

The Chowla conjecture and Landau–Siegel zeroes

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Abstract

Let $k \geq 2$ be an integer and let λ be the Liouville function. Given k non-negative distinct integers h_1, \dots, h_k , the Chowla conjecture claims that $\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) = o(x)$. An unconditional answer to this conjecture is yet to be found, and in this paper, we take a conditional approach. More precisely, we establish a non-trivial bound for the sums $\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k)$ under the existence of a Landau–Siegel zero for x in an interval that depends on the modulus of the character whose Dirichlet series corresponds to the Landau–Siegel zero. Our work constitutes an improvement over the previous related results of Germán and Kátai, Chinis and Tao and Teräväinen.

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1. Introduction

The Liouville function λ is the completely multiplicative function that is defined as $\lambda(n) = (-1)^{\Omega(n)}$ for all $n \in \mathbb{N}$, where $\Omega(n) = \sum_{p^\alpha \parallel n} \alpha$. The prime number theorem implies that

$$\sum_{n \leq x} \lambda(n) = o(x), \quad \text{as } x \rightarrow \infty,$$

which means that the sign of $\lambda(n)$ has to change frequently as n grows. Chowla expected a more general version of this asymptotic to hold, and in 1965 [2], he stated a conjecture which can be extended to the case of the Liouville function in the following way.

CONJECTURE 1.1. *For distinct fixed integers $h_1, h_2, \dots, h_k \geq 0$, we have that*

$$\sum_{n \leq x} \lambda(n + h_1) \lambda(n + h_2) \cdots \lambda(n + h_k) = o(x), \quad \text{as } x \rightarrow \infty.$$

Chowla’s conjecture remains open till this day. So far, there has only been partial progress on it. Let us now take a look at the the main results that this progress evolved through. In 1985, Harman, Pintz and Wolke [4] studied the binary case with $h_1 = 0$ and $h_2 = 1$ and proved that

$$-\frac{1}{3} + O\left(\frac{\log x}{x}\right) \leq \frac{1}{x} \sum_{n \leq x} \lambda(n) \lambda(n + 1) \leq 1 - O_\varepsilon\left(\frac{1}{(\log x)^{7+\varepsilon}}\right)$$

for $\varepsilon > 0$. About 30 years later, in 2016, Matomäki and Radziwiłł [11, Corollary 2] improved this by showing that for any integer $h \geq 1$, there exists $\delta(h) > 0$ such that

$$\frac{1}{x} \left| \sum_{n \leq x} \lambda(n)\lambda(n+h) \right| \leq 1 - \delta(h),$$

for all sufficiently large x . An averaged version of the conjecture was established by Matomäki, Radziwiłł and Tao [12] in 2015. The three of them proved [12, (1.3)] that for every integer $k \geq 2$ and $H \in [10, x]$, we have

$$\sum_{1 \leq h_2, \dots, h_k \leq H} \left| \sum_{n \leq x} \lambda(n)\lambda(n+h_2) \cdots \lambda(n+h_k) \right| \ll k \left(\frac{\log \log H}{\log H} + \frac{1}{(\log x)^{1/3000}} \right) H^k x.$$

Finally, a weaker version of Chowla’s conjecture, the so-called logarithmically averaged Chowla conjecture which claims that

$$\sum_{n \leq x} \frac{\lambda(n+h_1) \cdots \lambda(n+h_k)}{n} = o(\log x), \quad \text{as } x \rightarrow \infty,$$

has been accomplished in stages for odd k and $k = 2$ in the works of Tao [16], Tao and Teräväinen [17, 18], Helfgott and Radziwiłł [6], and Pilatte [13].

Despite the aforementioned partial advances towards Chowla’s conjecture, the full conjecture still seems to be out of reach. So, one may wish to approach it conditionally, and in this paper, we are doing this by assuming the existence of Landau–Siegel zeroes.

A *Landau–Siegel zero*, often just called a *Siegel zero*, is a real number β associated to a real primitive Dirichlet character χ modulo q such that $L(\beta, \chi) = 0$ and

$$\beta = 1 - \frac{1}{\eta \log q}$$

for some $\eta > 0$. The number η is called the *quality* of the Landau–Siegel zero β and from Siegel’s theorem we have the ineffective bound

$$\eta \ll_{\varepsilon} q^{\varepsilon}, \tag{1.1}$$

for any $\varepsilon > 0$. It is widely believed that Siegel zeroes do not exist, as their existence would come in contrast to the Generalised Riemann Hypothesis. However, the unlikely presence of Siegel zeroes has proven to have some interesting consequences. For example, in 1983, Heath-Brown [5] proved that if there exist infinitely many Siegel zeroes $\beta_j = 1 - 1/(\eta_j \log q_j)$ such that $\eta_j \rightarrow +\infty$, then the Twin Prime Conjecture must be true.

The first result of the literature on Chowla’s conjecture under Landau–Siegel zeroes comes from a work of Germán and Kátai from 2010. More precisely, in [3, Theorem 2], they proved that there exists an absolute constant $c > 0$ such that if χ is a primitive quadratic character modulo q for which $L(\cdot, \chi)$ has a Landau–Siegel zero $\beta = 1 - 1/(\eta \log q)$ with $\eta > \exp(\exp(30))$, then

$$\frac{1}{x} \left| \sum_{n \leq x} \lambda(n)\lambda(n+1) \right| \leq \frac{c}{\log \log \eta} + o(1),$$

for all $x \in [q^{10}, q^{(\log \log \eta)^3}]$. Under the same assumptions, Chinis [1, Theorem 1.2] in 2021 extended the work of Germán and Kátai to $k \geq 2$ distinct shifts and showed that

$$\frac{1}{x} \sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll_{k, h_1, \dots, h_k} \frac{1}{(\log \log \eta)^{1/2} (\log \eta)^{1/12}},$$

for all $x \in [q^{10}, q^{(\log \log \eta)^3}]$. A year later, in 2022, Tao and Teräväinen [19, Corollary 1.8 (ii)] improved the bound of Chinis as well as the q -range of x that his bound holds for. In particular, under the existence of a Siegel zero $\beta = 1 - 1/(\eta \log q)$ with $\eta \geq 10$, they showed that for a fixed $\varepsilon \in (0, 1)$, one has

$$\frac{1}{x} \sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll_{\varepsilon, k, h_1, \dots, h_k} \frac{1}{(\log \eta)^{1/10}},$$

for all x in $[q^{1/2+\varepsilon}, q^{\eta^{1/2}}]$. Until the publication of the present paper, the result of Tao and Teräväinen has provided the best bound on the sums $\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k)$ with the best q -range for x .

In this work, we improve their result in both directions. As we can see in Corollary 1.3 below, the following theorem, which is our main result, leads to stronger bounds on the k -point correlations $\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k)$ over a slightly wider q -range.

THEOREM 1.2. *Let $q \geq 2$ be a positive integer and let χ be a primitive quadratic character modulo q such that $L(\cdot, \chi)$ has a real zero $\beta = 1 - 1/(\eta \log q)$ with $\eta \geq 10$. We also fix an integer $k \geq 2$, distinct non-negative integers h_1, \dots, h_k , and $\varepsilon \in (0, 1/2)$. There exists a constant $c = c(\varepsilon, k) > 0$ such that for $x = q^V$ with $V \in [1/2 + \varepsilon, \eta]$, we have*

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll \frac{x^V}{\eta} + x \exp(-c\sqrt{V \log \eta}).$$

COROLLARY 1.3. *Let q, η, k and ε be as in the statement of Theorem 1.2. Let also c be the constant from the bound of Theorem 1.2 and consider distinct fixed integers $h_1, \dots, h_k \geq 0$.*

(i) *There exists a constant $c' = c'(\varepsilon, k) > 0$ such that*

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll x \exp(-c' \sqrt{\log \eta}),$$

for every $x \in [q^{1/2+\varepsilon}, q^{c^{-2} \log \eta}]$.

(ii) *For every $\delta \in (0, 1)$, we have that*

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll_{\delta} \frac{x}{\eta^{1-\delta}},$$

for all $x \in (q^{c^{-2} \log \eta}, q^{\eta^{\delta}}]$.

The implicit constants in Theorem 1.2 and 1.3 depend on ε, k and the shifts h_1, \dots, h_k , but we did not include them in the \ll notation because the parameters $\varepsilon, k, h_1, \dots, h_k$ are all

fixed in their statements. The same applies to the bounds in the proof of Theorem 1.2. The reason for this choice is to alleviate the notation for the reader.

We prove Theorem 1.2 in Section 5. A major input in its proof is Lemma 3.3 which is a technical version of the fundamental lemma of sieve theory. Tao and Teräväinen did not use the beta-sieve which basically gives rise to Lemma 3.3. They used a Selberg sieve and this technically complicated the matters in their paper [19]. This constitutes the main reason that we could improve their bound. However, it could be possible to somehow alter their arguments and obtain bounds comparable to ours with a Selberg sieve.

