THE WEAK-STAR CLOSURE OF THE UNIT BALL IN A HYPERPLANE

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1. Introduction

Let X be a normed linear space. We regard X as a subspace of its bidual X^{**}. Polars will always be evaluated in the pair (X^{**}, X^*). We denote the closed unit ball in X by U, so that U^0, U^{00} are the closed unit balls in X^*, X^{**} respectively. The weak topology induced by X on X^* (the "weak-star" topology) will be denoted by $\sigma(X)$, and cl( ) will denote $\sigma(X)$-closure.

Let ϕ be an element of X^{**} that is not in X, and let K be the kernel of ϕ. Then K is, of course, $\sigma(X)$-dense in X^*. When X is complete, the Krein-Šmul'yan theorem tells us that $K \cap U^0$ is not $\sigma(X)$-closed, but it gives no further information about the set $\mathrm{cl}(K \cap U^0)$. The purpose of this note is to determine $\mathrm{cl}(K \cap U^0)$ as accurately as possible (in doing so, we shall obtain incidentally a very simple proof of the Krein-Šmul'yan theorem for hyperplanes). The radius of the largest ball contained in $\mathrm{cl}(K \cap U^0)$ is known to be

$$\inf\{\| x - \lambda\phi \| : x \in X, \| x \| = 1, \lambda \text{ scalar}\}$$

((1), ch. IV, § 5, ex. 14). This last statement applies, in fact, when K is any $\sigma(X)$-dense linear subspace of X^* (with elements of K^0 replacing the multiples $\lambda\phi$), and is generalised to arbitrary linear subspaces in (3). However, $\mathrm{cl}(K \cap U^0)$ is clearly not just a multiple of U^0. Denoting by $d(\phi, X)$ the norm-distance from ϕ to X, we shall prove that

$$A(\phi) \subseteq \mathrm{cl}(K \cap U^0) \subseteq B(\phi),$$

where

$$A(\phi) = \left\{ f \in X^* : \| f \| + \frac{|\langle \phi, f \rangle|}{d(\phi, X)} \leqq 1 \right\},$$

$$B(\phi) = \{ f \in U^0 : |\langle \phi, f \rangle| \leqq 2d(\phi, X) \}.$$

In the particular case $X = c_0$, we show that $\mathrm{cl}(K \cap U^0)$ is always $A(\phi)$. In general, however, $\mathrm{cl}(K \cap U^0)$ can be either $A(\phi)$, $B(\phi)$ or something between the two.

The appearance of the ratio $|\langle \phi, f \rangle| / d(\phi, X)$ in the description is not as unreasonable as may at first seem, as the following considerations suggest:

(1) One would expect f to have a better chance of being in $\mathrm{cl}(K \cap U^0)$ if $|\langle \phi, f \rangle|$ is small.

(2) K is unchanged if ϕ is multiplied by a non-zero scalar. Hence the factor $|\langle\phi, f\rangle|$ will need to be balanced by something else that is multiplied by $|\lambda|$ when ϕ is replaced by $\lambda\phi$.

(3) If $d(\phi, X)$ is small, then ϕ is not far from being $\sigma(X)$-continuous. Consequently, one might expect $\mathrm{cl}(K\cap U^0)$ to be small.

The author is indebted to the referee for suggesting a better proof of Theorem 1, and for the comment in Note 3 to Theorem 1.

2. The theorems

Theorem 1. *If* $d(\phi, X)>0$, *then* $\mathrm{cl}(K\cap U^0)$ *contains* $A(\phi)$.

Proof. For the moment, let K be any linear subspace of X^*. It follows at once from the Hahn-Banach theorem, by extending the restriction to K, that

$$(K\cap U^0)^0 = K^0+U^{00}.$$

Hence $\mathrm{cl}(K\cap U^0)$ is the polar (in X^*) of $X\cap(K^0+U^{00})$. If K is now the kernel of ϕ, then K^0 is the linear span of ϕ. The result follows if we show that, for any f in X^*,

$$\sup\{|\langle x, f\rangle|:\ x\in X\cap(K^0+U^{00})\} \leqq \|f\|+\frac{|\langle\phi, f\rangle|}{d(\phi, X)}.$$

Let x be in $X\cap(K^0+U^{00})$. Then there exist ψ in U^{00} and a scalar λ such that $x=\lambda\phi+\psi$. Then $\|\lambda\phi-x\|\leqq 1$, so $|\lambda|\leqq 1/d(\phi, X)$, and

$$|\langle x, f\rangle| = |\langle\lambda\phi+\psi, f\rangle| \leqq |\lambda|\,.\,|\langle\phi, f\rangle|+\|f\|,$$

giving the required inequality.

