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SOME GROUPS ALL OF WHOSE PROPER VERBAL QUOTIENTS ARE RELATIVELY FREE

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For Bernhard Neumann with best wishes on his eightieth birthday

A class of blocking pairs (A, B) for which B/A^B is infinite cyclic is constructed and these are used to construct groups all of whose proper verbal quotients are relatively free.

1. INTRODUCTION

This note has three purposes. The first, of course, is to congratulate Bernhard on reaching this latest milestone and to express the wish that in his next decade he may enjoy life and give pleasure to the rest of us as he always has.

The second purpose is to record that a conjecture I made in [4] is false. In that paper I introduced the notion of a blocking pair (and for convenience the definition is repeated below) and I conjectured that a group can be written as a quotient B/A^B, where (A, B) is a blocking pair, if and only if it is countable, and locally free and has no infinite cyclic free factor. It is trivial that B/A^B is countable and that it is locally free is proved in [4], but I was wrong to suppose that it can have no infinite cyclic free factor. In fact, in Section 2 below I construct a class of blocking pairs (A, B) for which B/A^B is itself infinite cyclic.

The third purpose of this note is the one alluded to in the title. The point of introducing blocking pairs is that they are an ingredient in a recipe for making countably free groups of cardinality ω_1 which are not strongly countably free, and therefore not free. In Section 3 I show that if some of the pairs constructed in Section 2 are used in this recipe, the group G that emerges has the property that, for every non-trivial verbal functor v, the group $G/v(G)$ is relatively free of rank ω_1 in the variety determined by $v = 1$. Because it is countably free, G is residually nilpotent and therefore parafree. Thus G may be compared with parafree groups constructed by Gilbert Baumslag [1] (see also Dunwoody [3]), all of whose proper verbal quotients are relatively free of rank r, where r is a fixed positive integer; and with the parafree group of Baumslag and Urs Stammbach [2], which is countably free of cardinality ω_1 and all of whose lower central quotients are relatively free of rank ω.

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2. Blocking pairs with cyclic quotients

By definition, a blocking pair is a pair (A, B) of groups such that

(i) each of A, B is a free group of countably infinite rank;

(ii) A is a proper subgroup of B, but is not contained in a proper free factor of it;

(iii) every finitely generated free factor of A is a free factor of B.

Our object in this section is to construct a class of blocking pairs (A, B) for which the factor group B/A^B is infinite cyclic. For each natural number r we define A_r to be the free group freely generated by a_0, $a_1, \ldots, a_r$, and B_r to be the free group freely generated by a_0, $a_1, \ldots, a_r$, c_r. For each r we embed B_r in B_{r+1}, by identifying c_r with $w_r^{-1} c_{r+1} w_r$ where w_r is an element of $\langle a_r, a_{r+1}c_{r+1}\rangle$ chosen to satisfy

(2.1) $\quad w_r^{-1}c_{r+1}w_r$ does not belong to either of the double cosets $A_{r+1}c_{r+1}^{\pm 1}A_{r+1}$.

There is no difficulty in finding such elements; any non-trivial element of the derived subgroup of $\langle a_r, a_{r+1}c_{r+1}\rangle$ will do. We then set $A = \bigcup_r A_r$ and $B = \bigcup_r B_r$, and aim to prove:

(2.2) $\quad$ *(A, B) is a blocking pair such that B/A^B is infinite cyclic.*

Proof: Evidently A is freely generated by $a_0, a_1, a_2, \ldots$. Notice next that since w_r belongs to $\langle a_r, a_{r+1}c_{r+1}\rangle$ it belongs to $\langle B_r, a_{r+1}c_{r+1}\rangle$, whence so also do $c_{r+1}(= w_r c_r w_r^{-1})$ and a_{r+1}. That is $B_{r+1} = \langle B_r, a_{r+1}c_{r+1}\rangle$. Since B_{r+1} is a free group of finite rank one greater than the rank of B_r, this implies that B_{r+1} is the free product of B_r and $\langle a_{r+1}c_{r+1}\rangle$. Thus, for any r, B is freely generated by $a_0, \ldots, a_r, c_r, a_{r+1}c_{r+1}, a_{r+2}c_{r+2}, \ldots$. From this it follows not only that B is a free group of countably infinite rank, but also that every finitely generated free factor of A is a free factor of B (for it is a free factor of A_r for some r).