The use of Lemma 3.3 as a main component in the improvement of the previous results was inspired by the work of Matomäki and Merikoski in [10], where they assumed the existence of Siegel zeroes and refined a result of Tao and Teräväinen [19] on the binary “Hardy–Littlewood” type correlations $\sum_{n \leq x} \Lambda(n)\Lambda(n+h)$. As in our paper, Matomäki and Merikoski, who were the ones who proved [10, Lemma 3.3], strengthened the result of Tao and Teräväinen by applying Lemma 3.3 instead of a Selberg sieve that the latter used for the more general “Hardy–Littlewood–Chowla” type correlations

$$\sum_{n \leq x} \Lambda(n+h_1) \cdots \Lambda(n+h_k) \lambda(n+h'_1) \cdots \lambda(n+h'_k).$$

Notation

Let r be a positive integer. For $a_1, \dots, a_r \in \mathbb{N}$, we use the notation (a_1, \dots, a_r) for their greatest common divisor, and the notation $[a_1, \dots, a_r]$ for their least common multiple.

Throughout the text, we denote the largest and smallest prime factor of an integer $n > 1$ by $P^+(n)$ and $P^-(n)$, respectively. For $n = 1$, we conventionally have $P^+(1) = 0$ and $P^-(1) = +\infty$.

The capital Greek letters Φ and Ψ also appear in the paper and they correspond to the classical counting functions of the rough and smooth integers, respectively. More precisely, for $x \geq y \geq 2$, we have $\Phi(x, y) = \#\{n \leq x : P^-(n) > y\}$ and $\Psi(x, y) = \#\{n \leq x : P^+(n) \leq y\}$.

Given two arithmetic functions f and g , their Dirichlet convolution, denoted by $f * g$, is the arithmetic function which is defined as $(f * g)(n) = \sum_{ab=n} f(a)g(b)$ for all $n \in \mathbb{N}$.

For $m \in \mathbb{N}$, the symbol τ_m denotes the m -fold divisor function given as $\tau_m(n) = \sum_{d_1 \dots d_m=n} 1$ for all $n \in \mathbb{N}$. Finally, the lowercase Greek letters μ, τ and φ denote the Möbius, the divisor and the Euler functions, respectively.

2. The Core Ideas

The purpose of this section is to explain why the existence of Siegel zeroes is a useful hypothesis when approaching Chowla’s conjecture. Our explanation below also highlights the main steps in the proof of Theorem 1.2.

If $L(\cdot, \chi)$ has a zero close to 1 for some Dirichlet character χ of modulus q , then $L(1, \chi)$ will be small by continuity. Consequently, assuming that Landau–Siegel zeroes exist, we find a primitive quadratic character χ such that $L(\beta, \chi) = 0$ for some real number β close to 1. Then $L(1, \chi)^{-1}$ should be large. But,

$$L(1, \chi)^{-1} = \prod_p \left(1 - \frac{\chi(p)}{p} \right),$$

so the Euler product of the right-hand side has to be large, too. In conjunction with Mertens’ theorems, this suggests that $\chi(p) = -1 = \lambda(p)$ for many large primes. This intuitive conclusion is the basis of arguments that utilise the presence of Landau–Siegel zeroes.

Now, let $z > 1$ be some parameter. Following the work of Germán and Kátai [3], we define the completely multiplicative function λ_z which is fully determined by its prime values

$$\lambda_z(p) = \begin{cases} \lambda(p) & \text{if } p \leq z, \\ \chi(p) & \text{if } p > z. \end{cases} \tag{2.2}$$

According to the above discussion, in the presence of Siegel zeroes, the function λ_z seems to be a good approximation to the Liouville function λ for large z . Therefore, for a suitably chosen z , e.g. some small power of x , we expect

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \approx \sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k). \tag{2.3}$$

In fact, such an approximation is provided by Lemma 4.1 which constitutes the first step in the proof of Theorem 1.2.

Once the transition from the sums of λ to the sums of λ_z has been achieved with the introduction of an acceptable error, the study of the sums $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$ is tractable. Let us elaborate on why is that. Without loss of generality, we assume that h_1 is the smallest shift. Then the right-hand side of (2.3) can be written as

$$\sum_{\substack{a_1, \dots, a_k \leq x, \\ P^+(a_1 \cdots a_k) \leq z}} \lambda(a_1) \cdots \lambda(a_k) \quad \sum_{\substack{n_1 \leq (x+h_1)/a_1, \\ P^-(n_1 \cdots n_k) > z, \\ a_1 n_1 = a_i n_i + h_1 - h_i \\ \text{for all } i \in \{1, \dots, k\}}} \chi(n_1) \cdots \chi(n_k). \tag{2.4}$$

The reason that makes the sums $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$ easier to deal with lies on the innermost sum of (2.4), where we now have a primitive quadratic character χ instead of the Liouville function λ . This character χ possesses two crucial features that the Liouville function does not. The first one is its periodicity and the second one is the Weil bound that it satisfies (see Lemma 3.5 below).

Let us now explain how the Weil bound is useful for the estimation of the expression in (2.4). First, we can approximate the indicator function $\mathbb{1}_{P^-(\cdot) > z}$ by $1 * w$ for some suitable sieve weight. This removes the condition $P^-(n_1 \cdots n_k) > z$ from the innermost sum of (2.4) at the cost of inserting the terms $(1 * w)(n_1 \cdots n_k)$. After opening these convolutions, we can change the order of summation so that the innermost sums are character sums over polynomial values. So, provided that the contribution of the errors in the approximation $\mathbb{1}_{P^-(\cdot) > z} \approx 1 * w$ can be handled, the estimation of (2.4) ends up being a simple application of the Weil bound to the new innermost character sums. The arguments of this explanation are rigorously executed in Section 5.

To summarise, the existence of Siegel zeroes is helpful because it puts our focus on the sums $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$, which one can estimate by combining a sifting argument with the Weil bounds of χ .

3. Auxiliary results

In this section, we state and prove some preparatory results that are needed later for the proof of Theorem 1.2. We start with a lemma which will be used to bound the difference between the sums of the Liouville function λ and those of the functions λ_z .

LEMMA 3.1. *Let χ be a primitive quadratic character modulo $q \geq 2$. Assume that $L(\cdot, \chi)$ has a real zero β such that $\beta = 1 - 1/(\eta \log q)$ for some $\eta \geq 10$. Let $z = q^v$ for some $v > 0$. Then for any $x > z$, we have*

$$\sum_{z < p \leq x} \frac{1 + \chi(p)}{p} \ll \begin{cases} \left(\frac{1}{v^2 \eta^{v/2}} + \frac{1}{z}\right) \left(\frac{\log x}{\log z}\right)^3 & \text{for } v \in (0, 2), \\ \frac{\log x}{\eta \log q} & \text{for } v \geq 2. \end{cases}$$

Proof. The first branch of the estimate follows from [10, Lemma 2.2]. Indeed, since $(1 * \chi)(n) \geq 0$ for all $n \in \mathbb{N}$, for $v \in (0, 2)$, [10, Lemma 2.2] yields the following (note that in [10], the authors adopted the notation $\lambda = 1 * \chi$, whereas they denote the Liouville function by $\lambda_{\text{Liouville}}$):

$$\begin{aligned} \sum_{z < p \leq x} \frac{1 + \chi(p)}{p} &\leq \sum_{\substack{z < n \leq x \\ P^-(n) > z}} \frac{(1 * \chi)(n)}{n} \ll \left(\frac{1}{v^2 \eta^{v/2}} + \frac{v \log x}{\eta \log z} + \frac{1}{z}\right) \left(\frac{\log x}{\log z}\right)^2 \\ &\ll \left(\frac{1}{v^2 \eta^{v/2}} + \frac{v}{\eta} + \frac{1}{z}\right) \left(\frac{\log x}{\log z}\right)^3 \ll \left(\frac{1}{v^2 \eta^{v/2}} + \frac{1}{z}\right) \left(\frac{\log x}{\log z}\right)^3. \end{aligned}$$

The second branch comes from [19, Proposition 3.5].

The next lemma, which is found as exercise 6(b) in [20, end of Section III.5], allows us to restrict our attention to the summands of $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$ for which the z -smooth parts $\prod_{p \leq z, p^\alpha \parallel n + h_i} p^\alpha$ of $n + h_i$ are relatively small for all $i \in \{1, \dots, k\}$.

