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A NOTE ON ALMOST BALANCED BIPARTITIONS OF A GRAPH

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Abstract

Let G be a graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 4$. Arkin and Hassin ['Graph partitions with minimum degree constraints', *Discrete Math.* **190** (1998), 55–65] conjectured that there exists a bipartition S, T of $V(G)$ such that $\lfloor n/2 \rfloor - 2 \leq |S|, |T| \leq \lceil n/2 \rceil + 2$ and the minimum degrees in the subgraphs induced by S and T are at least two. In this paper, we first show that G has a bipartition such that the minimum degree in each part is at least two, and then prove that the conjecture is true if the complement of G contains no complete bipartite graph $K_{3,r}$, where $r = \lfloor n/2 \rfloor - 3$.

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1. Introduction

All graphs considered here are finite, simple and undirected graphs. Let $G = (V(G), E(G))$ be a graph. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraphs induced by S and $V(G) - S$, respectively. When $S = \{v\}$, we simplify $G - \{v\}$ to $G - v$. Let $N_S(v)$ be the set of the neighbours of a vertex v contained in S, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. A *k-vertex* is a vertex of degree k. We call k-vertices adjacent to v *k-neighbours* of v. The *minimum degree* of G is denoted by $\delta(G)$. Simply, we write $\delta(G[S])$ as $\delta(S)$. A complete bipartite graph of order $s + t$ is denoted by $K_{s,t}$. For $X, Y \subseteq V(G)$, define $(X, Y)_G = \{uv \in E(G) \mid u \in X, v \in Y\}$ and let $G[X, Y]$ be a graph with vertex set $X \cup Y$ and edge set $(X, Y)_G$. Let P be a path. We denote by $\overrightarrow{P}$ the path P with a given orientation and by $\overleftarrow{P}$ the path P with the reverse orientation. If $u, v \in V(P)$, then $u\overrightarrow{P}v$ ($u\overleftarrow{P}v$, respectively) denotes the consecutive vertices of P from u to v in the direction specified by $\overrightarrow{P}$ ($\overleftarrow{P}$, respectively). To *contract* an edge $e = uv$ of a graph G is to delete e, and then identify its ends and delete possible parallel edges. The resulting graph is

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denoted by G/e. A bipartition S, T of $V(G)$ is said to be *balanced* or *almost balanced* if $\lfloor n/2 \rfloor \le |S|, |T| \le \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor - 2 \le |S|, |T| \le \lceil n/2 \rceil + 2$. A bipartition S, T of $V(G)$ is said to be an (s,t)-*bipartition* if $\delta(S) \ge s$ and $\delta(T) \ge t$, where s, t are nonnegative integers.

In [6], Stiebitz showed that every graph with minimum degree at least $s + t + 1$ admits an (s,t)-bipartition. Kaneko [4] and Diwan [2] strengthened this result, proving that it suffices to assume that the minimum degree is at least $s + t$ or $s + t - 1$ ($s, t \ge 2$) if G contains no cycles shorter than four or five, respectively.

It is natural to ask an analogous question on balanced or almost balanced bipartitions. Let s and t be two nonnegative integers. Is there an integer k such that every graph with minimum degree at least k admits a balanced or an almost balanced (s,t)-bipartition? In [5], Maurer proved an interesting result, from which it is easy to see that every connected graph with minimum degree at least two admits a balanced $(1,1)$-bipartition.

Theorem 1.1. *Let G be a connected graph of order n with $\delta(G) \ge 2$. Then, for any positive integer l with $2 \le l \le n-2$, G admits a $(1,1)$-bipartition S, T such that $|S| = l$ and $|T| = n - l$.*

Arkin and Hassin [1] have given the following conjecture for graphs with minimum degree at least four.

Conjecture 1.2. Let G be a graph of order n with $\delta(G) \ge 4$. Then G admits an almost balanced $(2,2)$-bipartition.

