

NOTE ON ODD MULTIPERFECT NUMBERS

LI-XIA DAI, HAO PAN[✉] and CUI-E TANG

(Received 9 May 2012; accepted 26 June 2012; first published online 17 September 2012)

Abstract

For $k \geq 2$ and $r \geq 1$, we prove that the number of odd k -perfect numbers with r distinct prime factors is at most $4^{r^2}(k-1)^{2r^2+3}$.

2010 Mathematics subject classification: primary 11A05; secondary 11A25, 05A17.

Keywords and phrases: multiperfect number, multiplicative partition function.

For a positive integer n , let $\sigma(n)$ denote the sum of all divisors of n . We say that n is a perfect number if $\sigma(n) = 2n$. One of the most famous conjectures in number theory is that there exists no odd perfect number. However, no significant progress has been made toward this problem up to now. In 1913, Dickson [4] showed that for each $r \geq 1$, there are only finitely many odd perfect numbers n with $\omega(n) \leq r$, where $\omega(n)$ denotes the number of distinct prime factors of n . The first explicit upper bound for the number of odd perfect numbers with at most r distinct prime factors was given by Pomerance [9], who showed that any such n satisfies $n \leq (4r)^{(4r)^{2r^2}}$. Later, the bound was improved by Cook [3], Heath-Brown [5] and Nielsen [7]. Recently, with the help of an elementary discussion, Pollack [8] obtained the current best bound and showed that there exist at most 4^{r^2} odd perfect numbers n with $\omega(n) \leq r$.

Subsequently, Chen and Luo [1] extended Pollack's method to multiperfect numbers: a positive integer n is called α -perfect for rational α if $\sigma(n) = \alpha n$, and multiperfect or k -perfect when α is an integer k . They proved that the number of odd k -perfect numbers with at most r distinct prime factors is not greater than $(k-1) \cdot 4^{r^3}$. Unfortunately, Chen and Luo's result does not reduce to Pollack's when $k = 2$. The main purpose of this very short note is to fix this flaw.

The second author is supported by the National Natural Science Foundation of China (Grant No. 10901078).

© 2012 Australian Mathematical Publishing Association Inc. 0004-9727/2012 \$16.00

THEOREM 1. *Suppose that $k \geq 2$ and $r \geq 1$. Then the number of odd k -perfect numbers n with $\omega(n) \leq r$ is bounded by $4^{r^2}(k-1)^{2r^2+3}$.*

Notice that McCarthy [6] has shown that if n is k -perfect then $\omega(n) \geq k^2 - 1$. So our bound $4^{r^2}(k-1)^{2r^2+3} = (k-1) \cdot 4^{r^2+(r^2+1)\log_2(k-1)}$ is surely better than $(k-1) \cdot 4^{r^3}$ for any $k \geq 2$ and $r \geq k^2 - 1$.

PROOF OF THEOREM 1. Below we may assume that $k \geq 3$. For a positive integer n , we say that n is primitive if $d \nmid \sigma(d)$ for each proper divisor $d > 1$ of n with $(d, n/d) = 1$. Let

$$I_{\alpha,r}(x) = \{n \leq x : n \text{ is an odd primitive } \alpha\text{-perfect number and } \omega(n) \leq r\}.$$

In [1, Lemma 3], Chen and Luo proved that

$$|I_{\alpha,r}(x)| \leq \frac{2.62}{\alpha^2 - 1} (\log x)^r.$$

Furthermore, if α is an integer they showed that $|I_{\alpha,r}(x)| \leq 0.02(\log x)^r$.

Now suppose that $\sigma(n) = kn$. If n is not primitive, then let $1 < d_1 < n$ be the least divisor of n satisfying $(d_1, n/d_1) = 1$ and $d_1 \mid \sigma(d_1)$. Let $k_1 = \sigma(d_1)/d_1$. Clearly d_1 is an odd primitive k_1 -perfect number. Suppose that n/d_1 is still not primitive. Let $d_2 > 1$ be the least proper divisor of n/d_1 such that $(d_2, n/(d_1d_2)) = 1$ and $d_2 \mid \sigma(d_2)$, and let $k_2 = \sigma(d_2)/d_2$. Continue this process until $n/(d_1d_2 \cdots d_l)$ is primitive, where $k_i \geq 2$ is an integer and d_i is an odd primitive k_i -perfect number for every $1 \leq i \leq l$. Let $d_{l+1} = n/(d_1 \cdots d_l)$. Then d_{l+1} is an odd primitive $k/(k_1 \cdots k_l)$ -perfect number. Note that $d_{l+1} \geq 2$ and those d_i are co-prime. So

$$\begin{aligned} kn = \sigma(n) &= \sigma(d_1) \cdots \sigma(d_l)\sigma(d_{l+1}) = k_1d_1 \cdots k_ld_l \cdot \sigma(d_{l+1}) \\ &> (k_1 \cdots k_l) \cdot (d_1 \cdots d_l) \cdot d_{l+1} = (k_1 \cdots k_l) \cdot n, \end{aligned}$$

that is, $k_1 \cdots k_l < k$. Since $k_i \geq 2$, we must have $l \leq \log_2(k-1)$.