LEMMA 3.2. *Consider the real numbers $u \geq 1$ and $x, z > 1$. If $x \geq z^u$, then*

$$\#\{n \leq x : \prod_{p \leq z, p^\alpha \parallel n} p^\alpha > z^u\} \ll x e^{-u/2}.$$

Proof. We observe that

$$\#\{n \leq x : \prod_{p \leq z, p^\alpha \parallel n} p^\alpha > z^u\} = \sum_{\substack{ab \leq x, z^u < a \\ P^+(a) \leq z < P^-(b)}} 1. \tag{3.5}$$

When $b = 1$, the remaining sum over a is bounded by $\Psi(x, z)$. When $b > 1$, the condition $P^-(b) > z$ implies that $b > z$, and so $a \leq x/b < x/z$, in this case. For each such a in (3.5), there are at most $\Phi(x/a, z)$ integers b . Consequently, the relation (3.5) gives

$$\#\{n \leq x : \prod_{p \leq z, p^\alpha \parallel n} p^\alpha > z^u\} \leq \Psi(x, z) + \sum_{\substack{z^u < a < x/z \\ P^+(a) \leq z}} \Phi\left(\frac{x}{a}, z\right).$$

For $t \geq z$, we have the bound $\Psi(t, z) \ll xe^{-\log t/(2 \log z)}$ [20, Theorem 1 in Section III.5.1]. Since $x/a > z$, we also have the standard estimate $\Phi(x/a, z) \ll x/(a \log z)$, which readily follows e.g. from [9, Theorem 14.2] with $f = \mathbb{1}_{P^-(\cdot) > z}$. Therefore,

$$\#\{n \leq x : \prod_{p \leq z, p^\alpha \parallel n} p^\alpha > z^u\} \ll xe^{-u/2} + \frac{x}{\log z} \sum_{\substack{z^u < a < x \\ P^+(a) \leq z}} \frac{1}{a}. \tag{3.6}$$

Using partial summation and the aforementioned bound on $\Psi(t, z)$ for $t \geq z$, it follows that the sum of the right-hand side of (3.6) is $\ll \log z \cdot e^{-u/2}$. This concludes the proof of the lemma.

The version of the fundamental lemma of sieve theory that follows provides an approximation of the indicator function $\mathbb{1}_{P^-(\cdot) > z}$ in terms of appropriate sieve weights. It is useful when we deal with sums of the form $\sum_{P^-(n) > z} \alpha_n$ where α_n alternate signs.

LEMMA 3.3. *There exists a constant $\beta \geq 2$ such that the following holds for any $u \geq \beta$.*

Let $z \geq 1$, and define

$$z_r := z^{((\beta-1)/\beta)^r}$$

for $r \in \mathbb{N}$. There exists an arithmetic function w such that:

- (i) $|w(n)| \leq 1$ for all $n \in \mathbb{N}$;
- (ii) $\text{supp}(w) \subseteq \{d \in \mathbb{N} : d \mid \prod_{p \leq z} p \text{ and } d \leq z^u\}$; and
- (iii) $\mathbb{1}_{P^-(n) > z} = (1 * w)(n) + O\left(\tau(n)^2 \sum_{r \geq u-\beta} \mathbb{1}_{P^-(n) > z_r} 2^{-r}\right)$ for all $n \in \mathbb{N}$.

Proof. See [10, proof of Lemma 3.2(i)] with $A = 1$, $\beta = \beta_0$ and a u which here plays the role of the product $u\theta$ for some fixed $\theta \in (0, 1/3)$.

In the proof of Theorem 1.2, we also make use of the following extension of Shiu’s theorem [15, Theorem 1] which is due to Henriot [7, 8].

LEMMA 3.4. *Let k be a positive integer and let $Q_1, \dots, Q_k \in \mathbb{Z}[X]$ be k pairwise coprime irreducible polynomials. Set $Q = Q_1 \cdots Q_k$, and denote the degree and determinant of Q by g and D , respectively. For $n \in \mathbb{N}$ and $j \in \{1, \dots, k\}$, we let $\rho_j(n)$ (respectively $\rho(n)$) denote the number of zeroes of Q_j (respectively Q) modulo n . Assume that Q has no fixed prime divisor, and consider real numbers $A \geq 1$, $B \geq 1$, $\delta \in (0, 1)$, and $\varepsilon \in (0, 1/(100g(g + 1/\delta)))$. Suppose that $F: \mathbb{N}^k \rightarrow \mathbb{R}$ is a non-negative function such that*

- (i) $F(m_1 n_1, \dots, m_k n_k) = F(m_1, \dots, m_k) F(n_1, \dots, n_k)$ whenever $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$, and
- (ii) $F(n_1, \dots, n_k) \leq \min\{A^{\Omega(n_1 \cdots n_k)}, B(n_1 \cdots n_k)^\varepsilon\}$ for all $(n_1, \dots, n_k) \in \mathbb{N}^k$.

There exists a constant $c_0 > 0$, depending at most on g, A, B and δ , such that

$$\begin{aligned} & \sum_{n \leq x} F(|Q_1(n)|, \dots, |Q_k(n)|) \\ & \ll x \prod_{p|D} \left(1 + \sum_{\substack{0 \leq \mu_j \leq \deg Q_j \\ \text{for all } j \in \{1, \dots, k\}, \\ (\mu_1, \dots, \mu_k) \neq (0, \dots, 0)}} F(p^{\mu_1}, \dots, p^{\mu_k}) \right) \prod_{p \leq x} \left(1 - \frac{\rho(p)}{p} \right) \\ & \times \sum_{\substack{n_1 \cdots n_k \leq x \\ (n_1 \cdots n_k, D) = 1}} F(n_1, \dots, n_k) \frac{\rho_1(n_1) \cdots \rho_k(n_k)}{n_1 \cdots n_k}, \end{aligned}$$

for all $x \geq c_0 \|Q\|^\delta$, where $\|Q\|$ denotes the sum of the absolute values of the coefficients of Q . The implicit constant in the bound depends at most on g, A, B and δ .

Proof. It follows from [7, Theorem 3] upon noticing that we can trivially bound the Euler product Δ_D in its statement by the simpler product

$$\prod_{p|D} \left(1 + \sum_{\substack{0 \leq \mu_j \leq \deg Q_j \\ \text{for all } j \in \{1, \dots, k\}, \\ (\mu_1, \dots, \mu_k) \neq (0, \dots, 0)}} F(p^{\mu_1}, \dots, p^{\mu_k}) \right).$$

We will not need Lemma 3.4 in its full generality. In fact, we can only think of F as a product of divisor functions with a coprimality condition, whereas Q_1, \dots, Q_k are simply going to be linear polynomials.

Another useful result for the proof of Theorem 1.2 is the following generalisation of the classical Weil bound for character sums [14, Theorem 2C’].

LEMMA 3.5. *Let χ be a primitive quadratic character modulo $q \geq 2$, and consider a polynomial $f \in \mathbb{Z}[X]$. Let also M and $N \geq 0$ be two integers. If q^* is the product of the odd prime factors p of q for which f is a constant multiple of a square polynomial modulo p , then*

$$\sum_{M < n \leq M+N} \chi(f(n)) \ll_\varepsilon q^\varepsilon \sqrt{q^*} \left(\frac{N}{\sqrt{q}} + \sqrt{q} \right).$$

Proof. See the discussion preceding the statement of Lemma 3.7 in [19, Subsection 3.4]

We close the section with an inequality which was stated without proof in [1, Subsection 4.2].

LEMMA 3.6. *Let k be a positive integer. For given $m_1, \dots, m_k \in \mathbb{N}$, we have*

$$[m_1, \dots, m_k] \geq \frac{m_1 \cdots m_k}{\prod_{1 \leq i < j \leq k} (m_i, m_j)}.$$

Proof. For $k = 1$ the lemma is trivial and we may assume that $k \geq 2$. We start by making use of the following elementary observation; For positive integers a, b and c for which $b \mid c$, it is true that $(a, b) \leq (a, c)$. We apply this property and obtain

$$(m_k, [m_1, \dots, m_{k-1}]) \leq (m_k, m_1 \cdots m_{k-1}). \tag{3.7}$$

It is also easy to note that $(ab, c) \leq (a, c)(b, c)$ for $a, b, c \in \mathbb{N}$. Hence,

$$(m_k, m_1 \cdots m_{k-1}) \leq (m_1, m_k) \cdots (m_{k-1}, m_k). \tag{3.8}$$

Now, we put (3.7) and (3.8) together and deduce that

$$[m_1, \dots, m_k] = [m_k, [m_1, \dots, m_{k-1}]] = \frac{m_k [m_1, \dots, m_{k-1}]}{(m_k, [m_1, \dots, m_{k-1}])} \geq \frac{m_k [m_1, \dots, m_{k-1}]}{\prod_{1 \leq i \leq k-1} (m_i, m_k)}.$$

Based on this inequality, the proof is completed by induction on k .

4. Preparations for the Proof of Theorem 1.2

Throughout this and the following section, we assume that ε, k and h_1, \dots, h_k are fixed. Hence, any dependence of the implicit constants on any of these integers is ignored. Moreover, recall that for $z > 1$, the completely multiplicative function λ_z is defined by (2.2), where the Dirichlet character χ in (2.2) is now the primitive quadratic character modulo q from the statement of Theorem 1.2.

