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POLYNOMIAL APPROXIMATION OF AN ENTIRE FUNCTION AND RATE OF GROWTH OF TAYLOR COEFFICIENTS

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1. Introduction and results

The best uniform approximation of a function f on $[-1,1]$ by real algebraic polynomials satisfies

$$\lim_{n\to\infty} \{E_n[f]\}^{1/n} = 0, \tag{1.1}$$

if and only if f is the restriction to $[-1,1]$ of an entire function (Bernstein [**2**], p. 113, see also [**12**], pp. 83–85). For such functions f the rate of best approximation has been characterized by Varga [**24**], Reddy [**14**], Shah [**21**], and Kapoor and Nautiyal [**10**] in terms of order and type of f, lower order and type, and in terms of more general concepts of order. On the other hand, order and type of f are connected with the Taylor coefficients, i.e. with the rate of growth of the sequence $\{f^{(k)}(0)\}_{k\in\mathbb{N}}$ (see [**23**], p. 41 or [**3**], pp. 11/12; cf. also [**19**], [**20**], [**6**], [**7**], [**8**]) and this has been extended to iterated orders by Schönhage [**17**], Sato [**16**], Reddy [**14**], Juneja, Kapoor, and Bajpai [**9**] (also [**22**], [**13**]), and to generalized orders by Seremeta [**18**], Bajpai, Gautam, and Bajpai [**1**] as well as Kapoor and Nautiyal [**10**]. Combining the two kinds of characterizations (as done, e.g., by Reddy [**15**], p. 105) approximation theorems in terms of the sequence $\{f^{(k)}(0)\}_{k\in\mathbb{N}}$ are obtained. But in such results the rate of best approximation is always described by a limit relation, e.g. of the form $\limsup_{n\to\infty} n(E_n[f])^{\rho/n} = \tau\rho e 2^{-\rho}$, and this causes a considerable loss of precision, as will be discussed in more detail in Section 3 (in this respect cf. also the remark by Bernstein [**2**], pp. 114/115).

The purpose of this paper is to derive sharper results for part of the classes of functions considered in the above papers, including functions of order $\leqq 2$ and zero order, without employing some concept of order as an intermediate step. Also the sequence of maximum norms of $f^{(k)}$ on $[-1,1]$ as well as the Fourier Chebychev coefficients will be used for further characterizations.

The following notations will be needed. Let $C[-1,1]$ denote the space of continuous functions on the interval $[-1,1]$, with maximum norm, and $E_n[f] = \inf_{p\in\mathscr{P}_n} \|f-p\|$, where $\mathscr{P}_n$ is the set of polynomials of degree at most n. For rates of best approximation the elements φ of the following classes Ω_β will be admitted.

$$\Omega_\beta = \{\varphi;\, \varphi \in C^1(x_\beta, \infty) \text{ for some } x_\beta > 1, \varphi(x) > 0,$$
$$(\log\varphi)'(x) \geqq \beta \log x \text{ for each } x > x_\beta\}. \tag{1.2}$$

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Here β is a positive number, and $C^1(x_\beta, \infty)$ denotes the set of functions which have a continuous derivative on (x_β, ∞). Roughly speaking, Ω_β consists of functions φ which increase at least as rapidly as $c\exp\{\beta x(\log x - 1)\}$ for some constant $c > 0$.

Setting $h_0 = (1/\pi)^{1/2}, h_k = (2/\pi)^{1/2}$ for $k \in \mathbb{N}$ and $T_k(x) = \cos(k \arccos x)$, $x \in [-1, 1]$, $k \in \mathbb{P} = \{0, 1, 2, \dots\}$, the Fourier Chebychev coefficients of a function $f \in C[-1, 1]$ are defined by

$$c_k(f) = h_k \int_{-1}^{1} f(x) T_k(x)(1-x^2)^{-1/2} dx \quad (k \in \mathbb{P}).$$

Our main results are as follows.