Notes

(1) In particular, if $d(\phi, X)>0$, then $K\cap U^0$ is not $\sigma(X)$-closed. Hence we have proved the Krein-Šmul'yan theorem for hyperplanes: if X is complete and $K\cap U^0$ is $\sigma(X)$-closed, then K is $\sigma(X)$-closed.

(2) Similar reasoning can be applied to the common kernel of a finite number of functionals. Let $\phi_1, \ldots, \phi_m$ be elements of X^{**} such that

$$\inf\left\{\left\|\sum_{i=1}^{m}\lambda_i\phi_i-x\right\|:\ x\in X,\ \sum_{i=1}^{m}|\lambda_i|=1\right\}=r>0,$$

and let $K=\bigcap_{i=1}^{m}\ker\phi_i$. Then $\mathrm{cl}(K\cap U^0)$ contains

$$\left\{f\in X^*:\ \|f\|+\frac{1}{r}\max|\langle\phi_i, f\rangle|\leqq 1\right\}.$$

For if L is the linear span of $\phi_1, \ldots, \phi_m$, and x is in $X\cap(L+U^{00})$, then it is easily seen that

$$|\langle x, f\rangle|\leqq\|f\|+\frac{1}{r}\max|\langle\phi_i, f\rangle|.$$

(3) Let K be a linear subspace of X^*. Using the weak compactness of U^{00}, it is easy to show that

$$X \cap (K^0 + U^{00}) = \{x \in X: d(x, K^0) \leqq 1\}.$$

Hence $\mathrm{cl}(K \cap U^0)$ is precisely the dual unit ball when X is given the seminorm p, where $p(x) = d(x, K^0)$. This, of course, is the seminorm induced on X by the quotient norm in X/K^0. It is a norm when K is $\sigma(X)$-dense in X^*.

(4) The author's original proof of Theorem 1 was similar to the proof of the related result Corollary II, 4, 3 in (**2**). The proof given above was suggested by the referee.

Theorem 2. *$\mathrm{cl}(K \cap U^0)$ is contained in $B(\phi)$.*

Proof. $\mathrm{cl}(K \cap U^0)$ is contained in U^0, since U^0 is $\sigma(X)$-closed. Write $d(\phi, X) = r$. Suppose that $f \in U^0$ and $|\langle \phi, f \rangle| > 2r$: let $|\langle \phi, f \rangle| = 2r + 3\alpha$. Take $x_0 \in X$ such that $\| \phi - x_0 \| < r + \alpha$. Then $|\langle \phi - x_0, f \rangle| < r + \alpha$, since $\| f \| \leqq 1$, so $|\langle x_0, f \rangle| > r + 2\alpha$. For g in U^0, $|\langle \phi - x_0, g \rangle| < r + \alpha$, so if

$$|\langle x_0, g \rangle| > r + \alpha,$$

then $g \notin K$. Hence $\{g \in X^*: |\langle x_0, g - f \rangle| \leqq \alpha\}$ is disjoint from $K \cap U^0$.

Corollary. *If $d(\phi, X) = 0$, then $K \cap U^0$ is $\sigma(X)$-closed.*

Hence if X is an incomplete normed space, and ϕ is an element of $X^{**} \setminus X$ with $d(\phi, X) = 0$, then $K \cap U^0$ is $\sigma(X)$-closed, though K is $\sigma(X)$-dense in X^* (a fact noted by Kerr (**4**)).

The set $B(\phi)$ is not necessarily $\sigma(X)$-closed (cf. examples below).

3. Two particular cases

(i) $X = c_0$. We show that, in this case, $\mathrm{cl}(K \cap U^0)$ is always equal to $A(\phi)$.

Identify c_0^* with l_1 and c_0^{**} with m. We use the notation $x(n)$ for the nth term of a sequence x; we continue to use the notation $\langle\, ,\, \rangle$ for the evaluation of functionals. It is sufficient to consider $\phi \in m$ with $d(\phi, c_0) = 1$. Take an element f of l_1 that is not in $A(\phi)$: then

$$\| f \| + |\langle \phi, f \rangle| = 1 + 3\alpha$$

for some $\alpha > 0$. Since $d(\phi, c_0) = 1$, there exists N such that $|\phi(i)| \leqq 1 + \alpha$ for all $i > N$. Choose N so that, also,

$$\sum_{i=1}^{N} |f(i)| + \left| \sum_{i=1}^{N} \phi(i) f(i) \right| > 1 + 2\alpha.$$