4.1. Going from λ to λ_z

As was explained in Section 2, our first objective is to make a transition from the sums $\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k)$ to the sums $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$. To this end, we prove the following.

LEMMA 4.1 *Let $k, q \geq 2$ be natural numbers, and let h_1, \dots, h_k be distinct non-negative integers. Let also χ be as in the statement of Theorem 1.2 and write $z = q^v$ for some $v > 0$. For $x > z$, we have that*

$$\sum_{n \leq x} \left(\prod_{j=1}^k \lambda(n + h_j) - \prod_{j=1}^k \lambda_z(n + h_j) \right) \ll \begin{cases} x \left(\frac{1}{v^2 \eta^{v/2}} + \frac{1}{z} \right) \left(\frac{\log x}{\log z} \right)^3 & \text{for } v \in (0, 2), \\ \frac{x \log x}{\eta \log q} & \text{for } v \geq 2. \end{cases}$$

Proof. The beginning of the proof is similar to those of [1, Lemma 3.1] and [3, Section 3, relation (3.3)], but for the sake of completeness we write it down here.

We will need the following inequality that one can prove by induction; If $m \in \mathbb{N}$ and $w_1, \dots, w_m, u_1, \dots, u_m$ are complex numbers of modulus at most 1, then

$$|w_1 \cdots w_m - u_1 \cdots u_m| \leq \sum_{i=1}^m |w_i - u_i|. \tag{4.9}$$

Using this inequality, we infer that

$$\begin{aligned} \sum_{n \leq x} \left| \prod_{j=1}^k \lambda(n + h_j) - \prod_{j=1}^k \lambda_z(n + h_j) \right| &\leq \sum_{n \leq x} \sum_{j=1}^k |\lambda(n + h_j) - \lambda_z(n + h_j)| \\ &= \sum_{j=1}^k \sum_{n \leq x} |\lambda(n + h_j) - \lambda_z(n + h_j)| \\ &= k \sum_{n \leq x} |\lambda(n) - \lambda_z(n)| + O(1). \end{aligned} \tag{4.10}$$

In order to bound $\sum_{n \leq x} |\lambda(n) - \lambda_z(n)|$, we are exploiting the inequality (4.9) again. We combine it with the definition of λ_z and get

$$\begin{aligned}
 \sum_{n \leq x} |\lambda(n) - \lambda_z(n)| &\leq \sum_{n \leq x} \sum_{\substack{p^\alpha \parallel n \\ p > z}} |\lambda(p^\alpha) - \chi(p^\alpha)| \\
 &\leq \sum_{\substack{p^\alpha \leq x \\ p > z}} |\lambda(p^\alpha) - \chi(p^\alpha)| \sum_{\substack{n \leq x \\ p^\alpha \mid n}} 1 \\
 &\ll x \sum_{\substack{p^\alpha \leq x \\ p > z}} \frac{|\lambda(p^\alpha) - \chi(p^\alpha)|}{p^\alpha} \\
 &\leq x \sum_{z < p \leq x} \frac{1 + \chi(p)}{p} + 2x \sum_{p > z} \sum_{\alpha \geq 2} \frac{1}{p^\alpha} \\
 &\ll x \sum_{z < p \leq x} \frac{1 + \chi(p)}{p} + x \sum_{p > z} \frac{1}{p^2} \\
 &\ll x \sum_{z < p \leq x} \frac{1 + \chi(p)}{p} + \frac{x}{z}.
 \end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we deduce that

$$\sum_{n \leq x} \left(\prod_{j=1}^k \lambda(n + h_j) - \prod_{j=1}^k \lambda_z(n + h_j) \right) \ll x \sum_{z < p \leq x} \frac{1 + \chi(p)}{p} + \frac{x}{z}.$$

The proof is now completed by an application of Lemma 3.1. In the case $v < 2$, Lemma 3.1 completes the proof immediately because x/z is smaller than $x(\log x)^3 / (z(\log z)^3)$ for $x > z$. When $v \geq 2$, the extra term x/z is absorbed by the bound $x \log x / (\eta \log q)$, because in this case $\log x \geq \log q$ and (1.1) implies that $\eta \ll q \leq z$.

4.2. Restricting the z -smooth parts

Once Lemma 4.1 is used and the transition from the sums of λ to the sums of λ_z is completed, one’s interest turns to the study of $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$. For technical reasons, we seek to exclude the terms of $\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k)$ for which $\prod_{p \leq z, p^\alpha \parallel n + h_i} p^\alpha$ is relatively large for some $i \in \{1, \dots, k\}$. To this end, in the current subsection, we assess the contribution stemming from these “undesirable” summands.

Before doing so, we introduce the following notation for convenience; For $m \in \mathbb{N}$ and $z > 1$, we write

$$m_z := \prod_{p^\alpha \parallel m, p \leq z} p^\alpha.$$

LEMMA 4.2. *Let $x \geq z > 1$ and $u \geq 1$. If $x \geq z^u$, then*

$$\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) = \sum_{\substack{n \leq x, (n+h_i)_z \leq z^u \\ \text{for all } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) + O(xe^{-u/2} + 1).$$

Proof. We readily see that

$$\begin{aligned} \sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) &= \sum_{\substack{n \leq x, \\ (n+h_i)_z \leq z^u \\ \text{for all } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) \\ &+ \sum_{\substack{n \leq x, \\ (n+h_i)_z > z^u \\ \text{for some } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k). \end{aligned} \tag{4.12}$$

Note that

$$\begin{aligned} \sum_{\substack{n \leq x, \\ (n+h_i)_z > z^u \\ \text{for some } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) &\leq \sum_{i=1}^k \sum_{\substack{n \leq x \\ (n+h_i)_z > z^u}} 1 \leq \sum_{i=1}^k \sum_{\substack{h_i < m \leq x+h_i \\ m_z > z^u}} 1 \\ &= \sum_{i=1}^k \#\{m \leq x : \prod_{p^\alpha \parallel m, p \leq z} p^\alpha > z^u\} + O(1). \end{aligned}$$

Therefore, by using Lemma 3.2, we deduce that

$$\sum_{\substack{n \leq x, \\ (n+h_i)_z > z^u \\ \text{for some } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) \ll xe^{-u/2} + 1. \tag{4.13}$$

The proof now follows by inserting (4.13) into (4.12).

4.3. Rewriting the “main term” in Lemma 4.2

Let x, z and u be as in the statement of Lemma 4.2. We are going to rewrite the sum

$$S := \sum_{\substack{n \leq x, \\ (n+h_i)_z \leq z^u \\ \text{for all } i \in \{1, \dots, k\}}} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) \tag{4.14}$$

in a form that we use in the upcoming section. For the sequel, we assume without loss of generality that $h_1 = \min_{1 \leq i \leq k} h_i$. We write $n + h_i = a_i n_i$ with $P^+(a_i) \leq z$ and $P^-(n_i) > z$, and so

$$S = \sum_{\substack{a_1, \dots, a_k \leq z^u, \\ P^+(a_1 \cdots a_k) \leq z, \\ (a_i, a_j) \mid (h_i - h_j) \\ \text{for all } i, j \in \{1, \dots, k\} \text{ with } i \neq j}} \lambda(a_1 \cdots a_k) \sum_{\substack{n_1 \leq \frac{x+h_1}{a_1}, \\ P^-(n_1 \cdots n_k) > z, \\ a_1 n_1 = a_i n_i + h_1 - h_i \\ \text{for all } i \in \{2, \dots, k\}}} \chi(n_1 \cdots n_k). \tag{4.15}$$

We denote the inner sum in (4.15) by Σ , that is,

$$\Sigma := \sum_{\substack{n_1 \leq \frac{x+h_1}{a_1}, \\ P^-(n_1 \cdots n_k) > z, \\ a_1 n_1 = a_i n_i + h_1 - h_i \\ \text{for all } i \in \{2, \dots, k\}}} \chi(n_1 \cdots n_k).$$

Upon writing $h_i^* = h_i - h_1$ for $i \in \{1, \dots, k\}$, we see that

$$\Sigma = \sum_{\substack{n \leq \frac{x+h_1}{a_1}, \\ P^-\left(\prod_{i=1}^k \frac{a_1 n + h_i^*}{a_i}\right) > z, \\ a_1 n \equiv -h_i^* \pmod{a_i} \\ \text{for all } i \in \{2, \dots, k\}}} \chi\left(\prod_{i=1}^k \frac{a_1 n + h_i^*}{a_i}\right). \tag{4.16}$$

