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# ANALYTIC CYCLES AND GENERICALLY FINITE HOLOMORPHIC MAPS

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We study the behaviour of analytic cycles under generically finite holomorphic mappings between compact analytic spaces and prove that if two compact and normal complex analytic spaces have the same analytic homology groups, then any generically one to one holomorphic map between them must be a biholomorphic mapping. This generalises an old theorem of Ax and Borel.

## 1. INTRODUCTION AND STATEMENT OF RESULT

Ax [1] and Borel [2] have shown that an injective morphism from a variety $X$ to $X$ is necessarily surjective. In the case when $X$ is normal, this implies that an injective morphism is actually an automorphism. In this note, we give a generalisation of their result when $X$ is a compact complex analytic space.

**THEOREM.** (See Theorem 2.) *Suppose $X$ and $Z$ are reduced, compact complex analytic spaces satisfying*

(i) $H_*^{an}(X,Q) \cong H_*^{an}(Z,Q)$;
(ii) *the fundamental class of every closed analytic subspace of $X$ is nonzero;*
(iii) *$Z$ is normal and irreducible.*

*Then any generically one to one holomorphic mapping $f : X \to Z$ is a biholomorphic mapping.*

This theorem is obtained as a corollary of a more general result about the behaviour of analytic cycles under a generically finite morphism. It is known that all complex analytic spaces are triangulable (see [5]), so every compact complex analytic space $X$ has finite dimensional singular homology groups. A closed analytic subspace $Y$ defines a homology class with rational coefficients $y \in H_{2p}(X,Q)$, where $p$ is the complex dimension of $Y$. The element $y$ is called the fundamental class of $Y$. The vector subspace of $H_{2p}(X,Q)$ generated by all these fundamental classes will be called the $p$-dimensional analytic homology group. We denote this subspace by $H_{2p}^{an}(X,Q)$. Just like Chow groups in the category of algebraic varieties, $H_{2p}^{an}(X,Q)$ are the most appropriate analytic invariants to study in the context of analytic spaces.

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The Stein factorisation theorem deals with the structure of proper analytic maps between complex analytic spaces. It says that any proper holomorphic map $f : X \to Z$ has a unique factorisation as a composition of maps

$$X \xrightarrow{g} Y \xrightarrow{h} Z,$$

where $Y$ is a complex analytic space; $g$ is proper, surjective holomorphic map with connected fibres; $h$ is finite (see [**3**]).

This theorem enables one to give a better description of the resulting maps on analytic cycles.

## 2. Generically finite analytic maps

Definition 1: A holomorphic map of complex analytic spaces $\phi : X \to Z$ is said to be generically finite if there is a proper closed analytic subspace $V \subset Z$ such that for every $z \in Z - V$, the fibre $\phi^{-1}(z)$ is finite.

If $V = \emptyset$ and $\phi$ is a closed map, then $\phi$ is called a finite map.

Remark. In the case of a generic finite map, it can be proved that for generic $z \in Z$, the cardinality of $f^{-1}(z)$ is independent of $z$. This cardinality is called the degree of the map.

In the case when both $X$ and $Z$ are algebraic varieties, the degree of $\phi$ is simply the degree of the finite extension of algebraic function fields of $X$ and $\phi(X)$.

**Proposition 1.** *If $f : X \to Y$ is a surjective, finite holomorphic map, then the induced map $H_*^{an}(X,Q) \to H_*^{an}(Y,Q)$ is surjective.*

Proof: In fact, let $c$ be the fundamental class of a closed analytic subspace $C \subset Y$. Let $p = \dim(C)$. Then $f^{-1}(C)$ is a closed analytic subspace of $X$ of the same dimension and the induced map from $f^{-1}(C)$ to $C$ is again a finite map. We have the following digram:

$$\begin{array}{ccc} H_{2p}(f^{-1}(C),Q) & \longrightarrow & H_{2p}(C,Q) \\ \downarrow & & \downarrow \\ H_{2p}(X,Q) & \longrightarrow & H_{2p}(Y,Q) \end{array}$$

The map of the top arrow is simply multiplication by $deg(f)$. Thus the fundamental class of $f^{-1}(C)$ maps $deg(f) \cdot c$. From this it is clear that $c$ lies in the image of $H_{2p}^{an}(X,Q)$. This finishes the proof. ∎

**Theorem 1.** *Suppose that $X$ and $Z$ are compact analytic spaces satisfying that*

(i) $H_*^{an}(X,Q) \cong H_*^{an}(Z,Q)$;

(ii) *the fundamental class of every closed analytic subspace of $X$ is nontrivial.*

*Then any generically finite, surjective holomorphic map from $X$ to $Z$ is finite.*

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REMARK. If $X$ is a compact Kähler manifold or projective algebraic variety, then condition (ii) is satisfied.