Now let us denote images of elements of B in the natural map of B to B/A^B by a bar. Since B is generated by a_0, $a_1, \ldots, c_0$, $c_1, \ldots$ subject to the defining relations, $c_r = w_r^{-1}c_{r+1}w_r$ for $r = 0, 1, 2, \ldots$, and the elements a_0, $a_1, \ldots$ generate A, B/A^B is generated by $\bar{a}_0$, $\bar{a}_1, \ldots, \bar{c}_0$, $\bar{c}_1, \ldots$ subject to the defining relations $\bar{a}_0 = \bar{a}_1 = \ldots = 1$, $\bar{c}_r = \bar{w}_r^{-1}\bar{c}_{r+1}\bar{w}_r$, for $r = 0, 1, 2, \ldots$. But w_r is an element of $\langle a_r, a_{r+1}c_{r+1}\rangle$, so that $\bar{w}_r$ is a power of $\bar{c}_{r+1}$, and the second block of relations becomes $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \ldots$. Thus B/A^B is infinite cyclic, and, in particular, A is a proper subgroup of B.

Thus to complete the proof of (2.2) we have only to show that A is not contained in a proper free factor of B. Suppose then that B is the free product $X \star Y$ where $A \leqslant X$ and $Y \neq 1$. Then Y is a free group and is isomorphic to B/X^B, which is a homomorphic image of the cyclic group B/A^B. So Y is cyclic, say $Y = \langle b\rangle$.

Since $B = \bigcup_r B_r$, b belongs to B_r for large enough r. By the Kurosh subgroup theorem $B_r = (X \cap B_r) \star (Y \cap B_r) \star Z$ for some group Z. Now b belongs to B_r, so $Y \cap B_r = Y \neq 1$, so that $X \cap B_r$ is a proper free factor of B_r. But $X \cap B_r$ contains A_r which is a free factor of B_r of co-rank 1, hence a maximal proper free factor. It follows that $X \cap B_r = A_r$ and that $Z = 1$. That is, B_r is freely generated by $a_0, a_1, \ldots, a_r, b$ as well as by $a_0, a_1, \ldots, a_r, c_r$; which in turn implies that the double cosets $A_r b A_r$ and $A_r c_r^{\pm 1} A_r$ are the same, for one or other choice of sign. But, of course, if $b \in B_r$ then certainly also $b \in B_{r+1}$, so that the same argument proves that $A_{r+1} b A_{r+1}$ coincides with one of $A_{r+1} c_{r+1}^{\pm 1} A_{r+1}$. But then c_r belongs to one or other of $A_{r+1} c_{r+1}^{\pm 1} A_{r+1}$, whereas by (2.1) this is not so. This contradiction completes the proof of (2.2). ∎

3. Non-free groups with relatively free verbal quotients

The blocking pairs used in this section will be constructed by the method of Section 2; but we shall need to impose a stronger condition than (2.1) on the choice of the elements w_r. Let $\{v_i,\ i = 0, 1, \ldots\}$ be a listing of the non-trivial elements of a free group F of countable infinite rank, and assume that v_0 is a commutator. We shall use the symbols v_i also to denote the corresponding verbal functors. That is, for any group G, $v_i(G)$ will denote the subgroup of G generated by the images of v_i under homomorphisms of F into G. If F_2 is a free group of rank 2 then, of course, $v_i(F_2) \neq 1$ for any i, because F_2 contains an isomorphic copy of F; and $\bigcap_{i=0}^{r} v_i(F_2) \neq 1$, because the intersection of a finite set of non-trivial normal subgroups of a free group is non-trivial. The stronger condition on the elements w_r that we need is:

(3.1) $$w_r \text{ is a non-trivial element of } \bigcap_{i=0}^{r} v_i(\langle a_r, a_{r+1} c_{r+1} \rangle).$$