Theorem 1. *Let $f \in C[-1,1]$ and $\varphi \in \Omega_\beta$ for some $\beta \geqq 1$. The following are equivalent.*

(i) $E_n[f] = \mathcal{O}(1/\varphi(n+1)), n \to \infty$,

(ii) $\|f^{(r)}\| = \mathcal{O}\left(\dfrac{2^r r!}{\varphi(r)}\right), r \to \infty$,

(iii) $|f^{(r)}(0)| = \mathcal{O}\left(\dfrac{2^r r!}{\varphi(r)}\right), r \to \infty$.

If the assertion (ii) is omitted, the restriction on β can be relaxed somewhat:

Theorem 2. *Let $f \in C[-1,1]$ and $\varphi \in \Omega_\beta$ for some $\beta \geqq 1/2$. Then conditions* (i) *and* (iii) *of Theorem* 1 *are equivalent.*

2. Proofs

We need three elementary Lemmas.

Lemma 1. *Let $f \in C[-1,1]$ and suppose that* (1.1) *holds. The Chebychev coefficients $c_k(f)$ can be expressed in terms of the Taylor coefficients $a_k = f^{(k)}(0)/k!$ and vice versa, as follows.*

$$c_k(f) = h_k \pi 2^{-k} \sum_{j=0}^{\infty} \binom{k+2j}{j} a_{k+2j} 2^{-2j} \qquad (k \in \mathbb{P}), \tag{2.1}$$

$$a_k = \frac{2^k c_k(f)}{\pi h_k} + \frac{2^k}{\sqrt{2\pi}} \sum_{j=1}^{\infty} (-1)^j \left(1 + \frac{j}{j+k}\right) \binom{j+k}{j} c_{2j+k}(f) \qquad (k \in \mathbb{P}). \tag{2.2}$$

Proof. Equation (2.1) was given, e.g., by Bernstein [**2**], p. 116, and equation (2.2) follows by observing that, in view of (1.1) and [**23**], p. 245, the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} h_k c_k(f) T_k(x)$$

converge for each $x \in [-1, 1]$, then inserting

$$T_0(x)=1,\; T_n(x)=\frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k} \qquad (n\in\mathbb{N}),$$

(cf. [11], p. 297 (6)) and comparing coefficients.

Lemma 2. *For each $k\in\mathbb{N}, j\in\mathbb{P}$ one has*

$$\binom{k+2j}{j}\leqq 2^{2j}\binom{k+j}{j}. \tag{2.3}$$

Proof. Setting $a(k,j)=\binom{k+2j}{j}/\binom{k+j}{j}$, one has to show that $a(k,j)\leqq 2^{2j}$ for each $j\in\mathbb{P}$. Since $a(k+1,j)\leqq a(k,j)$ for each $k\in\mathbb{N}, j\in\mathbb{P}$, as is easily seen, it suffices to prove that

$$a(1,j)\leqq 2^{2j} \qquad (j\in\mathbb{P}). \tag{2.4}$$

Now

$$a(1,j)=\frac{2j+1}{(j+1)^2}\frac{(2j)!}{(j!)^2}\leqq\frac{(2j)!}{(j!)^2}\leqq 2^{2j} \qquad (j\in\mathbb{P}),$$

where the last inequality follows by induction, and the proof is complete.

Assertion (2.7) of the following lemma is a known characterization of condition (i) of Theorems 1, 2 in terms of Fourier Chebychev coefficients. It is a slightly modified version of a result of Bernstein (see, e.g., [5], p. 107 or [12], Theorem 74), where the hypothesis is $\sum_{j=0}^{\infty}|c_{j+n+1}(f)|=\mathcal{O}(|c_{n+1}(f)|), n\to\infty$, instead of (2.6).