PROOF: First of all, since $X$ is compact, any holomorphic map from $X$ to $Z$ is proper. Using the Stein Factorisation Theorem (see [**3**, p.213]) we can factor $f$ uniquely as the composition of $g$ and $h$: $f : X \xrightarrow{g} Y \xrightarrow{h} Z$, where $g$ is a proper surjective holomorphic mapping with connected fibres, and $h$ is a finite, surjective holomorphic mapping. This implies that $g$ is generically finite of degree 1. What we need to show is that $g$ is injective.

Let $A \subset X$ be the locus of those points $x \in X$ for which $\dim\left(g^{-1}(g(x))\right) > 0$. It is well-known that $A$ is a closed analytic subspace and hence by Remmert's proper mapping theorem its image $B = g(A)$ is an analytic subspace of $X$.

CLAIM. $A$ is empty.

Otherwise we have $\dim(A) > \dim(B)$. Let $p = \dim(A)$ and $q = \dim(B)$. Consider the induced homomorphism $H_{2p}^{an}(X,Q) \to H_{2p}^{an}(Y,Q)$. Since $g : X \to Y$ is a generically one to one, surjective holomorphic mapping, each $p$-dimensional complex analytic cycle on $Y$ is the image of its pull-back cycle on $X$, that is, $H_{2p}^{an}(X,Q) \to H_{2p}^{an}(Y,Q)$ is surjective. On the other hand, since $h : Y \to Z$ is a finite, surjective mapping, we see the induced homomorphism $H_{2p}^{an}(Y,Q) \to H_{2p}^{an}(Z,Q)$ is surjective by Proposition 1. Using assumption (i), we see $H_{2p}^{an}(X,Q) \cong H_{2p}^{an}(Y,Q) \cong H_{2p}^{an}(Z,Q)$. However, the fundamental cycle of $A$ is mapped to 0 because $\dim(B) < \dim(A)$. This contradicts the assumption (ii). The claim is proved.

Finally, since $f$ is the composition of two finite map, it is finite. ∎

THEOREM 2. *Suppose $X$ and $Z$ are reduced, compact complex analytic spaces satisfying*

(i) $H_*^{an}(X,Q) \cong H_*^{an}(Z,Q)$;
(ii) *the fundamental class of every closed analytic subspace of $X$ is nonzero.*
(iii) *$Z$ is normal and irreducible;*

*Then any generically one to one holomorphic mapping $f : X \to Z$ is a biholomorphic mapping.*

PROOF: As in the proof of the Theorem 1, we factorise $f$ as a composition of $g$ and $h$. Then $g$ is bijective. We claim that $h$ must be a bijective map also. Indeed, $f(X) = h(Y)$ is a closed analytic subspace of the same dimension as $Z$, so the irreducibility of $Z$ implies that $f(X) = h(Y) = Z$. Namely, $h$ is surjective. We have to show $h$ is injective. Otherwise, there will exist a point $Q \in Z$ such that $h^{-1}(Q) = \{P_1, P_2, \cdots, P_s\}$, $s > 1$. Taking pair wise disjoint open neighbourhoods $U_1, U_2, \cdots, U_n$ for $P_1, P_2, \cdots, P_n$ respectively, we can form the intersection $h(U_1) \cap h(U_2) \cap \cdots \cap h(U_n)$. It is known that a finite mapping to a normal complex space is open (see [**3**, p.107]), so this intersection is an open neighbourhood of $P$ each of whose points has $s$ inverse image points. But this is impossible because $f$ is generically one to one.

Thus $f$ is bijective and open, thus a homeomorphism.

Finally, since $Z$ is normal, any homeomorphism from $X$ to $Z$ is a biholomorphic mapping. (See [4, p.310].) $\square$

REMARK. If one assumes merely that $Z$ is maximal, the last statement in the theorem remains true. For a discussion of maximal complex structures on complex analytic spaces, see [**4**].

REMARK. The theorem implies that after proper modification or blowing-up the analytic homology groups of a complex analytic space changes. Actually, if $M$ is a compact complex manifold of dimension $n$ and $M'$ is the blowing-up of $M$ at a point, then $H_{2n-2}(M', Q) \cong H_{2n-2}(M, Q) \oplus Q$.

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