Note that if a graph G has a $(2,2)$-bipartition, then clearly $|G| \ge 6$, and so we need only consider whether Conjecture 1.2 is true for $n \ge 6$. In [3], El-Zahar established the following theorem.

Theorem 1.3 (El-Zahar [3]). *Let G be a graph of order $n = n_1 + n_2$ with $\delta(G) \ge \lceil n_1/2 \rceil + \lceil n_2/2 \rceil$; then G contains two vertex disjoint cycles of lengths n_1 and n_2, where $n_1, n_2 \ge 3$ are two integers.*

By Theorem 1.3, if $\delta(G) \ge 4$, then G contains two vertex disjoint cycles of lengths $n_1 = n_2 = 3$ if $n = 6$, $n_1 = 3$ and $n_2 = 4$ if $n = 7$ and $n_1 = n_2 = 4$ if $n = 8$. This is to say that Conjecture 1.2 is true for $6 \le n \le 8$. Up to now, no results were obtained on Conjecture 1.2 and so it is still open. In this paper, we first improve a result on $(2,2)$-bipartitions due to Stiebitz [6] by showing the following theorem.

Theorem 1.4. *Let G be a graph of order $n \ge 6$ with $\delta(G) \ge 4$. Then G admits a $(2,2)$-bipartition.*

By Theorem 1.4 and the definition of almost balanced bipartitions, it is easy to obtain the following corollary.

Corollary 1.5. *Let G be a graph of order $n \ge 6$ with $\delta(G) \ge 4$. If $n \le 11$, then G admits an almost balanced $(2,2)$-bipartition.*

Next we show that Conjecture 1.2 is true under some additional constraint. This is the main result of this paper.

THEOREM 1.6. *Let G be a graph of order $n \geq 6$ with $\delta(G) \geq 4$. If $\overline{G}$ contains no $K_{3,r}$, then G admits an almost balanced $(2,2)$-bipartition, where $r = \lfloor n/2 \rfloor - 3$.*

Obviously, our result supports the truth of Conjecture 1.2.

2. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemma.

LEMMA 2.1. *Let G be a graph of order $n \geq 6$ with $\delta(G) \geq 4$. Then G contains two vertex disjoint cycles.*

PROOF. If G is disconnected, then the result holds trivially. So, we may assume that G is connected. Let $P = v_1 v_2 \cdots v_p$ be any longest path of G. By the maximality of P, we have $N_G(v_1) \subseteq V(P)$ and $N_G(v_p) \subseteq V(P)$. Since $d_G(v_1) \geq 4$, we may assume that $v_i, v_j \in N_G(v_1)$, where $2 < i < j < p$. If $v_1 v_p \notin E(G)$, then, since $d_G(v_p) \geq 4$, there is either some k with $i+1 \leq k < p-1$ such that $v_k \in N_G(v_p)$ or some l, m with $1 < l < m \leq i-1$ such that $v_l, v_m \in N_G(v_p)$. Thus, $v_1 \overrightarrow{P} v_i v_1$ and $v_k \overrightarrow{P} v_p v_k$ in the former case and $v_1 v_i \overrightarrow{P} v_j v_1$ and $v_p v_l \overrightarrow{P} v_m v_p$ in the latter case are two vertex disjoint cycles of G. Therefore, we must have $v_1 v_p \in E(G)$. In this case, $p = n$. Noting that $v_{i-1} \overleftarrow{P} v_1 v_i \overrightarrow{P} v_p$ and $v_{j-1} \overleftarrow{P} v_1 v_j \overrightarrow{P} v_p$ are also longest paths of G, by similar arguments as before, we may assume that $v_{i-1}, v_{j-1} \in N(v_p)$. If $j > i+1$, then $v_1 \overrightarrow{P} v_i v_1$ and $v_p v_{j-1} \overrightarrow{P} v_p$ are two vertex disjoint cycles and hence $j = i+1$. If $i > 3$, then $v_2 \overrightarrow{P} v_i v_1 v_j \overrightarrow{P} v_p$ is a longest path and so $v_2 v_p \in E(G)$. Thus, $v_1 v_i v_j v_1$ and $v_2 \overrightarrow{P} v_{i-1} v_p v_2$ are two vertex disjoint cycles. If $i = 3$, then, since $p = n \geq 6$, we have $4 = j < p-1$, whence $v_{p-1} \overleftarrow{P} v_i v_p v_{i-1} v_1$ is a longest path and so $v_1 v_{p-1} \in E(G)$. Hence, $v_p v_{i-1} v_i v_p$ and $v_1 v_j \overrightarrow{P} v_{p-1} v_1$ are two vertex disjoint cycles. □

