $\psi \in L^2(a, \infty)$. Thus M is LP at ∞.

Returning again to Section 4 we now repeat the argument following (4.1) and (4.2) since both these conditions are now seen to hold but with $b = \infty$. We obtain

$$(5.4) \qquad \lim_{\infty} pfg' = 0 \quad (f, g \in \Delta_R)$$

and so M is SLP at ∞ as required.

It does not seem to be possible to avoid the condition $p^{-1} \notin L(a, \infty)$ in this argument but this is an open question. Note, however, that this condition is satisfied in the special case $p(x) = 1$ $(x \in [a, \infty))$.

(ii) Suppose now it is known only that

$$(5.5) \qquad \lim_{x \to a} \int_a^x qf^2 \quad \text{exists and is finite for all } f \in \Delta_R,$$

i.e. the integral is in general conditionally convergent only. We show that this condition implies that M is CD (and hence SLP) at ∞, i.e. that (1.10) holds.

From the identity (3.1) we obtain

$$\int_a^x pf'^2 = (pff')(x) - (pff')(a) - \int_a^x qf^2 + \int_a^x fM[f]$$

valid for all $x \in [a, \infty)$ and all $f \in \Delta_R$. Thus on using (5.5) it follows that if $p^{1/2}f' \notin L^2(a, \infty)$ then $\lim pff' = \infty$ at $+\infty$, $ff' > 0$ in some neighbourhood of $+\infty$ and this is inconsistent with $f \in L^2(a, \infty)$. Hence (5.5) implies that

$$(5.6) \qquad p^{1/2}f' \in L^2(a, \infty) \quad (f \in \Delta_R).$$

It now follows from (5.5), (5.6) and the identity (3.1) that (5.1) is satisfied. An application of the lemma of Section 2 then shows that (5.2) also is satisfied. Repeating the argument following (5.2) in part (i) above it follows that M is not LC at ∞, M is LP at ∞ and then M is SLP at ∞, i.e. (5.4) holds. Returning to the identity (3.1), and using (5.4) and (5.6) it now follows that

$$(5.7) \qquad \lim_{x \to \infty} \int_a^x qfg \quad \text{exists and is finite for all } f, g \in \Delta_R.$$

Taken together (5.6) and (5.7) show that M is CD at ∞ as required. Conversely, it is clear that (5.5) holds when M is CD at ∞.

(iii) If

$$(5.8) \qquad |q|^{1/2}f \in L^2(a, \infty) \quad (f \in \Delta_R)$$

then following the argument in part (ii) above it follows that (5.6) is satisfied and together this implies that M is D at ∞. Conversely (5.8) is satisfied directly from the definition of the D condition of M at ∞.

This completes the proof of Theorem 2.

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