Lemma 3. *Let $f\in C[-1,1]$ and $\varphi\in\Omega_\beta$ for some $\beta>0$. Then*

$$\varphi(r+j)/\varphi(r)\geqq r^{j\beta} \qquad (j\in\mathbb{P}, r>x_\beta), \tag{2.5}$$

$$\sum_{j=0}^{\infty}\frac{1}{\varphi(j+n+1)}=\mathcal{O}\left(\frac{1}{\varphi(n+1)}\right),\; n\to\infty. \tag{2.6}$$

Moreover, (2.6) *implies that*

$$E_n[f]=\mathcal{O}\left(\frac{1}{\varphi(n+1)}\right), n\to\infty \text{ if and only if } |c_k(f)|=\mathcal{O}\left(\frac{1}{\varphi(k)}\right),\; k\to\infty. \tag{2.7}$$

Proof. By (1.2) and the mean value theorem one has for each $r>x_\beta$, setting $g(x)=\log\varphi(x)$,

$$g(r+j)-g(r)=jg'(r+\delta j)\geqq j\beta\log r \qquad (\delta\in(0,1), j\in\mathbb{P}),$$

and thus (2.5). This implies

$$\sum_{j=0}^{\infty}\frac{1}{\varphi(j+n+1)}\leqq\frac{1}{\varphi(n+1)}\sum_{j=0}^{\infty}(n+1)^{-j\beta}=\mathcal{O}\left(\frac{1}{\varphi(n+1)}\right),\; n\to\infty,$$

i.e. (2.6). The one part of (2.7) is an immediate consequence of the standard inequality

$$|c_k(f)| \leqq \frac{2\sqrt{2}}{\sqrt{\pi}} E_{k-1}[f] \tag{2.8}$$

(cf., e.g. [5], p. 107, (8.41)). Conversely, if $|c_k(f)| = \mathcal{O}(1/\varphi(k)), k \to \infty$, the Fourier–Chebychev series of f is uniformly convergent on $[-1, 1]$ in view of (1.2), and (2.6) implies that

$$E_n[f] \leqq \left\| f(x) - \sum_{k=0}^{n} h_k c_k(f) T_k(x) \right\| \leqq \sum_{k=n+1}^{\infty} h_k |c_k(f)|$$

$$= \mathcal{O}\left(\sum_{j=0}^{\infty} \frac{1}{\varphi(j+n+1)} \right) = \mathcal{O}\left(\frac{1}{\varphi(n+1)} \right), \ n \to \infty.$$

Proof of Theorem 1. If (i) is assumed, it follows by (1.2) that (1.1) is satisfied. Thus (2.2) can be used. Inserting (2.8) and (i) into (2.2), and observing (2.5), we have

$$|a_k| \leqq \mathcal{O}\left(\frac{1}{\varphi(k)} \right) + \frac{M 2^{k+1}}{\sqrt{2\pi}\, \varphi(k)} \sum_{j=0}^{\infty} \binom{j+k}{j} k^{-2j\beta} \qquad (k > x_\beta),$$

and, by the binomial theorem,

$$\sum_{j=0}^{\infty} \binom{j+k}{j} k^{-2j\beta} = (1 - k^{-2\beta})^{-(k+1)} \qquad (k > x_\beta),$$

which remains bounded, as $k \to \infty$, if and only if $\beta \geqq \frac{1}{2}$. This proves the implication (i)⇒(iii).

If (iii) holds, (1.2) implies again that the Taylor expansion of f converges uniformly on $[-1, 1]$, so that, for each $r \in \mathbb{P}$,

$$\|f^{(r)}(x)\| = \left\| \sum_{k=r}^{\infty} \frac{f^{(k)}(0)}{(k-r)!} x^{k-r} \right\| \leqq M 2^r r! \sum_{j=0}^{\infty} \binom{r+j}{j} \frac{2^j}{\varphi(j+r)}.$$

Using (2.5), (2.9), and (2.10), we find for each $r > x_\beta$

$$\|f^{(r)}\| \leqq M \frac{2^r r!}{\varphi(r)} \sum_{j=0}^{\infty} \binom{r+j}{j} \left(\frac{2}{r^\beta} \right)^j = \mathcal{O}\left(\frac{2^r r!}{\varphi(r)} \right), \ r \to \infty,$$

where the last equation holds provided $\beta \geqq 1$. This proves (ii).

The implication (ii)⇒(i) is an immediate consequence of the well-known inequality (see e.g. [5], p. 103)

$$E_n[f] \leqq 2 \|f^{(n+1)}\| \frac{1}{2^n (n+1)!} \qquad (n \in \mathbb{P}),$$

and the proof is complete.