PROOF OF THEOREM 1.4. By Lemma 2.1, G has two vertex disjoint subgraphs H_1 and H_2 such that $\delta(H_i) \geq 2$ for $i = 1, 2$. Choose H_1 and H_2 such that $|H_1| + |H_2|$ is as large as possible. If $|H_1| + |H_2| < n$, set $H_3 = G - V(H_1) - V(H_2)$. By the choice of H_1 and H_2, we have $d_{H_i}(h) \leq 1$ for any $h \in V(H_3)$ and $i = 1, 2$, which implies that $\delta(H_3) \geq 2$ since $\delta(G) \geq 4$. In this case, H_1 and $H_2 \cup H_3$ satisfy $\delta(H_1) \geq 2$ and $\delta(H_2 \cup H_3) \geq 2$, which contradicts the choice of H_1 and H_2. □

3. Proof of Theorem 1.6

PROOF OF THEOREM 1.6. We will use induction on n. By Corollary 1.5, Theorem 1.6 holds for $6 \leq n \leq 11$. Now we assume that $n \geq 12$ and that the result holds for all small n. In the following, we let $r = \lfloor n/2 \rfloor - 3$.

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Firstly, we show that G admits a $(2,2)$-bipartition S,T such that $||S|-|T|| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$.

If there exists $x \in V(G)$ such that $\delta(G-x) \geq 4$, then $\delta(G') \geq 4$ and $\overline{G'}$ contains no $K_{3,r}$, as $\overline{G'} \subseteq \overline{G}$, where $G' = G - x$. Thus, G' admits an almost balanced $(2,2)$-bipartition S',T' by induction. Since $d_G(x) \geq 4$, we have $d_{S'}(x) \geq 2$ or $d_{T'}(x) \geq 2$, say $d_{S'}(x) \geq 2$. Let $S = S' \cup \{x\}$, $T = T'$; then S,T is a $(2,2)$-bipartition of G and $||S|-|T|| = ||S'|+1-|T'|| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$. So, we assume that

$$\delta(G-x) = 3 \quad \text{for any } x \in V(G). \tag{$*$}$$

By ($*$), we have $\delta(G) = 4$ and, if $d_G(x) = 4$, then x has a 4-neighbour.

Let u,v be two adjacent 4-vertices.

Claim 1. u,v have at most two common neighbours.

To prove our claim, suppose that u,v have three common neighbours w_1,w_2,w_3. Denote $X = V(G) - \{u,v,w_1,w_2,w_3\}$; then $N_{\overline{G}}(u) = N_{\overline{G}}(v) = X$. Since $\overline{G}$ contains no $K_{3,r}$, we have $|N_{\overline{G}}(w) \cap X| \leq r-1$ for any $w \in V(G) - \{u,v\}$. Then $|N_G(w_i) \cap X| \geq n-r-4$ for $1 \leq i \leq 3$ and $|N_G(x) \cap X| \geq n-r-5$ for any $x \in X$. Noting that $d_G(x) \geq 4$, we have $|N_G(x) \cap (V(G) - \{u,v,w_1,w_2\})| \geq 2$. If $n = 12$, then $|N_G(w_3) \cap (V(G) - \{u,v,w_1,w_2\})| \geq n-r-4 = 5$; thus, $\{u,v,w_1,w_2\}, V(G) - \{u,v,w_1,w_2\}$ is an almost balanced $(2,2)$-bipartition of G. If $n \geq 13$, let $x_0 \in X$; then $x_0u, x_0v \in E(\overline{G})$. Since $|N_G(x) \cap (X - x_0)| \geq n-r-6 = n - \lfloor n/2 \rfloor - 3 \geq 4$ for any $x \in X - x_0$ and $|N_G(w_i) \cap (X - x_0)| \geq n-r-5 = n - \lfloor n/2 \rfloor - 2 \geq 5$ for $1 \leq i \leq 3$, we have $\delta(G - x_0) \geq 4$, which contradicts ($*$). This proves our claim.