For any term (k_1, \dots, k_l) , by the auxiliary lemma of Chen and Luo, the number of odd primitive $k/(k_1 \cdots k_l)$ -perfect $d_{l+1} \leq x$ is at most

$$I_{k/(k_1 \cdots k_l),r}(x) \leq \frac{2.62(\log x)^r}{(k/(k_1 \cdots k_l))^2 - 1} \leq \frac{2.62(\log x)^r}{(k/(k-1))^2 - 1} = \frac{2.62(k-1)^2}{2k-1} (\log x)^r.$$

And for each $1 \leq i \leq l$, the number of odd primitive k_i -perfect $d_i \leq x$ is at most $0.02(\log x)^r$. Hence the number of all odd nonprimitive k -perfect $n \leq x$ is bounded by

$$\frac{2.62(k-1)^2}{2k-1} (\log x)^r \cdot (0.02(\log x)^r)^{\log_2(k-1)} \sum_{\substack{2 \leq k_1 \leq \dots \leq k_l \\ k_1 \cdots k_l < k}} 1.$$

On the other hand, for an integer $m \geq 2$, let the multiplicative partition function $\mathfrak{f}(m)$ denote the number of ways of factorising m into a product of integers greater than 1,

where the order of those factors is ignored. In [2], Chen proved that $\tilde{f}(m) \leq m/4 + 1$ for all m . Thus, for $k \geq 3$, using a simple induction,

$$\sum_{\substack{2 \leq k_1 \leq \dots \leq k_l \\ k_1 \dots k_l < k}} 1 = \sum_{2 \leq m \leq k-1} \left(\sum_{\substack{2 \leq k_1 \leq \dots \leq k_l \\ k_1 \dots k_l = m}} 1 \right) = \sum_{2 \leq m \leq k-1} \tilde{f}(m) \leq \frac{(k-1)^2 - 1}{2}.$$

So the number of odd k -perfect numbers in $[1, x]$ is not greater than

$$\begin{aligned} & \frac{2.62(k-1)^2}{2k-1} (\log x)^r \cdot (0.02(\log x)^r)^{\log_2(k-1)} \cdot \left(1 + \sum_{2 \leq m \leq k-1} \tilde{f}(m) \right) \\ & \leq \frac{(k-1)^2 + 1}{2} \cdot \frac{2.62(k-1)^2}{2k-1} \cdot (\log x)^{r(1+\log_2(k-1))} \\ & \leq (k-1)^3 \cdot (\log x)^{r(1+\log_2(k-1))}. \end{aligned} \tag{1}$$

Finally, Nielsen [7, Theorem 1] has proved that any odd k -perfect number n with $\omega(n) = r$ does not exceed 2^{4^r} . Substituting $x = 2^{4^r}$ in (1), we get that the number of odd k -perfect numbers with r distinct prime factors is bounded by

$$(k-1)^3 \cdot (\log 2^{4^r})^{r(1+\log_2(k-1))} \leq 4^{r^2} (k-1)^{2r^2+3}.$$

The proof is complete.

Acknowledgement

We are grateful to the anonymous referee for his/her very useful comments and helpful suggestions on our paper.

References

- [1] S.-C. Chen and H. Luo, ‘Bounds for odd k -perfect numbers’, *Bull. Aust. Math. Soc.* **84**(3) (2011), 475–480.
- [2] W.-L. Chen, ‘Upper bound of the number of multiplicative partitions’, *Acta Math. Sinica (Chin. Ser.)* **32** (1989), 604–609 (in Chinese).
- [3] R. J. Cook, ‘Bounds for odd perfect numbers’, in: *Number Theory (Ottawa, ON, 1996)*, 67–71, CRM Proc. Lecture Notes, 19 (American Mathematical Society, Providence, RI, 1999).
- [4] L. E. Dickson, ‘Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors’, *Amer. J. Math.* **35** (1913), 413–422.
- [5] D. R. Heath-Brown, ‘Odd perfect numbers’, *Math. Proc. Cambridge Philos. Soc.* **115** (1994), 191–196.
- [6] P. J. McCarthy, ‘Note on perfect and multiply perfect numbers’, *Portugal Math.* **16** (1957), 19–21.
- [7] P. P. Nielsen, ‘An upper bound for odd perfect numbers’, *Integers* **3** (2003), A14, 9pp (electronic).
- [8] P. Pollack, ‘On Dickson’s theorem concerning odd perfect numbers’, *Amer. Math. Monthly* **118** (2011), 161–164.
- [9] C. Pomerance, ‘Multiply perfect numbers, Mersenne primes, and effective computability’, *Math. Ann.* **226** (1977), 195–206.

LI-XIA DAI, School of Mathematical Sciences, Nanjing Normal University,
Nanjing 210046, PR China

e-mail: lilidainjnu@163.com

HAO PAN, Department of Mathematics, Nanjing University, Nanjing 210093,
PR China

e-mail: haopan79@yahoo.com.cn

CUI-E TANG, School of Mathematical Sciences, Nanjing Normal University,
Nanjing 210046, PR China

e-mail: tanguie2008@126.com