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Proof of Theorem 2. As has been noted in the above proof, the implication (i)⇒(iii) remains valid for $\beta \geqq 1/2$.

If (iii) holds, the Taylor expansion of f is uniformly convergent on $[-1,1]$, so that

$$E_n[f] \leqq \left\| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \right\| \leqq \sum_{k=n+1}^{\infty} \frac{|f^{(k)}(0)|}{k!} \leqq M 2^{n+1} \sum_{j=0}^{\infty} \frac{2^j}{\varphi(n+1+j)}.$$

Using (2.5), it follows as in the proof of (2.6) that $E_n[f] = \mathcal{O}(2^{n+1}/\varphi(n+1))$, $n \to \infty$, so that (1.1) is satisfied. Now (2.1) can be employed, and hence by (iii) we have for each $k \in \mathbb{N}$

$$|c_k(f)| \leqq \sqrt{2\pi}\, 2^{-k} \sum_{j=0}^{\infty} \binom{k+2j}{j} |a_{k+2j}| 2^{-2j} \leqq M\sqrt{2\pi} \sum_{j=0}^{\infty} \binom{k+2j}{j} \frac{1}{\varphi(k+2j)}.$$

Using (2.5) once more and Lemma 2, one has

$$|c_k(f)| \leqq M \frac{\sqrt{2\pi}}{\varphi(k)} \sum_{j=0}^{\infty} \binom{k+j}{j} \left(\frac{4}{k^{2\beta}}\right)^j \qquad (k > x_\beta),$$

and as in the proof of Theorem 1 it follows that the latter sum is bounded, provided $\beta \geqq 1/2$. Thus (i) follows in view of (2.7).

3. Remarks

In connection with Theorem 2, a result of Bernstein [2], p. 116 (cf. also [12], p. 89) is to be mentioned which states that under the condition $\lim_{n\to\infty} \sqrt{n}|a_n|^{1/n} = 0$ there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{E_{n_k}[f]^{2n_k}}{|a_{n_k+1}|} = 1.$$

Bernstein's hypothesis is slightly more restrictive than our requirement $\beta \geqq 1/2$.

We further compare the above results with known characterizations in terms of order and type of an entire function ([14], Thm. 3, [3], p. 11/12). For $f \in C[-1,1]$, $0 < \rho < \infty$, $0 \leqq \tau < \infty$, the following are equivalent:

$$f \text{ is the restriction to } [-1,1] \text{ of an entire function of order } \rho \text{ and type } \tau, \tag{3.1}$$

$$\limsup_{n\to\infty} n(E_n[f])^{\rho/n} = \tau\rho e 2^{-\rho}, \tag{3.2}$$

$$\limsup_{n\to\infty} n(|f^{(n)}(0)|/n!)^{\rho/n} = \tau\rho e. \tag{3.3}$$

In particular, setting

$$\varphi_1(x) = \left(\frac{x 2^\rho}{\rho e \tau}\right)^{x/\rho},\ \varphi_2(x) = e^{x^\alpha}\varphi_1(x),\ \varphi_3(x) = e^{-x^\alpha}\varphi_1(x) \tag{3.4}$$

for some $\alpha \in (0,1)$, all functions f with the property that

$$E_n[f] = 1/\varphi_j(n) \qquad (n \in \mathbb{N}), \tag{3.5}$$

for some $j = 1,2,3$ satisfy (3.2) with same values of τ and ρ. Similarly, all f with the property that

$$|f^{(n)}(0)| = n!2^n/\varphi_j(n) \qquad (n \in \mathbb{N}), \tag{3.6}$$

for some $j = 1,2,3$ satisfy (3.3) with same τ and ρ. Thus the above characterization in terms of ρ and τ does not allow to distinguish between different values of j. If we restrict ρ and τ to $0 < \rho \leqq 2$, $0 \leqq \tau < 2^\rho/\rho$, however, the above φ_j are in Ω_β with $\beta = 1/\rho$, so that Theorem 2 associates the cases $j = 1,2,3$ in (3.5) and (3.6) to each other in the right order.