Claim 2. If u,v have a common neighbour w, then $d_G(w) \geq 5$.

To prove our claim, suppose to the contrary that $d_G(w) = 4$. Then $|V(G) - N_G[u] - N_G[v] - N_G[w]| \geq n-9 \geq \lfloor n/2 \rfloor - 3 = r$ that is, there exist $x_1,x_2,\ldots,x_r \in V(G) - N_G[u] - N_G[v] - N_G[w]$ and $\overline{G}[\{u,v,w\},\{x_1,x_2,\ldots,x_r\}]$ is a $K_{3,r}$, which is a contradiction. This proves our claim.

Let $G' = G/uv$ and denote by w the vertex resulting from the contraction of e. By Claim 1, we have $|N_G(u) \cup N_G(v)| \geq 4$ and $d_{G'}(w) \geq 4$. For any $x \in V(G) - \{u,v\}$, if x is not a common neighbour of u,v, we have $d_{G'}(x) = d_G(x) \geq 4$; if x is a common neighbour of u,v, we have $d_{G'}(x) = d_G(x) - 1 \geq 4$ by Claim 2. Therefore, $\delta(G') \geq 4$. Since $\overline{G'}$ contains no $K_{3,r}$, by induction, G' admits an almost balanced $(2,2)$-bipartition S',T'. Assume without loss of generality that $w \in S'$. If $d_{S'-w}(u) = 0$, then we must have $d_{T'}(u) \geq 3$, as $d_G(u) \geq 4$. Since $d_{S'}(w) \geq 2$, we have $d_{S'-w}(v) \geq 2$. Let $S = (S'-w) \cup \{v\}$, $T = T' \cup \{u\}$; then S,T is a $(2,2)$-bipartition of G and $||S| - |T|| = ||S'| - |T'| - 1| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$. Thus, by symmetry of u and v, we may assume that $d_{S'-w}(u) \geq 1$ and $d_{S'-w}(v) \geq 1$. Let $X = (S'-w) \cup \{u,v\}$, $T = T'$; then S,T is a $(2,2)$-bipartition of G and $||S|-|T|| = ||S'|+1-|T'|| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$.

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Next, we show that G admits an almost balanced $(2,2)$-bipartition.

By the argument above, G admits a $(2,2)$-bipartition S,T with $||S|-|T|| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$. Assume without loss of generality that $|S| \geq |T| \geq 3$. If $|S|-|T| \leq \lceil n/2 \rceil - \lfloor n/2 \rfloor + 4$, then we see that S,T is an almost balanced $(2,2)$-bipartition. Therefore, we have $\lceil n/2 \rceil - \lfloor n/2 \rfloor + 5 \leq |S| - |T| \leq \lceil (n-1)/2 \rceil - \lfloor (n-1)/2 \rfloor + 5$; thus, n is even and $|S| = \lceil (n-1)/2 \rceil + 3 = \lfloor n/2 \rfloor + 3$ and $|T| = \lfloor (n-1)/2 \rfloor - 2 = \lfloor n/2 \rfloor - 3$. In the following, we will show that G admits an almost balanced $(2,2)$-bipartition under these conditions.