Since we have the condition $(a_i, a_j) | (h_i - h_j)$ in the outermost sum of (4.15), for every $i \in \{1, \dots, k\}$, one has that

$$\begin{aligned} a_1 n \equiv -h_i^* \pmod{a_i} &\iff \frac{a_1}{(a_1, a_i)} n \equiv -\frac{h_i^*}{(a_1, a_i)} \pmod{\frac{a_i}{(a_1, a_i)}} \\ &\iff n \equiv -\overline{a_{1,i}} \frac{h_i^*}{(a_1, a_i)} \pmod{a_i^*}, \end{aligned} \tag{4.17}$$

where $a_i^* := a_i(a_1, a_i)^{-1}$, $a_{1,i} := a_1(a_1, a_i)^{-1}$, and $\overline{a_{1,i}}$ is its inverse $\pmod{a_i^*}$. Now, if the system of linear congruences (4.17) is not soluble, then $\Sigma = 0$ and there is no contribution from such systems. Consequently, we may limit the outermost sum of (4.15) to those a_1, \dots, a_k for which the system of linear congruences (4.17) is soluble. In this case, by the Chinese Remainder Theorem, there exists an integer r^* such that:

- (i) $0 < r^* \leq [a_2^*, \dots, a_k^*]$;
- (ii) $n = \ell[a_2^*, \dots, a_k^*] + r^* - [a_2^*, \dots, a_k^*]$ for $\ell \in \mathbb{N}$; and
- (iii) $a_1 r^* \equiv -h_i^* \pmod{a_i}$ for all $i \in \{1, \dots, k\}$.

Applying these to (4.16), we derive that

$$\Sigma = \sum_{\substack{l \leq \frac{x+h_1-a_1 r^*}{a_1 [a_2^*, \dots, a_k^*]} + 1 \\ P^-(Q(\ell)) > z}} \chi(Q(\ell)), \tag{4.18}$$

where

$$Q(X) := \prod_{i=1}^k \left(a_{1,i} \frac{[a_2^*, \dots, a_k^*]}{a_i^*} X + \frac{a_1 r^* + h_i^*}{a_i} - a_{1,i} \frac{[a_2^*, \dots, a_k^*]}{a_i^*} \right)$$

defines a polynomial of $\mathbb{Z}[X]$. Since Σ denotes the inner sum of (4.15), by inserting (4.18) into (4.15), we obtain that

$$S = \sum'_{\substack{a_1, \dots, a_k \leq z^u, \\ P^+(a_1 \dots a_k) \leq z, \\ (a_i, a_j) | (h_i - h_j) \\ \text{for all } i, j \in \{1, \dots, k\} \text{ with } i \neq j}} \lambda(a_1 \dots a_k) \sum_{\substack{l \leq \frac{x+h_1-a_1 r^*}{a_1 [a_2^*, \dots, a_k^*]} + 1 \\ P^-(Q(\ell)) > z}} \chi(Q(\ell)), \tag{4.19}$$

where the prime $'$ indicates that the sum is taken over the integers a_1, \dots, a_k for which the system of congruences (4.17) is soluble.

Relation (4.19) provides the form of S that we will exploit.

5. Proof of Theorem 1.2

In this section, we establish Theorem 1.2. To achieve this, we bound S by starting from (4.19), and then, once the estimation of S is complete, we combine the definition (4.14) of S with Lemmas 4.1 and 4.2.

Let $z = q^v$ with

$$v = \min \left\{ \sqrt{\frac{V}{\log \eta}}, 2 \right\}.$$

We now assume that η is sufficiently large, otherwise $xV/\eta \gg x$, in which case, Theorem 1.2 would immediately follow from the trivial bound x . Since η is assumed to be large enough, Siegel’s theorem, that is (1.1), implies that q is also sufficiently large and that $v \gg 1/\sqrt{(\log q)}$. Therefore, $v \geq \log k / \log q$, and so $z = q^v \geq k$.

In (4.19), we have that $P^+(a_1 \cdots a_k) \leq z < P^-(Q(\ell))$, and so the inner sum Σ of S is taken over integers ℓ for which $(Q(\ell), a_1 \cdots a_k) = 1$. Moreover, since $z \geq k$, we also have that $P^-(Q(\ell)) > z \geq k$, and this imposes the condition $(Q(\ell), \prod_{p \leq k} p) = 1$ on Σ . According to the above, we can add the supplementary condition $(Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1$ to the inner sum Σ of (4.19). Therefore,

$$\Sigma = \sum_{\substack{\ell \leq \frac{x+h_1-a_1x^*}{a_1[a_2^* \cdots a_k^*]} + 1 \\ P^-(Q(\ell)) > z \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \chi(Q(\ell)). \tag{5.20}$$

We now detect the sifting condition $P^-(Q(\ell)) > z$ of (5.20) by applying Lemma 3.3 with $\beta = 25k$ and $u = \varepsilon V / (30kv)$. Since $V \geq \frac{1}{2} + \varepsilon$ and η is large, we observe that $u = \varepsilon V / (30vk) \geq 50k$. Hence,

$$\begin{aligned} \Sigma = & \sum_{\substack{d \leq z^u \\ d | \prod_{p \leq z} p \\ (d, a_1 \cdots a_k \prod_{p \leq k} p) = 1}} w(d) \sum_{\substack{\ell \leq \frac{x}{a_1[a_2^* \cdots a_k^*]} \\ d | Q(\ell) \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \chi(Q(\ell)) \\ & + O \left(z^u + \sum_{r \geq \frac{u}{2}} 2^{-r} \sum_{\substack{\ell \leq \frac{x}{a_1[a_2^* \cdots a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2 \right), \end{aligned} \tag{5.21}$$

where $z_r = z^{((25k-1)/(25k))^r}$, and w is the arithmetic function from the statement of Lemma 3.3.

5.1. Evaluation of the main term of Σ in (5.21)

We seek a bound for

$$\Sigma' := \sum_{\substack{d \leq z^u \\ d | \prod_{p \leq z} p \\ (d, a_1 \cdots a_k \prod_{p \leq k} p) = 1}} w(d) \sum_{\substack{\ell \leq \frac{x}{a_1[a_2^* \cdots a_k^*]} \\ d | Q(\ell) \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \chi(Q(\ell)).$$

We spot the condition $(Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1$ using Möbius inversion, so we obtain

$$\begin{aligned} \Sigma' &= \sum_{\substack{d \leq z^u, d | \prod_{p \leq k} p \\ (d, a_1 \cdots a_k \prod_{p \leq k} p) = 1 \\ d' | a_1 \cdots a_k \prod_{p \leq k} p}} w(d)\mu(d') \sum_{\substack{v \pmod{dd'} \\ Q(v) \equiv 0 \pmod{dd'}}} \sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]} \\ \ell \equiv v \pmod{dd'}}} \chi(Q(\ell)) \\ &= \sum_{\substack{d \leq z^u, d | \prod_{p \leq k} p \\ (d, a_1 \cdots a_k \prod_{p \leq k} p) = 1 \\ d' | a_1 \cdots a_k \prod_{p \leq k} p}} w(d)\mu(d') \sum_{\substack{v \pmod{dd'} \\ Q(v) \equiv 0 \pmod{dd'}}} \sum_{m \leq \frac{x}{dd' a_1 [a_2^* \cdots a_k^*]}} \chi(R_{d,d',v}(m)), \end{aligned} \tag{5.22}$$

where $R_{d,d',v} \in \mathbb{Z}[X]$ with $R_{d,d',v}(X) := Q(dd'X + v)$.

We want to apply Lemma 3.5 to bound the innermost character sum in (5.22). In order to do this with $q^* \ll (dd', q)$, we have to verify that $R_{d,d',v}$ is not a constant multiple of a square polynomial modulo p for primes $p \nmid dd'$ with $p > \max_{1 \leq i \leq k} h_i$ and $p | q$. We do this by considering the following complementary cases; $p \nmid a_i$ for all $i \in \{1, \dots, k\}$ and $p | a_i$ for some $i \in \{1, \dots, k\}$.

We first assume that $p \nmid a_i$ for all $i \in \{1, \dots, k\}$. In this case, if $R_{d,d',v}$ is a square, then its roots have even multiplicity, which implies that there exist distinct indices $i, j \in \{1, \dots, k\}$ such that

$$\begin{aligned} &\overline{dd'} \cdot \frac{a_1 [a_2^*, \dots, a_k^*]}{a_i} \cdot \left(\frac{a_1 r^* + h_i^*}{a_i} + \frac{(v-1)a_1 [a_2^*, \dots, a_k^*]}{a_i} \right) \\ &\equiv \overline{dd'} \cdot \frac{a_1 [a_2^*, \dots, a_k^*]}{a_j} \cdot \left(\frac{a_1 r^* + h_j^*}{a_j} + \frac{(v-1)a_1 [a_2^*, \dots, a_k^*]}{a_j} \right) \pmod{p}. \end{aligned}$$

We multiply both sides of this congruence by $dd'(a_1 [a_2^*, \dots, a_k^*])^2 a_i^{-1} a_j^{-1}$ and infer that

$$\frac{a_1 [a_2^*, \dots, a_k^*]}{a_i a_j} (h_i^* - h_j^*) \equiv 0 \pmod{p}.$$

Since $p \nmid a_i$ for all $i \in \{1, \dots, k\}$, we get $h_i^* \equiv h_j^* \pmod{p}$, which is equivalent to $h_i \equiv h_j \pmod{p}$. But, since $h_i, h_j \leq \max_{1 \leq \kappa \leq k} h_\kappa < p$, we conclude that $h_i = h_j$, which is a contradiction for $i \neq j$ as the shifts h_1, \dots, h_k are distinct. Consequently, $R_{d,d',v}$ is not a constant multiple of a square polynomial modulo p in this case.