There is a $\sigma(c_0)$-neighbourhood V of f such that, for $g \in V$,

$$\sum_{i=1}^{N} |g(i)| + \left| \sum_{i=1}^{N} \phi(i) g(i) \right| > 1 + \alpha. \tag{1}$$

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If $g \in V$ and $\langle \phi, g \rangle = 0$, then

$$\left| \sum_{N+1}^{\infty} \phi(i)g(i) \right| = \left| \sum_{1}^{N} \phi(i)g(i) \right|.$$

But

$$\left| \sum_{N+1}^{\infty} \phi(i)g(i) \right| \leqq (1+\alpha) \sum_{N+1}^{\infty} |g(i)|. \tag{2}$$

By (1) and (2),

$$\| g \| = \sum_{1}^{N} |g(i)| + \sum_{N+1}^{\infty} |g(i)| > \frac{1+\alpha}{1+\alpha} = 1.$$

Hence V does not meet $K \cap U^0$.

An example is given in (1) (loc. cit.) of a $\sigma(c_0)$-dense linear subspace E of l_1 (necessarily not a hyperplane) such that $\text{cl}(E \cap U^0)$ contains no multiple of U^0.

(ii) $X = l_1$. We use the following (more or less standard) notation: e denotes the sequence having 1 in each place, and e_n denotes the sequence having 1 in place n and 0 elsewhere.

Let L be a linear functional on m such that $L(c_0) = \{0\}$, $L(e) = 1$ and $\| L \| = 1$ (i.e. an "extended limit"; the existence of such functionals is guaranteed by the Hahn-Banach theorem). We show that $d(L, l_1) = 1$. Choose $x \in l_1$. For any $\varepsilon > 0$, there exists N such that

$$\sum_{N+1}^{\infty} |x(n)| \leqq \varepsilon.$$

Let $f_N = e - (e_1 + \ldots + e_N) \in m$. Then $\| f_N \| = 1$, $L(f_N) = 1$, and $|\langle x, f_N \rangle| \leqq \varepsilon$. Hence $\| L - x \| \geqq 1 - \varepsilon$.

First let $\phi = L$. Then K contains c_0, from which it follows that

$$\text{cl}(K \cap U^0) = U^0$$

(in general, U is $\sigma(X^*)$-dense in U^{00}). In this case, $\text{cl}(K \cap U^0)$ is $B(\phi)$.

Now take $k > 1$, and let $\phi = L + ke_1$. Then $d(\phi, l_1) = 1$. We show that:

(*a*) There exists f_0 in $\text{cl}(K \cap U^0)$ with $\langle \phi, f_0 \rangle = 2$.

(*b*) Given $\varepsilon > 0$, there exists $f_\varepsilon \notin \text{cl}(K \cap U^0)$ with $|\langle \phi, f_\varepsilon \rangle| \leqq \varepsilon$ and $\| f_\varepsilon \| = 1$.

Roughly speaking, this means that $\text{cl}(K \cap U^0)$ goes out as far as $B(\phi)$ in some directions, and only as far as $A(\phi)$ in others.

(*a*) Let

$$f_0 = e - \left(1 - \frac{1}{k}\right) e_1 = \left(\frac{1}{k}, 1, 1, \ldots\right).$$

Then $\langle \phi, f_0 \rangle = 2$, and f_0 is the $\sigma(l_1)$-limit of the sequence (f_n), where

$$f_n = \left(\frac{1}{k}, 1, \ldots, 1, -1, -1, \ldots\right) = -e + \left(1 + \frac{1}{k}\right) e_1 + \sum_{j=2}^{n} 2e_j.$$

Each f_n is in $K \cap U^0$.

(*b*) We may assume that $\varepsilon \leqq k-1$. Let

$$f_\varepsilon = \left(\frac{1+\varepsilon}{k}, -1, -1, \ldots\right) = \left[1 + \frac{1}{k}(1+\varepsilon)\right] e_1 - e.$$

Then $\langle \phi, f_\varepsilon \rangle = \varepsilon$. If $\| g \| \leqq 1$ and $g(1) > \dfrac{1}{k}\left(1 + \dfrac{\varepsilon}{2}\right)$, then $\langle ke_1, g \rangle > 1 + \dfrac{\varepsilon}{2}$, while $|\langle L, g \rangle| \leqq 1$, so $\langle \phi, g \rangle \neq 0$. Hence $f_\varepsilon \notin \mathrm{cl}(K \cap U^0)$.

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