If $\delta(S) = 2$, we assume that H is any component of the subgraph induced by the 2-vertices of $G[S]$ and $V(H) = \{x_1, \ldots, x_p\}$. Clearly, H is a path or a cycle. Let $P = x_1 x_2 \cdots x_p$ be a path in H. Since $d_S(x_i) = 2$ and $d_G(x_i) \geq 4$, we have $d_T(x_i) \geq 2$ for $1 \leq i \leq p$. If $p \geq 3$, then $|S - \bigcup_{i=1}^{3} N_S(x_i)| \geq |S| - 5 = \lfloor n/2 \rfloor - 2 \geq r$. Let $y_1, y_2, \ldots, y_r \in S - \bigcup_{i=1}^{3} N_S(x_i)$. Then $\overline{G}[\{x_1, x_2, x_3\}, \{y_1, y_2, \ldots, y_r\}]$ is a $K_{3,r}$, which is a contradiction. Therefore, $p \leq 2$. If $p = 2$ and x_1, x_2 have a common neighbour x in S, then $d_S(x) \geq 4$. Otherwise, $|S - N_S[x] - N_S[x_1] - N_S[x_2]| \geq |S| - 4 = \lfloor n/2 \rfloor - 1 > r$, that is, there exist $y_1, y_2, \ldots, y_r \in S - N_S[x] - N_S[x_1] - N_S[x_2]$; then $\overline{G}[\{x, x_1, x_2\}, \{y_1, y_2, \ldots, y_r\}]$ is a $K_{3,r}$, which is a contradiction. Thus, $\delta(S - \{v_1, \ldots, v_p\}) \geq 2$. Let $S^* = S - V(P)$, $T^* = T \cup V(P)$. Noting that $\delta(T) \geq 2$, $d_T(x_i) \geq 2$, we have $\delta(T \cup V(P)) \geq 2$. Since $||S^*| - |T^*|| = |S| - |T| - 2p \leq 4$, we see that S^*, T^* is an almost balanced $(2,2)$-bipartition of G.

Now we assume that $\delta(S) \geq 3$. Denote $X = \{x \mid x \in S \text{ and } d_T(x) \geq 1\}$. Then X contains all 3-vertices of $G[S]$. Since $|T| = \lfloor n/2 \rfloor - 3 = r$ and $\overline{G}$ contains no $K_{3,r}$, we have $(S,T)_G \neq \emptyset$, that is, $X \neq \emptyset$. Choose $x \in X$ such that $d_S(x)$ is as small as possible and let x_1 be a neighbour of x in T. Since $\overline{G}$ contains no $K_{3,r}$, we have $(N_S(x), T)_G \neq \emptyset$. Choose $y \in N_S(x) \cap X$ such that $d_S(y)$ is as small as possible and let y_1 be a neighbour of y in T. If $\delta(S - \{x,y\}) = 1$, then $\delta(S) = 3$ and x, y have a common neighbour z of degree three in S. Noting that $z \in X$, by the choice of x, y, we have $d_S(x) = d_S(y) = 3$. Since $|S - N_S[x] - N_S[y] - N_S[z]| \geq |S| - 6 = \lfloor n/2 \rfloor - 3 = r$, there exist $w_1, w_2, \ldots, w_r \in S - N_S[x] - N_S[y] - N_S[z]$. But then $\overline{G}[\{x,y,z\}, \{w_1, w_2, \ldots, w_r\}]$ is a $K_{3,r}$, which is a contradiction. Therefore, we have $\delta(S - \{x,y\}) \geq 2$. Noting that $\delta(T) \geq 2$, $\{x_1, y\} \subseteq N_{T \cup \{x,y\}}(x)$ and $\{x, y_1\} \subseteq N_{T \cup \{x,y\}}(y)$, we have $\delta(T \cup \{x,y\}) \geq 2$. Let $S^* = S - \{x,y\}$, $T^* = T \cup \{x,y\}$. Since $||S^*| - |T^*|| = |S| - |T| - 4 = 2$, we see that S^*, T^* is an almost balanced $(2,2)$-bipartition of G.

Therefore, the proof of Theorem 1.6 is complete. □

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