We now assume that $p | a_i$ for some $i \in \{1, \dots, k\}$. This implies that $p \nmid a_j$ for all $j \neq i$, otherwise $p | (a_i, a_j)$ for some $j \neq i$, which, combined with the condition $(a_i, a_j) | (h_i - h_j)$ of (4.19), would imply the divisibility relation $p | (h_i - h_j)$ which contradicts the condition $p > \max_{1 \leq i \leq k} h_i$ that we are working under. By the same argument, if $i \neq 1$, then we can deduce that $p | a_i^*$. Combining both our conclusions, we observe that p divides $a_{1,j} [a_2^*, \dots, a_k^*] a_j^{*-1} = a_1 [a_2^*, \dots, a_k^*] a_j^{-1}$ for $j \neq i$. Note also that

$$a_{1,i} [a_2^*, \dots, a_k^*] a_i^{*-1} \mid \prod_{j=1, j \neq i}^k a_j,$$

because $[a_2^*, \dots, a_k^*]$ divides $a_2^* \cdots a_k^*$, which in turn divides $a_2 \cdots a_k$. So, $p \nmid a_{1,i} [a_2^*, \dots, a_k^*] a_i^{*-1}$, whereas $p | a_{1,j} [a_2^*, \dots, a_k^*] a_j^{*-1}$ for $j \neq i$. We thus see that $R_{d,d',v}(X) \equiv$

$c_1X + c_0 \pmod p$ for some $c_0, c_1 \in \mathbb{N}$, which shows that $R_{d,d',v}$ is not a constant multiple of a square polynomial modulo p in this case either.

Now that we have concluded that $R_{d,d',v}$ is not a constant multiple of a square polynomial modulo p when p is a prime such that $p|q, p \nmid dd'$ and $p > \max_{1 \leq i \leq k} h_i$, Lemma 3.5 yields

$$\sum_{m \leq \frac{x}{dd' a_1 [a_2^* \dots a_k^*]}} \chi(R_{d,d',v}(m)) \ll (d, q)^{1/2} (d', q)^{1/2} \left(\frac{x}{dd' a_1 [a_2^* \dots a_k^*] q^{\frac{1-\varepsilon}{2}}} + q^{\frac{1+\varepsilon}{2}} \right). \tag{5.23}$$

Using Lemma 3.6 and the condition $(a_i, a_j) | (h_i - h_j)$ from the outermost sum of (4.19), we find that

$$[a_2^*, \dots, a_k^*] \geq a_2 \dots a_k \prod_{1 \leq i < j \leq k} (a_i, a_j)^{-1} \geq a_2 \dots a_k \left(\max_{1 \leq i \leq k} h_i \right)^{-\binom{k}{2}}. \tag{5.24}$$

We combine (5.23) and (5.24). Hence, since d and d' are coprime and the arithmetic function ρ defined as $\rho(m) := \#\{v \pmod m : Q(v) \equiv 0 \pmod m\}$ for all $m \in \mathbb{N}$ is multiplicative (by the Chinese Remainder Theorem), (5.22) gives

$$\begin{aligned} \Sigma' &\ll \frac{x}{q^{(1-\varepsilon)/2} a_1 \dots a_k} \left(\sum_{\substack{d|\prod_{p \leq z} p \\ (d, a_1 \dots a_k) = 1}} \frac{\rho(d)(d, q)^{1/2}}{d} \right) \left(\sum_{d'|a_1 \dots a_k \prod_{p \leq k} p} \rho(d') \mu(d')^2 \right) \\ &\quad + q^{(1+\varepsilon)/2} \left(\sum_{d \leq z^u} \rho(d) \sqrt{d} \right) \left(\sum_{d'|a_1 \dots a_k \prod_{p \leq k} p} \rho(d') \sqrt{d'} \right). \end{aligned} \tag{5.25}$$

Using the trivial inequalities $\tau(m), \rho(m) \leq m$ for all $m \in \mathbb{N}$, we immediately derive the bound

$$\sum_{d \leq z^u} \rho(d) \sqrt{d} \ll z^{5u/2},$$

as well as the estimate

$$\sum_{d'|a_1 \dots a_k \prod_{p \leq k} p} \rho(d') \sqrt{d'} \ll (a_1 \dots a_k)^{3/2} \tau(a_1 \dots a_k) \leq (a_1 \dots a_k)^{5/2}.$$

Consequently, after recalling that $a_1, \dots, a_k \leq z^u$ in the expression (4.19) of S , (5.25) implies

$$\Sigma' \ll \frac{x}{q^{\frac{1}{4}} a_1 \dots a_k} \left(\sum_{\substack{d|\prod_{p \leq z} p \\ (d, a_1 \dots a_k) = 1}} \frac{\rho(d)(d, q)^{\frac{1}{2}}}{d} \right) \left(\sum_{d'|a_1 \dots a_k \prod_{p \leq k} p} \rho(d') \mu(d')^2 \right) + z^{\frac{5(k+1)u}{2}} q^{\frac{1+\varepsilon}{2}}. \tag{5.26}$$

We previously noted that any two roots of the polynomial $R_{d,d',v}$ cannot be congruent modulo p . By an almost identical approach, for the primes $p > \max_{1 \leq i \leq k} h_i$ that do not divide the product $a_1 \dots a_k$, one can show that the k roots

$$\overline{a_1 [a_2^*, \dots, a_k^*] a_i^{-1} (a_1 r^* + h_i^*) a_i^{-1} - 1}$$

of Q modulo p are distinct. Thus, $\rho(p) = k$ for every prime $p > \max_{1 \leq i \leq k} h_i$ which does not divide $a_1 \cdots a_k$. Furthermore, if $p \mid a_1 \cdots a_k$, then Q reduces to a constant or first degree polynomial modulo p . Therefore, for the primes $p \mid a_1 \cdots a_k$, we have that $\rho(p) \leq 1$. We are going to use these two observations to bound the sums at (5.26). Firstly, we have that

$$\begin{aligned} \sum_{\substack{d \mid \prod_{p \leq z} p \\ (d, a_1 \cdots a_k) = 1}} \frac{\rho(d)(d, q)^{1/2}}{d} &\ll \prod_{\substack{\max_{1 \leq i \leq k} h_i < p \leq z \\ p \nmid qa_1 \cdots a_k}} \left(1 + \frac{\rho(p)}{p}\right) \prod_{\substack{p \mid q, p \nmid a_1 \cdots a_k \\ p > \max_{1 \leq i \leq k} h_i}} \left(1 + \frac{\rho(p)}{\sqrt{p}}\right) \\ &\leq \prod_{\substack{\max_{1 \leq i \leq k} h_i < p \leq z \\ p \nmid qa_1 \cdots a_k}} \left(1 + \frac{k}{p}\right) \prod_{\substack{p \mid q, p \nmid a_1 \cdots a_k \\ p > \max_{1 \leq i \leq k} h_i}} (1 + k) \\ &\ll \tau_{k+1}(q)(\log z)^k \ll q^{1/20}(\log z)^k. \end{aligned} \tag{5.27}$$

We also have

$$\sum_{d' \mid a_1 \cdots a_k \prod_{p \leq k} p} \rho(d')\mu(d')^2 \ll \prod_{p \mid a_1 \cdots a_k} (1 + \rho(p)) \leq \tau(a_1 \cdots a_k) \leq \tau(a_1) \cdots \tau(a_k). \tag{5.28}$$

We now insert (5.27) and (5.28) into (5.26) to complete the estimation of Σ' . This way we obtain

$$\Sigma' \ll \frac{x(\log z)^k \tau(a_1) \cdots \tau(a_k)}{q^{1/5} a_1 \cdots a_k} + z^{5(k+1)u/2} q^{(1+\varepsilon)/2}. \tag{5.29}$$

5.2. Evaluation of the error term of Σ in (5.21)

Upon denoting the Big-Oh term of (5.21) by

$$\Sigma'' := z^u + \sum_{r \geq \frac{u}{2}} 2^{-r} \sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2, \tag{5.30}$$

the next step of the proof is to bound Σ'' . We basically achieve this with an application of Lemma 3.4 to the divisor sum of Σ'' .