Because we have assumed that v_0 is a commutator, (3.1) implies (2.1). Then we have

(3.2) *If the blocking pair (A, B) is constructred as in Section 2, with elements w_r satisfying (3.1), then, for any non-trivial verbal functor v, $B/v(B)$ is the free product of $A/v(A)$ and a free cyclic group, where the terms "free product" and "free cyclic" are to be interpreted in the variety $\mathcal{V}$ of groups G satisfying $v(G) = 1$.*

Proof: $B/v(B)$ is the group generated in $\mathcal{V}$ by $a_0, a_1, \ldots, c_0, c_1, \ldots$, subject to the relations $c_r = w_r^{-1} c_{r+1} w_r$. But v is non-trivial, so that, for some integer i, v_i belongs to $v(F)$, whence $v_i(G) \leqslant v(G)$ for any group G. Thus, (3.1) implies that in $\mathcal{V}$ we have $w_r = 1$ for $r \geqslant i$. Thus, the relations $c_r = w_r^{-1} c_{r+1} w_r$ for $r \geqslant i$ reduce to $c_i = c_{i+1} = c_{i+2} = \ldots$. If this element is denoted by c, the relations for $r < i$ simply define $c_0, \dots, c_{i-1}$ in terms of a_0, $a_1, \dots$ and c, so that $B/v(B)$ is free (in $\mathcal{V}$) on these generators. But a_0, $a_1, \dots$ generate $A/v(A)$ in $\mathcal{V}$, so we are home. ∎

Recall next, from [4], that a *blocking tower* is a tower $\{A_\alpha, \alpha < \omega_1\}$ of groups such that (i) for each finite or countable ordinal α, $(A_\alpha, A_{\alpha+1})$ is a blocking pair and (ii) the tower is smooth, that is, for limit ordinals α, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. The *type sequence* of a blocking tower is the sequence of length ω_1 of isomorphism types of blocking pairs, in which the α-th term is always the isomorphism type of $(A_\alpha, A_{\alpha+1})$. It is proved in [4] that every sequence of length ω_1 of isomorphism types of blocking pairs is the type sequence of a blocking tower, and that the union of the groups in a blocking tower is a countably free group of cardinality ω_1 which is not strongly countably free. As in [4], if (A, B) is a blocking pair, we shall write $G(A, B)$ for a group which is the union of the groups in a blocking tower whose type sequence is constant, being equal to the isomorphism type of (A, B) for all α. We shall prove:

(3.3) *If (A, B) is a blocking pair constructed by the method of Section 2 with elements w_r satisfying (3.1) then $G = G(A, B)$ is a countably free group of cardinality ω_1 which is not free, such that for every non-trivial verbal functor v, $G/v(G)$ is the free group of rank ω_1 in the variety of groups X satisfying $v(X) = 1$.*

Proof: It is part of the general theory that G is countably free of cardinality ω_1 but not strongly countably free. By assumption G is the union of a smooth tower $\{A_\alpha, \alpha < \omega_1\}$ where $(A_\alpha, A_{\alpha+1})$ is isomorphic to (A, B). By Corollary 2.11 of [4] each A_α is a local retract of G whence $v(A_\alpha) = v(G) \cap A_\alpha$, so that $A_\alpha/v(A_\alpha)$ embeds in $G/v(G)$ for each α. Thus $G/v(G)$ is the union of the smooth tower of groups $\{A_\alpha/v(A_\alpha), \alpha < \omega_1\}$. Now A_0 is an absolutely free group of countable rank, so that $A_0/v(A_0)$ is free of countable rank in $\mathcal{V}$, freely generated, say, by the countable set C. By (3.2), for each α, $A_{\alpha+1}/v(A_{\alpha+1})$ is the free product in $\mathcal{V}$ of $A_\alpha/v(A_\alpha)$ and the $\mathcal{V}$-free cyclic group $\langle c_\alpha \rangle$. This, with smoothness, guarantees that $G/v(G)$ is freely generated in $\mathcal{V}$ by the set $C \cup \{c_\alpha, \alpha < \omega_1\}$. We have proved the result. ∎

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