More refined characterizations than those in (3.1)–(3.3) were given by Reddy **[14]**, Seremeta **[18]** (who generalized results of Schönhage **[17]**) and S.M. Shah **[21]**. They used more general concepts of an order which make sense in cases where the usual order is infinite. But due to our definition of Ω_β there is no overlap between their results and the present paper.

There is, however, an overlap with results of Kapoor and Nautiyal **[10]** who defined as the generalized order of an entire function f the quantity

$$\rho(\alpha, \alpha, f) = \limsup_{r \to \infty} \frac{\alpha(\log M(r,f))}{\alpha(\log r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ and $\alpha(x)$ is a nonnegative, increasing function to be chosen from certain sets Ω, $\bar{\Omega}$ (see **[10]**, p. 65).

Setting $P(L) = \max\{1, L\}$ if $\alpha \in \Omega$ and $P(L) = 1 + L$ if $\alpha \in \bar{\Omega}$, and defining, for a given entire f, a strictly increasing sequence $\{\lambda_n\}_{n=0}^\infty$ of naturals such that $\lambda_0 = 0, f(z) = \sum_{n=0}^\infty a_n z^{\lambda_n}$ with $a_n \neq 0$ for all n, the results in **[10]** (Theorems 1 and 4) can be interpreted as follows (for the case $\alpha(x) = \log x$ cf. also Reddy **[14]**, Thm. 5 and **[15]**, La. 3).

Let f satisfy (1.1). The following are equivalent.

f is the restriction to $[-1,1]$ of an entire function of order $\rho(\alpha, \alpha, f) = \rho$, (3.7)

$$P\left(\limsup_{n \to \infty} \frac{\alpha(n)}{\alpha\left(\dfrac{1}{n} \log \dfrac{1}{E_n[f]}\right)}\right) = \rho, \tag{3.8}$$

$$P\left(\limsup_{n \to \infty} \frac{\alpha(\lambda_n)}{\alpha\left(\lambda_n^{-1} \log \dfrac{n!}{|f^{(n)}(0)|}\right)}\right) = \rho. \tag{3.9}$$

This can be applied, e.g., to $\alpha(x) = \log x$ (then $\alpha \in \bar{\Omega}$) and $f \in C[-1,1]$ with $E_n[f] = 1/\varphi_j(n)$ for each $n \in \mathbb{N}$, where $j = 4,5$ and

$$\varphi_4(x) = \exp(\gamma x^\tau), \varphi_5(x) = \exp(\delta x^\tau(1 + \log x)) \qquad (\gamma, \delta > 0, \tau > 1), \tag{3.10}$$

with the result that (3.8) is always satisfied with $\rho=\tau/(\tau-1)$. Thus condition (3.8) is not suited to distinguish between φ_4 and φ_5. The same holds with respect to property (3.9).

The situation is similar in case $\alpha_k(x)=\log_{k+1}(x)$, for some $k\in\mathbb{N}$, where $\log_1(x)=\log x, \log_{k+1}(x)=\log(\log_k(x))$, and $f_k\in C[-1,1]$ with $E_n[f_k]=1/\psi_k(n)$ for each $n\in\mathbb{N}$, where

$$\psi_k(x)=\exp\{\gamma x \exp_k[(\log_k(x))^{1/L}]\} \tag{3.11}$$

for x large enough, $L>1$, $\gamma>0$ and $\exp_1(x)=\exp(x)$, $\exp_{k+1}(x)=\exp(\exp_k(x))$. For each $k\in\mathbb{N}$, condition (3.8) is satisfied with $\rho(\alpha_k,\alpha_k,f_k)=L$, so that, again, the choice of γ has no influence upon the generalized order.

An application of Theorem 1, however, will produce sharp results for the above examples, i.e. to different φ,ψ in (3.10), (3.11) different rates of increase of $\{f^{(r)}(0)\}_{r\in\mathbb{N}}$ are assigned.

The above phenomena are due to the fact that in the definitions of order, type, and generalized orders, the maximum modulus $M(r)$ is compared with a very special set of reference functions only. So the lack of precision there does not imply that $M(r)$ itself would be useless for characterizing rates of best approximation. In this respect see also the forthcoming paper [**4**].

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