The inequality $\tau(ab) \leq \tau(a)\tau(b)$ which holds for any $a, b \in \mathbb{N}$ implies that

$$\sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2 \leq \sum_{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]}} F(Q_1(\ell), \dots, Q_k(\ell)), \tag{5.31}$$

where

$$\begin{aligned} F(n_1, \dots, n_k) &:= \prod_{i=1}^k \tau(n_i)^2 \mathbb{1}_{P^-(n_i) > z_r} \mathbb{1}_{(n_i, a_1 \cdots a_k) = 1}, \quad \text{and} \\ Q_i(X) &:= \frac{a_1 [a_2^*, \dots, a_k^*]}{a_i} X + \frac{a_1 r^* + h_i^* - a_1 [a_2^*, \dots, a_k^*]}{a_i}, \quad \text{for } i \in \{1, \dots, k\}. \end{aligned}$$

We now aim to apply Lemma 3.4 to the right-hand side of (5.31) as this will lead to a bound for Σ'' . In order to use Lemma 3.4, we verify that all of its conditions are met. First of all, the polynomials Q_1, \dots, Q_k are irreducible and pairwise coprime because they are linear and have distinct roots. Now, suppose that $Q = Q_1 \cdots Q_k$ has a fixed prime divisor p , or that we equivalently have that $\rho(p) = p$. If $p|a_1 \cdots a_k$, then the fixed prime divisor p of Q divides a_1, \dots, a_k . However, in this case, the condition $(Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1$ in (5.30) is not satisfied for any $\ell \in \mathbb{N}$, and $\Sigma'' = 0$. If $p \nmid a_1 \cdots a_k$, then Q is a polynomial of degree k . This implies that $p = \rho(p) \leq k$, and so, in this case, Q has a fixed prime divisor $p \leq k$. Then the condition $(Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1$ in (5.30) is not met for any $\ell \in \mathbb{N}$ and Σ'' vanishes again. Hence, if Q has a fixed prime divisor, then $\Sigma'' = 0$ and there is nothing to bound. Therefore, we may assume that Q has no fixed prime factors. Finally, by combining $z^{30ku} = x^\varepsilon$, which is an immediate consequence of the definition of u , with the inequalities $r^* \leq [a_2^*, \dots, a_k^*] \leq a_2 \cdots a_k$ and $a_1, \dots, a_k \leq z^u$, we observe that

$$\frac{x}{a_1 [a_2^*, \dots, a_k^*]} \gg \|Q\|^{\frac{1}{k}},$$

where $\|Q\|$ denotes the sum of the absolute values of the coefficients of the polynomial Q .

Now that we finished verifying that Henriot’s bound, namely Lemma 3.4, is applicable, we apply it to the right-hand side of (5.31) and obtain that

$$\begin{aligned} \sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^*, \dots, a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2 &\ll \frac{x}{a_1 [a_2^*, \dots, a_k^*]} \prod_{p \leq z} \left(1 - \frac{\rho(p)}{p}\right) \\ &\times \prod_{\substack{p|D \\ p \nmid a_1 \cdots a_k}} \left(1 + \sum_{\substack{\mu_1, \dots, \mu_k \in \{0,1\} \\ (\mu_1, \dots, \mu_k) \neq (0, \dots, 0)}} \tau(p^{\mu_1})^2 \cdots \tau(p^{\mu_k})^2\right) \\ &\times \sum_{\substack{m_1, \dots, m_k \leq x \\ P^-(m_1 \cdots m_k) > z_r \\ (m_1 \cdots m_k, a_1 \cdots a_k) = 1}} \frac{\tau(m_1)^2 \cdots \tau(m_k)^2}{m_1 \cdots m_k}, \end{aligned} \tag{5.32}$$

where D is the discriminant of Q .

The value of D can be easily evaluated by the roots of Q and a simple calculation shows that

$$D = \frac{[a_2^*, \dots, a_k^*]^{k(k-1)}}{(a_1 \cdots a_k)^{2(k-1)}} \prod_{1 \leq i < j \leq k} (h_i - h_j)^2.$$

Since we are working with a fixed set of non-negative integers h_1, \dots, h_k , we see from this that

$$\prod_{\substack{p|D \\ p \nmid a_1 \cdots a_k}} \left(1 + \sum_{\substack{\mu_1, \dots, \mu_k \in \{0,1\} \\ (\mu_1, \dots, \mu_k) \neq (0, \dots, 0)}} \tau(p^{\mu_1})^2 \cdots \tau(p^{\mu_k})^2\right) \ll 1. \tag{5.33}$$

We saw that $\rho(p) = k$ for the primes $p > \max_{1 \leq i \leq k} h_i$ which do not divide $a_1 \cdots a_k$. Therefore, by Bernoulli's inequality,

$$\begin{aligned} \prod_{p \leq z} \left(1 - \frac{\rho(p)}{p}\right) &\leq \prod_{\substack{\max_{1 \leq i \leq k} h_i \leq p \leq z \\ p \nmid a_1 \cdots a_k}} \left(1 - \frac{\rho(p)}{p}\right) \leq \prod_{\substack{\max_{1 \leq i \leq k} h_i \leq p \leq z \\ p \nmid a_1 \cdots a_k}} \left(1 - \frac{k}{p}\right) \\ &\leq \prod_{\substack{\max_{1 \leq i \leq k} h_i \leq p \leq z \\ p \nmid a_1 \cdots a_k}} \left(1 - \frac{1}{p}\right)^k \ll (\log z)^{-k} \prod_{p \mid a_1 \cdots a_k} \left(1 - \frac{1}{p}\right)^{-k} \\ &\ll (\log z)^{-k} \prod_{j=1}^k \prod_{p \mid a_j} \left(1 - \frac{1}{p}\right)^{-k} = (\log z)^{-k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k. \end{aligned} \tag{5.34}$$

Furthermore,

$$\begin{aligned} \sum_{\substack{m_1, \dots, m_k \leq x \\ P^-(m_1 \cdots m_k) > z_r \\ (m_1 \cdots m_k, a_1 \cdots a_k) = 1}} \frac{\tau(m_1)^2 \cdots \tau(m_k)^2}{m_1 \cdots m_k} &\leq \left(\sum_{\substack{m \leq x \\ P^-(m) > z_r}} \frac{\tau(m)^2}{m}\right)^k \\ &\ll \prod_{z_r < p \leq x} \left(1 + \frac{4}{p} + O\left(\frac{1}{p^2}\right)\right)^k \ll \exp\left(\sum_{z_r < p \leq x} \frac{4k}{p}\right) \\ &\ll \left(\frac{\log x}{\log z_r}\right)^{4k} = \left(\frac{25k}{25k-1}\right)^{4kr} \left(\frac{\log x}{\log z}\right)^{4k}. \end{aligned} \tag{5.35}$$

We insert the bounds (5.24), (5.33), (5.34) and (5.35) into (5.32) and obtain

$$\sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2 \ll \frac{x}{a_1 \cdots a_k (\log z)^k} \left(\frac{25k}{25k-1}\right)^{4kr} \left(\frac{\log x}{\log z}\right)^{4k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k.$$

But, we note that $25k - 1 > 24k > 8k / (\log 4 - 1)$, and so

$$\left(\frac{25k}{25k-1}\right)^{4kr} = \exp\left\{4kr \log\left(1 + \frac{1}{25k-1}\right)\right\} \leq \exp\left(\frac{4kr}{25k-1}\right) < 2^r e^{-r/2}.$$

Hence,

$$\sum_{\substack{\ell \leq \frac{x}{a_1 [a_2^* \cdots a_k^*]} \\ P^-(Q(\ell)) > z_r \\ (Q(\ell), a_1 \cdots a_k \prod_{p \leq k} p) = 1}} \tau(Q(\ell))^2 \ll \frac{2^r e^{-r/2} x}{a_1 \cdots a_k (\log z)^k} \left(\frac{\log x}{\log z}\right)^{4k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k,$$

and then in virtue of (5.30), we obtain

$$\begin{aligned} \Sigma'' &\ll z^u + \frac{x}{a_1 \cdots a_k (\log z)^k} \left(\frac{\log x}{\log z}\right)^{4k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k \sum_{r \geq u/2} e^{-r/2} \\ &\ll z^u + \frac{x}{a_1 \cdots a_k (\log z)^k} \left(\frac{\log x}{\log z}\right)^{4k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k e^{-u/4}. \end{aligned} \tag{5.36}$$

5.3 Completing the estimation of S in (4.14)

We now combine (5.29) with (5.36), and so (5.21) gives

$$\Sigma \ll \frac{x(\log z)^k \tau(a_1) \cdots \tau(a_k)}{q^{1/5} a_1 \cdots a_k} + z^{5(k+1)u/2} q^{(1+\varepsilon)/2} + \frac{xe^{-u/4}}{a_1 \cdots a_k (\log z)^k} \left(\frac{\log x}{\log z}\right)^{4k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k.$$

We finally insert this bound into (4.19) and obtain

$$S \ll \frac{x(\log z)^k}{q^{1/5}} \sum_{\substack{a_1, \dots, a_k \\ P^+(a_1 \cdots a_k) \leq z}} \frac{\tau(a_1) \cdots \tau(a_k)}{a_1 \cdots a_k} + z^{(7k+5)u/2} q^{(1+\varepsilon)/2} + \frac{xe^{-u/4}}{(\log z)^k} \left(\frac{\log x}{\log z}\right)^{4k} \sum_{\substack{a_1, \dots, a_k \\ P^+(a_1 \cdots a_k) \leq z}} \frac{1}{a_1 \cdots a_k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k. \tag{5.37}$$

But,

$$\begin{aligned} & \sum_{\substack{a_1, \dots, a_k \\ P^+(a_1 \cdots a_k) \leq z}} \frac{1}{a_1 \cdots a_k} \prod_{j=1}^k \left(\frac{a_j}{\varphi(a_j)}\right)^k \leq \left(\sum_{\substack{a \\ P^+(a) \leq z}} \frac{1}{a} \left(\frac{a}{\varphi(a)}\right)^k \right)^k \\ & \leq \prod_{p \leq z} \left(1 + \left(\frac{p}{p-1}\right)^k \frac{1}{p} + O\left(\frac{1}{p^2}\right) \right)^k \ll \exp \left(\sum_{p \leq z} \left(\frac{p}{p-1}\right)^k \frac{k}{p} \right) \\ & = \exp \left\{ \sum_{p \leq z} \left(\frac{k}{p} + O\left(\frac{1}{p^2}\right) \right) \right\} \ll (\log z)^k. \end{aligned} \tag{5.38}$$

In a similar fashion one can show that

$$\sum_{\substack{a_1, \dots, a_k \\ P^+(a_1 \cdots a_k) \leq z}} \frac{\tau(a_1) \cdots \tau(a_k)}{a_1 \cdots a_k} \ll (\log z)^{2k}.$$

Upon recalling that $z = q^v$ for some $v > 0$ and that $z^{30ku} = x^\varepsilon$, the last bound, (5.37), and (5.38) imply that

$$S \ll \frac{x(\log z)^{3k}}{q^{1/5}} + xe^{-u/4} \left(\frac{\log x}{\log z}\right)^{4k} \ll \frac{xv^{3k}}{q^{1/6}} + xe^{-u/5}. \tag{5.39}$$

5.4. The completion of the proof

Since $u = \varepsilon V / (30kv)$, when we put the bound (5.39) together with the definition (4.14) of S and Lemma 4.2, we infer that

$$\sum_{n \leq x} \lambda_z(n + h_1) \cdots \lambda_z(n + h_k) \ll \frac{xv^{3k}}{q^{1/6}} + x \exp \left(-\frac{\varepsilon V}{150kv} \right). \tag{5.40}$$

Now, if $V \geq 4 \log \eta$, then $v = 2$, and so a combination of (5.40) with (1.1) and Lemma 4.1 yields

$$\begin{aligned} \sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) &\ll \frac{xV}{\eta} + \frac{x}{q^{1/6}} + x \exp\left(-\frac{\varepsilon V}{300k}\right) \\ &\ll \frac{xV}{\eta} + x \exp\left(-\frac{\varepsilon \sqrt{V \log \eta}}{150k}\right). \end{aligned} \tag{5.41}$$

On the other hand, if $V < 4 \log \eta$, then $v = \sqrt{V/(\log \eta)} < 2$, so by putting the estimate (5.40) together with Lemma 4.1, it follows that

$$\begin{aligned} \sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) &\ll \frac{x}{q^{1/6}} + x \exp\left(-\frac{\varepsilon \sqrt{V \log \eta}}{150k}\right) \\ &\quad + x \left(\frac{\log \eta}{V} \eta^{-\sqrt{\frac{v}{4 \log \eta}}} + q^{-\sqrt{\frac{v}{\log \eta}}}\right) \left(V \sqrt{\frac{\log \eta}{V}}\right)^3 \\ &\ll \frac{x}{q^{1/6}} + x \exp\left(-c_0 \sqrt{V \log \eta}\right) \\ &\quad + x(V \log \eta)^{3/2} \exp\left(-\log q \sqrt{\frac{V}{\log \eta}}\right), \end{aligned}$$

where $c_0 \in (0, \varepsilon/(150k))$ is some constant.

According to (1.1), we have that $xq^{-1/6} \ll x/\eta \ll xV/\eta$ and that $\log \eta \ll \log q$. Consequently, the last bound gives

$$\sum_{n \leq x} \lambda(n + h_1) \cdots \lambda(n + h_k) \ll \frac{xV}{\eta} + x \exp\left(-c_1 \sqrt{V \log \eta}\right), \tag{5.42}$$

where $c_1 < c_0$ is a positive constant.

We now finish the proof of Theorem 1.2 by merging the bounds (5.41) and (5.42) from above.

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REFERENCES

[1] J. CHINIS. Siegel zeros and Sarnak’s conjecture. *Preprint*, ArXiv:2105.14653 (2021).
 [2] S. CHOWLA. *The Riemann hypothesis and Hilbert’s tenth problem. Mathematics and its Applications*, vol. 4. 9 Gordon and Breach Science Publishers, New York-London-Paris, 1965).
 [3] L. GERMÁN and I. KÁTAL. On multiplicative functions on consecutive integers. *Lith. Math. J.* **50**(1) (2010), 43–53.
 [4] G. HARMAN, J. PINTZ and D. WOLKE. A note on the Möbius and Liouville functions. *Studia Sci. Math. Hungar.* **20**(1–4) (1985), 295–299.
 [5] D. R. HEATH-BROWN. Prime twins and Siegel zeros. *Proc. London Math. Soc.* (3), **47**(2) (1983), 193–224.

- [6] H. HELFGOTT and M. RADZIWIŁŁ. Expansion, divisibility, and parity. *Preprint*, ArXiv:2103.06853 (2021).
- [7] K. HENRIOT. Nair–Tenenbaum bounds uniform with respect to the discriminant. *Math. Proc. Camb. Phil. Soc.* **152**(3) (2012), 405–424.
- [8] K. HENRIOT. Nair–Tenenbaum uniform with respect to the discriminant—erratum [mr2911138]. *Math. Proc. Camb. Phil. Soc.* **157**(2) (2014), 375–377.
- [9] D. KOUKOULOPOULOS. The distribution of prime numbers. Graduate Studies in Math., **203** (American Mathematical Society, Providence, RI, 2019).
- [10] K. MATOMÄKI and J. MERIKOSKI. Siegel zeros, twin primes, Goldbach’s conjecture, and primes in short intervals. *Int. Math. Res. Not. IMRN* **23** (2023), 20337–20384.
- [11] K. MATOMÄKI and M. RADZIWIŁŁ. Multiplicative functions in short intervals. *Ann. of Math. (2)* **183**(3) (2016), 1015–1056.
- [12] K. MATOMÄKI, M. RADZIWIŁŁ and T. TAO. An averaged form of Chowla’s conjecture. *Algebra and Number Theory* **9**(9) (2015), 2167–2196.
- [13] C. PILATTE. Improved bounds for the two-point logarithmic Chowla conjecture. *Preprint*, ArXiv:2310.19357 (2023).
- [14] W. M. SCHMIDT. Equations over finite fields. An elementary approach. *Lecture Notes in Math.*, vol. 536 (Springer-Verlag, Berlin-New York, 1976).
- [15] P. SHIU. A Brun–Titchmarsh theorem for multiplicative functions. *J. Reine Angew. Math.* **313** (1980), 161–170.
- [16] T. TAO. The logarithmically averaged Chowla and Elliott conjectures for two-point correlations. *Forum Math. Pi* **4** (2016), e8, 36pp.
- [17] T. TAO and J. TERÄVÄINEN. Odd order cases of the logarithmically averaged Chowla conjecture. *J. Théor. Nombres Bordeaux* **30**(3) (2018), 997–1015.
- [18] T. TAO and J. TERÄVÄINEN. The structure of logarithmically averaged correlations of multiplicative functions, with applications to the Chowla and Elliott conjectures. *Duke Math J.* **168**(11) (2019), 1977–2027.
- [19] T. TAO and J. TERÄVÄINEN. The Hardy–Littlewood–Chowla conjecture in the presence of a Siegel zero. *J. London Math. Soc. (2)* **106**(4) (2022), 3317–3378.
- [20] G. TENENBAUM. *Introduction to analytic and probabilistic number theory*. Translated from the second French edition (1995) by C. B. Thomas. Stud. Adv. Math. **46** (Cambridge University Press, Cambridge, 1995).